Maximal Semigraphoids

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Purpose

Semigraphoids are combinatorial structures arising from models of probabilistic conditional independence. We investigate the classification of maximal semigraphoids on n random variables to better understand these sets.

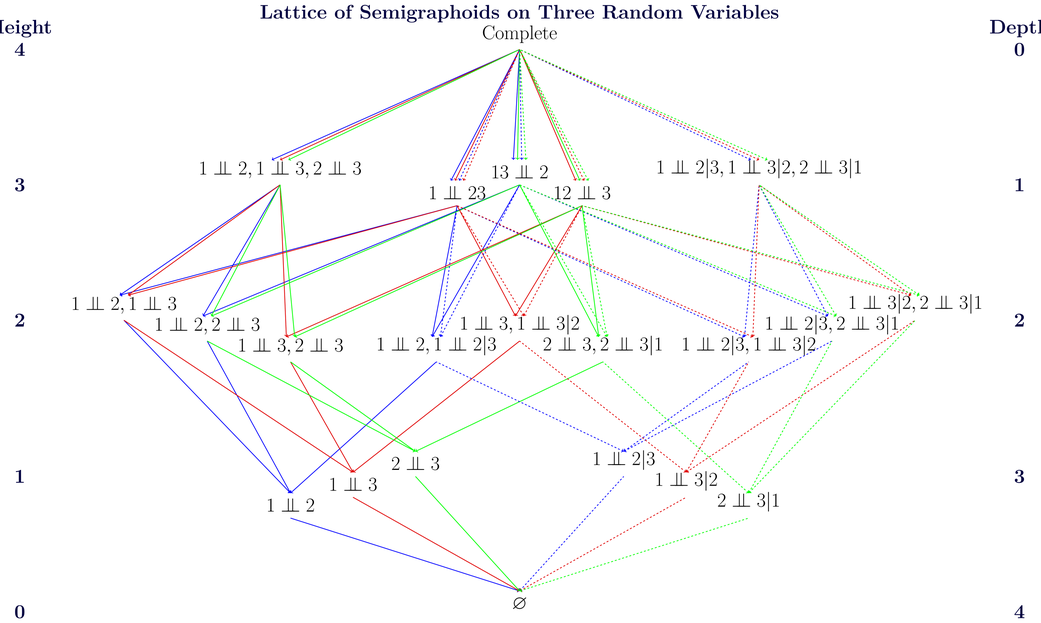
- Let A, B, C be pairwise disjoint subsets of the set $[n] = \{1, \ldots, n\}$. We regard statements of the form $A \perp\!\!\!\perp B|C$ as triplets over $[n]$.

What is a Semigraphoid?

A *semigraphoid* $\mathcal{S}$ is a set of triplets that satisfy the following for every pairwise disjoint $A, B, C, D \subset [n]$:

1. Symmetry: If $A \perp\!\!\!\perp B|C \in \mathcal{S}$, then $B \perp\!\!\!\perp A|C \in \mathcal{S}$.
2. Decomposition: If $A \perp\!\!\!\perp BC|D \in \mathcal{S}$, then $A \perp\!\!\!\perp B|D \in \mathcal{S}$.
3. Weak Union: If $A \perp\!\!\!\perp BC|D \in \mathcal{S}$, then $A \perp\!\!\!\perp B|CD \in \mathcal{S}$.
4. Contraction: If $\{A \perp\!\!\!\perp B|CD,\ A \perp\!\!\!\perp C|D\} \subset \mathcal{S}$, then $A \perp\!\!\!\perp BC|D \in \mathcal{S}$. [1,4]

Lattice of Semigraphoids on Three Random Variables



Semigraphoids as a Lattice

- With the binary relation $\subseteq$, the set of all semigraphoids on n random variables is a lattice [4]. Let $\mathcal{S}$ and $\mathcal{T}$ be semigraphoids. Then the join of $\mathcal{S}$ and $\mathcal{T}$ is $\langle \mathcal{S} \cup \mathcal{T} \rangle$ and the meet is precisely $\mathcal{S} \cap \mathcal{T}$.
- Statements of the form $a \perp\!\!\!\perp b|C$ where $a, b \in [n]$ and $C \subseteq [n] \setminus \{a, b\}$ are the building blocks of the lattice of semigraphoids on n random variables. They are the *atoms*.
- The lattice of semigraphoids is *atomic*. That is, every semigraphoid is generated by a join of atoms [3].
- We define an atom $a \perp\!\!\!\perp b|C$ to be a *k-atom* if $|C| = k$. The set $\{k - \text{atoms}\}$ is the set of all k-atoms in $CI(n)$ for a certain cardinality k.

Maximality of non-k-atoms

Theorem. If $A \subseteq \langle \{l - \text{atoms} : l < k\} \rangle$ and $B \subseteq \langle \{l - \text{atoms} : l > k\} \rangle$, then $\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle$.

Corollary. Given any integer $k \leq n - 2$ and generating set of atoms $\mathcal{C}$. A semigraphoid $\mathcal{S} = \langle \mathcal{C} \rangle$ contains a k-atom if and only if there is at least one k-atom in $\mathcal{C}$.

Theorem. The semigraphoid $\langle \cup_{l \in \{1,\ldots,n-2\} \setminus k} \{l - \text{atoms}\} \rangle$ is maximal for any $k \in \{1, \ldots, n-2\}$. This semigraphoid is notated $\langle \text{non} - k - \text{atoms} \rangle$ and called the *non-k-atoms*.

- Let $a \perp\!\!\!\perp b|C$ be any k-atom for $k < n - 2$. Then we have $\langle a \perp\!\!\!\perp b|C \cup \{(k+1) - \text{atoms}\} \rangle = \langle \{x \perp\!\!\!\perp y|C : \text{for all } x, y \in [n] \setminus C\} \cup \{(k+1) - \text{atoms}\} \rangle$.
- Let $a \perp\!\!\!\perp b|C$ be any k-atom for $k < n - 2$. Then we have $\langle a \perp\!\!\!\perp b|C \cup \{(k-1) - \text{atoms}\} \rangle = \langle \{x \perp\!\!\!\perp y|Z : xyZ = abC\} \cup \{(k-1) - \text{atoms}\} \rangle$.

Direct Sums

Corollary. Any maximal semigraphoid that is not equal to $\langle \text{non} - k - \text{atoms} \rangle$ for any k, must have at least one l-atom for every $0 \leq l \leq n - 2$.

Let A and B be two disjoint sets of random variables. Given two semigraphoids S, T such that $S \subseteq CI(A)$ and $T \subseteq CI(B)$, we define the direct sum [3] of S and T to be the semigraphoid:

$$S \oplus T = \langle S \cup T \cup \{A \perp\!\!\!\perp B\} \rangle.$$

The direct sum operation is associative. For any semigraphoids $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ on disjoint sets we have

$$\mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U}) = (\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U}.$$

Lemma. Let N be an index set and $A \subset N$. The direct sum of the complete model on A and the complete model on $N \setminus A$ is $CI(N)$.

Theorem. Let N be an index set and $A \subset N$. The direct sum of a maximal semigraphoid on A and the complete model on $N \setminus A$ is a maximal semigraphoid on N.

The following direct sums are maximal on four random variables:

1. $\langle a \perp\!\!\!\perp bc \rangle \oplus \langle \varnothing \rangle_{\{d\}} = \langle \varnothing \rangle_{\{a\}} \oplus \langle \varnothing \rangle_{\{b,c\}} \oplus \langle \varnothing \rangle_{\{d\}} = \langle a \perp\!\!\!\perp d \rangle \oplus \langle \varnothing \rangle_{\{b,c\}}$
2. $\langle a \perp\!\!\!\perp b, a \perp\!\!\!\perp c, b \perp\!\!\!\perp c \rangle \oplus \langle \varnothing \rangle_{\{d\}}$
3. $\langle a \perp\!\!\!\perp b|c, a \perp\!\!\!\perp c|b, b \perp\!\!\!\perp c|a \rangle \oplus \langle \varnothing \rangle_{\{d\}}$

Maximal on Four Random Variables

All maximal semigraphoids on four random variables up to permutation (for arbitrary $a, b, c, d \in [4]$):

For $k \in \{0, 1, 2\}$, the first three maximal semigraphoids are the non-k-atoms.

4. $\langle \cup_{k \in \{0,1\}} \{k - \text{atoms}\} \rangle$
5. $\langle \cup_{k \in \{0,2\}} \{k - \text{atoms}\} \rangle$
6. $\langle \cup_{k \in \{1,2\}} \{k - \text{atoms}\} \rangle$

The last four follow definite patterns:

7. $\langle CI(3) \cup \{2 - \text{atoms}\} \rangle = \langle \begin{bmatrix} a \perp\!\!\!\perp b \\ a \perp\!\!\!\perp c \\ b \perp\!\!\!\perp c \end{bmatrix}, \begin{bmatrix} a \perp\!\!\!\perp b|c \\ a \perp\!\!\!\perp c|b \\ b \perp\!\!\!\perp c|a \end{bmatrix}, \{2 - \text{atoms}\} \rangle$
8. $\langle \{0 - \text{atoms}\}, \begin{bmatrix} a \perp\!\!\!\perp b|c \\ a \perp\!\!\!\perp c|b \\ b \perp\!\!\!\perp c|a \end{bmatrix}, \begin{bmatrix} a \perp\!\!\!\perp b|cd \\ a \perp\!\!\!\perp c|bd \\ b \perp\!\!\!\perp c|ad \end{bmatrix} \rangle$

Observe that these last two are formed by strategically choosing a *chain* of atoms without consequences.

	0-atoms	1-atoms	2-atoms
9	$\langle\ a \perp\!\!\!\perp b,$	$\sim \begin{bmatrix} a \perp\!\!\!\perp x\|b \\ b \perp\!\!\!\perp x\|a \end{bmatrix}_{x \in [4] \setminus ab},$	$c \perp\!\!\!\perp d\|ab\ \rangle$
10	$\langle\ \sim a \perp\!\!\!\perp b,$	$\begin{bmatrix} a \perp\!\!\!\perp x\|b \\ b \perp\!\!\!\perp x\|a \end{bmatrix}_{x \in [4] \setminus ab},$	$\sim c \perp\!\!\!\perp d\|ab\ \rangle$

Algorithm Sketch

An algorithm for finding all maximal semigraphoids on n random variables begins with the non-k-atoms. For every set of non-k-atoms, we add a k-atom to the set while simultaneously choosing either $(k-1)$- or $(k+1)$-atoms to remove that *interact* with that k-atom (have consequences). Unless our maximal semigraphoid is a chain of atoms that do not interact with one another, they will be composed of sets of k-atoms for every k except two consecutive, say l and $l+1$.

Outlook

- A *graphoid* $\mathcal{G}$ is a semigraphoid that satisfies a fifth axiom for every pairwise disjoint $A, B, C, D \subset [n]$: Intersection: If $\{A \perp\!\!\!\perp B|CD, A \perp\!\!\!\perp C|BD\} \subset \mathcal{G}$, then $A \perp\!\!\!\perp BC|D \in \mathcal{G}$. [1]
- As graphoids are semigraphoids by definition, we extend our study to these structures. Out of the ten types of maximal semigraphoids classified above, the semigraphoids of type 1,2, and 4 are all of the maximal graphoids.
- Semigraphoids on n discrete random variables define algebraic varieties [1,2]. We plan to further investigate the conditional independence ideals of these maximal sets. Addressing computational complications that the contraction axiom presents.

References

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