Maximal Semigraphoids

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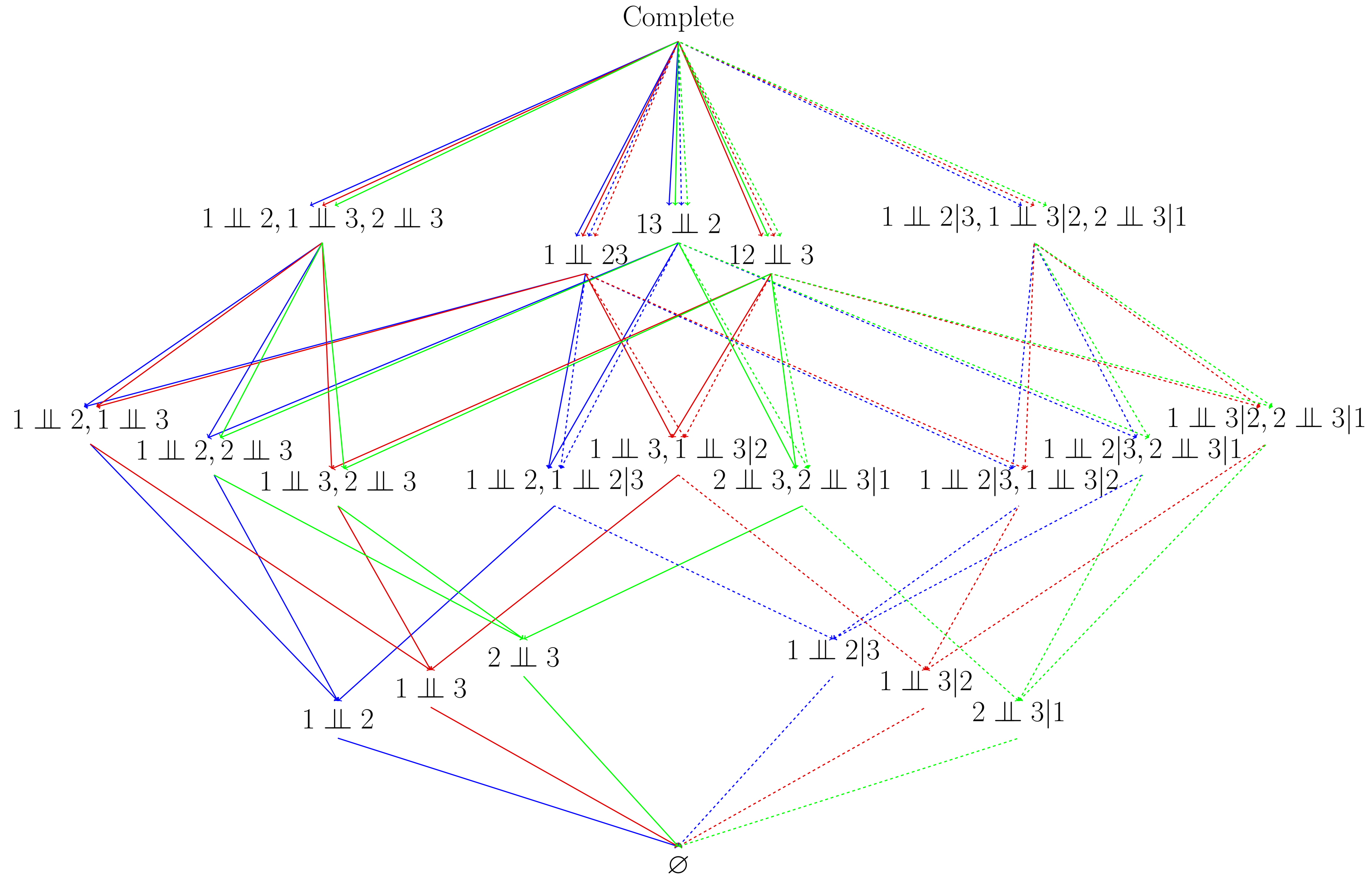
Purpose

Semigraphoids are combinatorial structures arising from models of probabilistic conditional independence. By studying the characterizations of maximal semigraphoids, we highlight the construction of two very different classes.

What is a Semigraphoid?

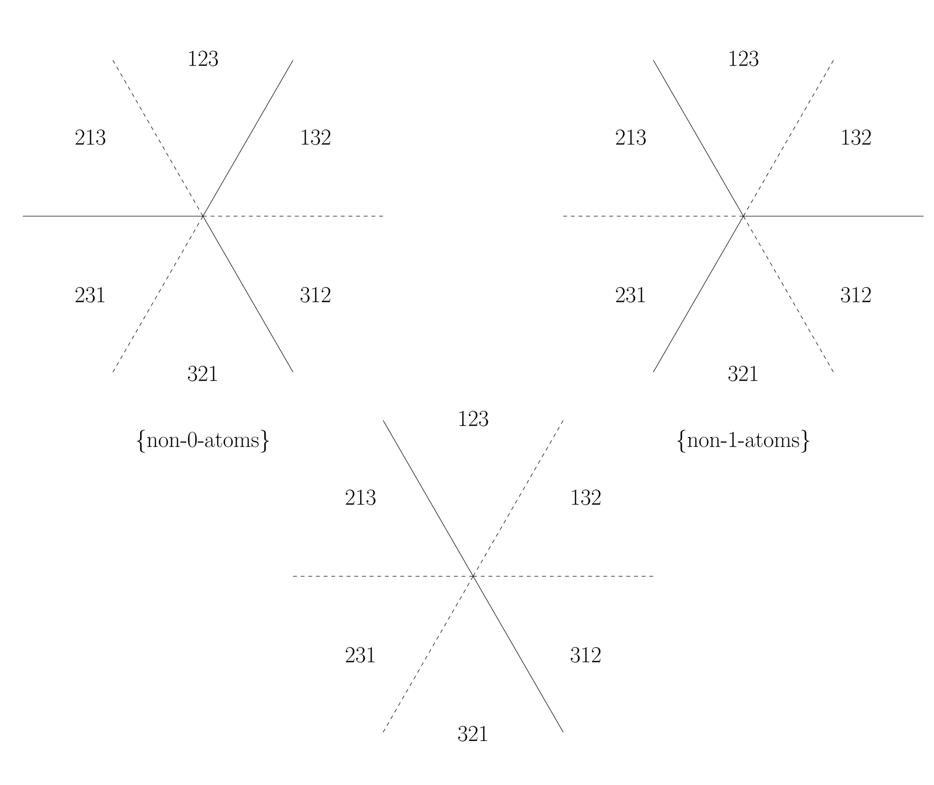
Let $a, b, c \in [n] = \{1, \dots, n\}$. A *semigraphoid* $\mathcal{S}$ is a set of triplets of the form $a \perp\!\!\!\perp b | K$, for $K \subseteq [n] \setminus ab$, that satisfy the following axiom:

$$\{a \perp\!\!\!\perp b|cK, a \perp\!\!\!\perp c|K\} \subset \mathcal{S} \Leftrightarrow \{a \perp\!\!\!\perp c|bK, a \perp\!\!\!\perp b\} \subset \mathcal{S}.$$



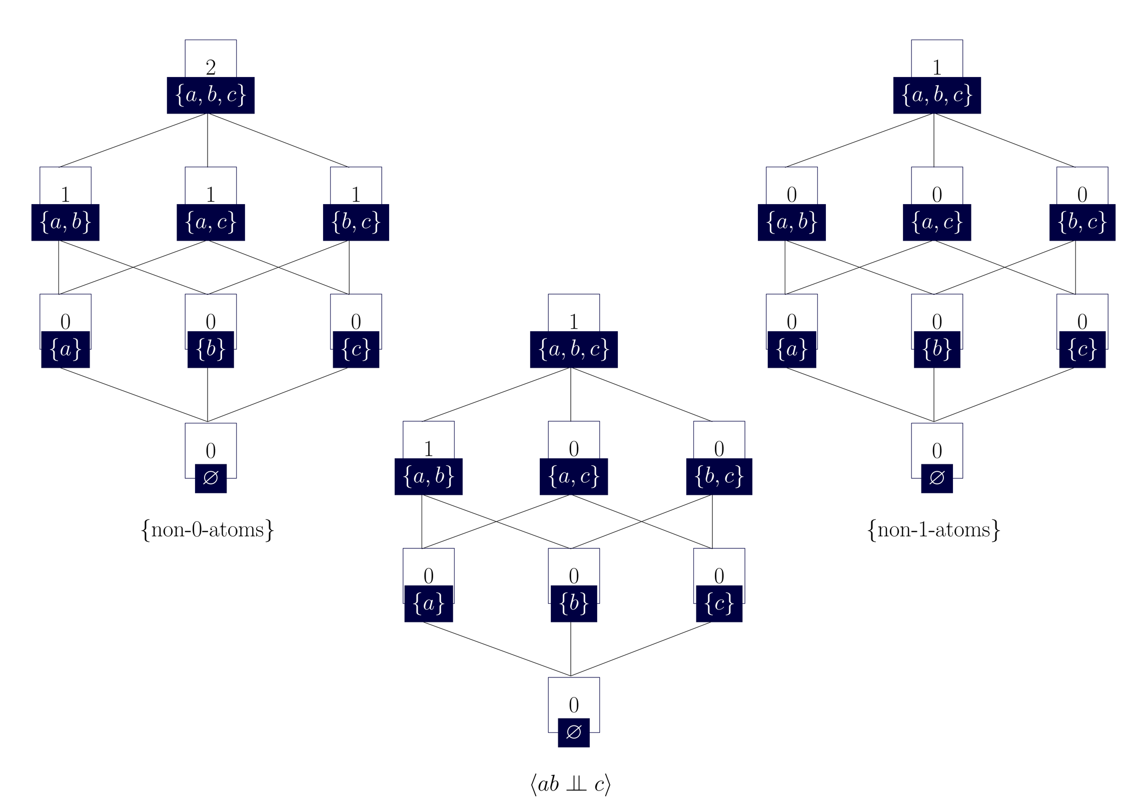
Convex Rank Tests

- The S_n-*fan* in $\mathcal{R}^n$ is defined by the planes $x_i = x_j$.
- A *convex rank test* gives a coarsening of the S_n-fan. A fan achieved by erasing rays of the S_n-fan.
- There is a bijection between convex rank tests and semigraphoids. [2]



Structural Imsets

- An *imset over* $[n]$ is an element of $\mathbb{Z}^{\mathcal{P}([n])}$.
- The imset $u_{a \perp\!\!\!\perp b|K} = \delta_{abK} + \delta_K - \delta_{aK} - \delta_{bK}$.
- For $m, u \in \mathbb{Z}^{\mathcal{P}([n])}$, the l-standardized imset of the semigraphoid u is $\langle m, u \rangle = \sum_{s \subseteq [n]} m_s u_s$. [3]
- The semigraphoid $\mathcal{M}^m$ is defined to be $\mathcal{M}^m = \{u_{a \perp\!\!\!\perp b|K} : \langle m, u_{a \perp\!\!\!\perp b|K} \rangle = 0\}$.



Classification on Four Random Variables

- *Proposition.* For any $k \in [n-2]$, the semigraphoid $\cup_{l \in \{1,\dots,n-2\} \setminus k} \{l - \text{atoms}\}$ is maximal. We call this semigraphoid the *non-k-atoms.*

1. non-0-atoms $= \cup_{k \in \{1,2\}} \{k\text{-atoms}\}$
2. non-1-atoms $= \cup_{k \in \{0,2\}} \{k\text{-atoms}\}$
3. non-2-atoms $= \cup_{k \in \{0,1\}} \{k\text{-atoms}\}$

- Let $A \subseteq [n]$ and $B = [n] \setminus A$. Given two semigraphoids $\mathcal{S} \subseteq CI(A), \mathcal{T} \subseteq CI(B)$, we define the *direct sum* of S and T to be the semigraphoid:
  $\mathcal{S} \oplus \mathcal{T} = \langle \mathcal{S} \cup \mathcal{T} \cup \{a \perp\!\!\!\perp b|K : a \in A, b \in B, K \subset AB \setminus ab\} \rangle.$
- *Proposition.* Let N be an index set and $A \subset N$. The direct sum of a maximal semigraphoid on A and the complete model on $N \setminus A$ is maximal on N.

4. $\langle ab \perp\!\!\!\perp c \rangle \oplus \langle \varnothing \rangle_{\{d\}}$
5. $\langle a \perp\!\!\!\perp b, a \perp\!\!\!\perp c, b \perp\!\!\!\perp c \rangle \oplus \langle \varnothing \rangle_{\{d\}}$
6. $\langle a \perp\!\!\!\perp b|c, a \perp\!\!\!\perp c|b, b \perp\!\!\!\perp c|a \rangle \oplus \langle \varnothing \rangle_{\{d\}}$

Complementary Maximals

- Given a semigraphoid $\mathcal{S}$, let $\mathcal{S}_k = \mathcal{S} \cap \{k\text{-atoms}\}$ for each k. Define its *complement* $\mathcal{S}^C$ by
  $$(\mathcal{S}^C)_k = \begin{cases} \{k\text{-atoms}\} & \text{if } \mathcal{S}_k = \{k\text{-atoms}\} \\ \{k\text{-atoms}\} \setminus \mathcal{S}_k & \text{else.} \end{cases}$$
- *Proposition.* The complement of the direct sum $\{\text{non-}k\text{-atoms}\} \oplus \varnothing_n$ is maximal.

7. $\left\langle \begin{Bmatrix} a \perp\!\!\!\perp b \\ a \perp\!\!\!\perp c \\ b \perp\!\!\!\perp c \end{Bmatrix}, \begin{Bmatrix} a \perp\!\!\!\perp b|c \\ a \perp\!\!\!\perp c|b \\ b \perp\!\!\!\perp c|a \end{Bmatrix}, \{2-\text{atoms}\} \right\rangle$
8. $\left\langle \{0-\text{atoms}\}, \begin{Bmatrix} a \perp\!\!\!\perp b|c \\ a \perp\!\!\!\perp c|b \\ b \perp\!\!\!\perp c|a \end{Bmatrix}, \begin{Bmatrix} a \perp\!\!\!\perp b|cd \\ a \perp\!\!\!\perp c|bd \\ b \perp\!\!\!\perp c|ad \end{Bmatrix} \right\rangle$

- The last two are complements and can be found by strategically choosing atoms without consequences.

	0-atoms	1-atoms	2-atoms
9.	$\langle \; a \perp\!\!\!\perp b,$	$\sim\begin{Bmatrix} a \perp\!\!\!\perp x|b \\ b \perp\!\!\!\perp x|a \end{Bmatrix}_{x \in [4] \setminus ab},$	$c \perp\!\!\!\perp d|ab \; \rangle$
10.	$\langle \; \sim\{a \perp\!\!\!\perp b\},$	$\begin{Bmatrix} a \perp\!\!\!\perp x|b \\ b \perp\!\!\!\perp x|a \end{Bmatrix}_{x \in [4] \setminus ab},$	$\sim\{c \perp\!\!\!\perp d|ab\} \; \rangle$

Connections with Imsets

- *Proposition.* The imset for $\{\text{non-}k\text{-atoms}\}$ is
  $$\sum_{|s| \geq k+2} \delta_s(|s| - k - 1).$$
- *Proposition.* The direct sum of $\{\text{non-}k\text{-atoms}\}$ on $[n-1]$ and $\varnothing_n$ is:
  $$\{\text{non-}k\text{-atoms}\}_{[n-1]} \oplus \varnothing_n = CI(n) \setminus \cup_{a \perp\!\!\!\perp b|K} \{a \perp\!\!\!\perp b|K, a \perp\!\!\!\perp b|nK\}.$$
- *Proposition.* Given the semigraphoid $\{\text{non-}k\text{-atoms}\}$ on $[n]$, we have:
  $$\{\text{non-}k\text{-atoms}\}_{[n]} \oplus \varnothing_{n+1} \cdots \oplus \varnothing_{n+r} = CI(n+r) \setminus \cup_{a \perp\!\!\!\perp b|K, k\text{-atom on } [n]} \{a \perp\!\!\!\perp b|KT : T \subseteq \{n+1, \dots, n+r\}\}.$$
- *Proposition.* Suppose $\mathcal{M}^m$ is maximal. Then $\mathcal{M}^m \oplus \varnothing_n = \mathcal{M}^{\overline{m}}$ where $\overline{m} = \sum_{s \subseteq [n-1]} m_s(\delta_s + \delta_{sn})$.

Five Random Variables

- From [3], there are 1319 maximal submodular semigraphoids up to permutation. We have found:
- 10 direct sums
- 4 sets of non-k-atoms
- 3 complements of direct sums on non-k-atoms
- 4 non-submodular that include the set $\{0\text{-atoms}, 2\text{-atoms}\}$.

References

[1] F. Matus, *Towards Classification of Semigraphoids*, Discrete Mathematics, **277**, pp. 115-145, 2004.

[2] J. Morton, L.Pachter, A. Shiu, B. Sturmfels, O. Wienand, *Convex Rank Tests and Semigraphoids*, SIAM Journal on Discrete Mathematics, **23**, pp. 1117-1134, 2009.

[3] M. Studeny, *Probabilistic Conditional Independence Structures*, Springer Series in Information Science and Statistics, Springer-Verlag, London, UK, 2005.

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