Existence over a predicate
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An innocent question

Throughout this talk, L is a countable first-order language. Suppose that T is an L-complete theory with a predicate P. Let $\mathfrak{C}$ be a monster model for T. The *induced structure on* P is the structure $P(\mathfrak{C})_{\text{ind}}$ whose universe is $P(\mathfrak{C})$ with the language L_P consisting of a relation $R_{\phi(x)}$ for each L-formula $\phi(x)$, where $R_{\phi(x)}^{P_{\text{ind}}} = \{a \in P(\mathfrak{C})^x \mid \mathfrak{C} \vDash \phi(a)\}$. Let T_P be the complete theory of $P(\mathfrak{C})_{\text{ind}}$ in the language L_P. Clearly, if $M \vDash T$ then $P(M)_{\text{ind}}$ is an L_P-structure and $P(M)_{\text{ind}} \vDash T_P$.

Question (Existence)

Suppose that $N \vDash T_P$, is there some $M \vDash T$ such that $P(M)_{\text{ind}} = N$?

Example

Let $T = \text{Th}(\mathbb{R}, \mathbb{Q}, <)$ where P is a predicate for the rationals. Then T_P is essentially DLO and $(\mathbb{R}, <) \vDash T_P$. However, there is no model $M \vDash T$ such that $P(M) = (\mathbb{R}, <)$.

Stability, NIP, and existence

If T is stable, then the answer to the existence question over P is positive (using the density of locally isolated types). Similarly, if T is NIP and P is stable (equivalently T_P is stable), then the answer to the existence question is positive (using density of compressible types, [BKS25]).
Instead of assuming stability of P, what if we assume just that P is *stably embedded* in T?

Definition

P is (uniformly) *stably embedded* in T if for $a \in \mathfrak{C}$, the $\mathrm{tp}(a/P)$ is definable. In other words, for every formula $\phi(x, y)$, there is a formula $\psi(y, z)$ such that for every $a \in \mathfrak{C}^x$, there is some $d \in P^y$ such that $\phi(a, P) = \psi(P, d)$.

Question

Suppose that P is stably embedded. Does existence over P hold? What if we assume that T is NIP?

A counterexample

The answer to both questions is negative.
If N is a countable model of T_P, then the existence property holds by the omitting types theorem. A slightly more complicated argument, still using the omitting type theorem shows that the same is true if N is of size $\leq \aleph_1$. So any counterexample must be of size at least $\aleph_2$.

Example (Hrushovski)

Let $L = \{H, C, Q, f, g\}$ where H, C, Q are unary predicates and f, g are unary function symbols, and let T be the model completion of: H is the universe, $f : [H]^2 \to C$ with no homogeneous triple and $g : H \to Q$. Let $P = Q \cup C$. Then, P is stably embedded and the induced structure $P(\mathfrak{C})_{\text{ind}}$ is just equality. However, by Erdös-Rado, we cannot find a model $M \vDash T$ such that $|H(M)| = \aleph_0$ and $|Q(M)| = (2^{\aleph_0})^+$.

This example is not NIP, and not simple but is NSOP_1.

Question

What if we assume that T is simple?

An NIP example

Here is a counterexample to the existence question over a stably embedded predicate in an NIP theory. Let $L = \{R, S, K, H, C, f, h, c, \wedge, \leq\}$ where R, S, K, H, C are unary predicates, and $f : R \times R \to K$, $h : R \to H$, and $c : S \to C$ are functions. On R and S there is a structure of a meet tree with a predicate for the top S. On H there is a linear order, and h is increasing: $x < y \Rightarrow h(x) < h(y)$ in the tree. The function f satisfies: it is only defined on pairs where $x < y$ and if $x < y < z$ then $f(x, y) = f(x, z)$, and: if $a \neq b$ in P are not comparable then $f(a \wedge b, a) \neq f(a \wedge b, b)$. On the function c there are no restrictions. Let T be the model companion of this theory. Let $P = C \cup H \cup K$. The induced structure on P is just equality on C, K and the (dense without endpoints) order on H. Now, if N is a model of the theory of the predicate, where $|K| = |H| = \aleph_0$ and $|C| = (2^{\aleph_0})^+$, then S^N has size $> 2^{\aleph_0}$. We can put a coloring on pairs of distinct elements from S. The color of a pair $\{a, b\}$ is the pair $(h(a \wedge b), \{f(a \wedge b, a), f(a \wedge b, b)\})$ (consisting of an element from H together with a pair of elements from K). Since there are only countably many colors, by Erdös-Rado there is a homogeneous set of size 3: a, b, c. Since $h(a \wedge b) = h(a \wedge c) = h(c \wedge b)$ and h is a height function, we can let $m := a \wedge b = a \wedge c = b \wedge c$. Then $f(m, a)$ is in $\{f(m, b), f(m, c)\}$, and in either case we get a contradiction to the condition on f.

The Gaifman conjecture

Let us move to positive results, starting with a conjecture of Gaifman and a theorem of Shelah.

Definition

T is *relatively categorical with respect to* P if whenever $M, N \models T$ and $\sigma : P(M) \to P(N)$ is an isomorphism (as L-substructures) then σ extends to an isomorphism from M to N.

Conjecture (Gaifman, 1974, [Gai74])

Let T be a complete theory with a predicate P defining a substructure. If T is relatively categorical with respect to P, then the answer to the existence question is positive.

Theorem (Shelah, 1986, [She86])

A slightly weaker version of the Gaifman conjecture holds: if T is countable and absolutely relatively categorical, meaning that it remains relatively categorical even in generic extension of the set-theoretic universe, then the answer to the existence question is positive.

The proof of this theorem has two highly non-trivial ingredients: a structure side and a non-structure side.

P-niceness

Let T be a complete theory in a countable language L.
Let P be a stably embedded predicate. Write P^{eq} for $\mathrm{dcl}^{\mathrm{eq}}(P)$.
For a set A, let $P(A) := P^{\mathrm{eq}} \cap \mathrm{acl}^{\mathrm{eq}}(A)$.

Definition

T is *P-nice*™ if for any b and b-indiscernible sequence $(a_i)_{i\in\mathbb{Q}}$,
$P(ba_0) \subseteq P(P(ba_{<0}) \cup P(a_{\leq 0}))$.

Roughly, the idea is that the predicate behaves nicely with respect to algebraic closure.

Theorem (Bays, K., Simon)

If T is P-nice, then the answer to the existence question is positive.

Theorem (Bays, K., Simon)

Every Rosy theory is P-nice for every stably embedded predicate P, and thus the same is true for all simple theories.

Thank you!
Happy 80th birthday, Saharon! May you live to be 130!

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