

# MORE ON CARDINAL INVARIANTS OF ANALYTIC P-IDEALS

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ABSTRACT. Given an ideal  $\mathcal{I}$  on  $\omega$  let  $\mathfrak{a}(\mathcal{I})$  ( $\bar{\mathfrak{a}}(\mathcal{I})$ ) be minimum of the cardinalities of infinite (uncountable) maximal  $\mathcal{I}$ -almost disjoint subsets of  $[\omega]^\omega$ . We show that  $\mathfrak{a}(\mathcal{I}_h) > \omega$  if  $\mathcal{I}_h$  is a summable ideal; but  $\mathfrak{a}(\mathcal{Z}_{\bar{\mu}}) = \omega$  for any tall density ideal  $\mathcal{Z}_{\bar{\mu}}$  including the density zero ideal  $\mathcal{Z}$ . On the other hand, you have  $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$  for any analytic  $P$ -ideal  $\mathcal{I}$ , and  $\bar{\mathfrak{a}}(\mathcal{Z}_{\bar{\mu}}) \leq \mathfrak{a}$  for each density ideal  $\mathcal{Z}_{\bar{\mu}}$ .

For each ideal  $\mathcal{I}$  on  $\omega$  denote  $\mathfrak{b}_{\mathcal{I}}$  and  $\mathfrak{d}_{\mathcal{I}}$  the unbounding and dominating numbers of  $\langle \omega^\omega, \leq_{\mathcal{I}} \rangle$  where  $f \leq_{\mathcal{I}} g$  iff  $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$ . We show that  $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$  and  $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$  for each analytic  $P$ -ideal  $\mathcal{I}$ .

Given a Borel ideal  $\mathcal{I}$  on  $\omega$  we say that a poset  $\mathbb{P}$  is  $\mathcal{I}$ -*bounding* iff  $\forall x \in \mathcal{I} \cap V^{\mathbb{P}} \exists y \in \mathcal{I} \cap V x \subseteq y$ .  $\mathbb{P}$  is  $\mathcal{I}$ -*dominating* iff  $\exists y \in \mathcal{I} \cap V^{\mathbb{P}} \forall x \in \mathcal{I} \cap V x \subseteq^* y$ .

For each analytic  $P$ -ideal  $\mathcal{I}$  if a poset  $\mathbb{P}$  has the Sacks property then  $\mathbb{P}$  is  $\mathcal{I}$ -bounding; moreover if  $\mathcal{I}$  is tall as well then the property  $\mathcal{I}$ -bounding/ $\mathcal{I}$ -dominating implies  $\omega^\omega$ -bounding/adding dominating reals, and the converses of these two implications are false.

Using David Fremlin's results we prove that if  $\mathbb{P}$  adds a slalom capturing all ground model reals then  $\mathbb{P}$  is  $\mathcal{I}$ -dominating for each analytic  $P$ -ideal  $\mathcal{I}$ , and that a poset  $\mathbb{P}$  is  $\mathcal{Z}$ -bounding iff it has the Sacks property.

## 1. INTRODUCTION

In this paper we investigate some properties of some cardinal invariants associated with analytic  $P$ -ideals. Moreover we analyze related “bounding” and “dominating” properties of forcing notions.

Let us denote  $\mathfrak{fin}$  the Frechet ideal on  $\omega$ , i.e.  $\mathfrak{fin} = [\omega]^{<\omega}$ . Further we always assume that if  $\mathcal{I}$  is an ideal on  $\omega$  then the ideal is *proper*, i.e.  $\omega \notin \mathcal{I}$ , and  $\mathfrak{fin} \subseteq \mathcal{I}$ , so especially  $\mathcal{I}$  is *non-principal*. Write  $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$  and  $\mathcal{I}^* = \{\omega \setminus X : X \in \mathcal{I}\}$ .

An ideal  $\mathcal{I}$  on  $\omega$  is *analytic* if  $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$  is an analytic set in the usual product topology.  $\mathcal{I}$  is a *P-ideal* if for each countable  $\mathcal{C} \subseteq \mathcal{I}$  there is an  $X \in \mathcal{I}$  such that  $Y \subseteq^* X$  for each  $Y \in \mathcal{C}$ , where  $A \subseteq^* B$  iff  $A \setminus B$  is finite.  $\mathcal{I}$  is *tall* (or *dense*) if each infinite subset of  $\omega$  contains an infinite element of  $\mathcal{I}$ .

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A function  $\varphi : \mathcal{P}(\omega) \rightarrow [0, \infty]$  is a *submeasure on  $\omega$*  iff  $\varphi(X) \leq \varphi(Y)$  for  $X \subseteq Y \subseteq \omega$ ,  $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$  for  $X, Y \subseteq \omega$ , and  $\varphi(\{n\}) < \infty$  for  $n \in \omega$ . A submeasure  $\varphi$  is *lower semicontinuous* iff  $\varphi(X) = \lim_{n \rightarrow \infty} \varphi(X \cap n)$  for each  $X \subseteq \omega$ . A submeasure  $\varphi$  is *finite* if  $\varphi(\omega) < \infty$ . Note that if  $\varphi$  is a lower semicontinuous submeasure on  $\omega$  then  $\varphi(\bigcup_{n \in \omega} A_n) \leq \sum_{n \in \omega} \varphi(A_n)$  holds as well for  $A_n \subseteq \omega$ . We assign the *exhaustive ideal*  $\text{Exh}(\varphi)$  to a submeasure  $\varphi$  as follows

$$\text{Exh}(\varphi) = \{X \subseteq \omega : \lim_{n \rightarrow \infty} \varphi(X \setminus n) = 0\}.$$

Solecki, [So, Theorem 3.1], proved that an ideal  $\mathcal{I} \subseteq \mathcal{P}(\omega)$  is an analytic  $P$ -ideal or  $\mathcal{I} = \text{Exh}(\varphi)$  for some lower semicontinuous finite submeasure. Therefore each analytic  $P$ -ideal is  $F_{\sigma\delta}$  (i.e.  $\Pi_3^0$ ) so a Borel subset of  $2^\omega$ . It is straightforward to see that if  $\varphi$  is a lower semicontinuous finite submeasure on  $\omega$  then the ideal  $\text{Exh}(\varphi)$  is tall iff  $\lim_{n \rightarrow \infty} \varphi(\{n\}) = 0$ .

Let  $\mathcal{I}$  be an ideal on  $\omega$ . A family  $\mathcal{A} \subseteq \mathcal{I}^+$  is  $\mathcal{I}$ -almost-disjoint ( $\mathcal{I}$ -AD in short), if  $A \cap B \in \mathcal{I}$  for each  $\{A, B\} \in [\mathcal{A}]^2$ . An  $\mathcal{I}$ -AD family  $\mathcal{A}$  is an  $\mathcal{I}$ -MAD family if for each  $X \in \mathcal{I}^+$  there exists an  $A \in \mathcal{A}$  such that  $X \cap A \in \mathcal{I}^+$ , i.e.  $\mathcal{A}$  is  $\subseteq$ -maximal among the  $\mathcal{I}$ -AD families.

Denote  $\mathfrak{a}(\mathcal{I})$  the minimum of the cardinalities of infinite  $\mathcal{I}$ -MAD families. In Theorem 2.2 we show that  $\mathfrak{a}(\mathcal{I}_h) > \omega$  if  $\mathcal{I}_h$  is a summable ideal; but  $\mathfrak{a}(\mathcal{Z}_{\bar{\mu}}) = \omega$  for any tall density ideal  $\mathcal{Z}_{\bar{\mu}}$  including the *density zero ideal*

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

On the other hand, if you define  $\bar{\mathfrak{a}}(\mathcal{I})$  as minimum of the cardinalities of uncountable  $\mathcal{I}$ -MAD families then you have  $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$  for any analytic  $P$ -ideal  $\mathcal{I}$ , and  $\bar{\mathfrak{a}}(\mathcal{Z}_{\bar{\mu}}) \leq \mathfrak{a}$  for each density ideal  $\mathcal{Z}_{\bar{\mu}}$  (see Theorems 2.6 and 2.8).

In Theorem 3.1 we prove under  $\text{CH}$  the existence of an uncountable Cohen-indestructible  $\mathcal{I}$ -MAD families for each analytic  $P$ -ideal  $\mathcal{I}$ .

A sequence  $\langle A_\alpha : \alpha < \kappa \rangle \subset [\omega]^\omega$  is a *tower* if it is  $\subseteq^*$ -descending, i.e.  $A_\beta \subseteq^* A_\alpha$  if  $\alpha \leq \beta < \kappa$ , and it has no *pseudointersection*, i.e. a set  $X \in [\omega]^\omega$  such that  $X \subseteq^* A_\alpha$  for each  $\alpha < \kappa$ . In Section 4 we show it is consistent that the continuum is arbitrarily large and for each tall analytic  $P$ -ideal  $\mathcal{I}$  there is towers of height  $\omega_1$  whose elements are in  $\mathcal{I}^*$ .

Given an ideal  $\mathcal{I}$  on  $\omega$  if  $f, g \in \omega^\omega$  write  $f \leq_{\mathcal{I}} g$  if  $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$ . As usual let  $\leq^* = \leq_{\text{fin}}$ . The unbounding and dominating numbers of the partially ordered set  $\langle \omega^\omega, \leq_{\mathcal{I}} \rangle$ , denoted by  $\mathfrak{b}_{\mathcal{I}}$  and  $\mathfrak{d}_{\mathcal{I}}$  are defined in the natural way, i.e.  $\mathfrak{b}_{\mathcal{I}}$  is the minimal size of a  $\leq_{\mathcal{I}}$ -unbounded family, and  $\mathfrak{d}_{\mathcal{I}}$  is the minimal size of a  $\leq_{\mathcal{I}}$ -dominating family. By these notations  $\mathfrak{b} = \mathfrak{b}_{\text{fin}}$  and  $\mathfrak{d} = \mathfrak{d}_{\text{fin}}$ . In Section 5 we show that  $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$  and  $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$  for each analytic  $P$ -ideal  $\mathcal{I}$ . We also prove, in Corollary 6.8, that for any analytic  $P$ -ideal  $\mathcal{I}$  a poset  $\mathbb{P}$  is  $\leq_{\mathcal{I}}$ -bounding iff it is  $\omega^\omega$ -bounding, and  $\mathbb{P}$  adds  $\leq_{\mathcal{I}}$ -dominating reals iff it adds dominating reals.

In Section 6 we introduce the  $\mathcal{I}$ -bounding and  $\mathcal{I}$ -dominating properties of forcing notions for Borel ideals:  $\mathbb{P}$  is  $\mathcal{I}$ -*bounding* iff any element of  $\mathcal{I} \cap V^{\mathbb{P}}$  is contained in some element of  $\mathcal{I} \cap V$ ;  $\mathbb{P}$  is  $\mathcal{I}$ -*dominating* iff there is an element in  $\mathcal{I} \cap V^{\mathbb{P}}$  which mod-finite contains all elements of  $\mathcal{I} \cap V$ .

In Theorem 6.2 we show that for each tall analytic P-ideal  $\mathcal{I}$  if a forcing notion is  $\mathcal{I}$ -bounding then it is  $\omega^\omega$ -bounding, and if it is  $\mathcal{I}$ -dominating then it adds dominating reals. Since the random real forcing is not  $\mathcal{I}$ -bounding for each tall summable and tall density ideal  $\mathcal{I}$  by Proposition 6.3, the converse of the first implication is false. Since a  $\sigma$ -centered forcing can not be  $\mathcal{I}$ -dominating for a tall analytic P-ideal  $\mathcal{I}$  by Theorem 6.4, the standard dominating real forcing  $\mathbb{D}$  witnesses that the converse of the second implication is also false.

We prove in Theorem 6.5 that the Sacks property implies the  $\mathcal{I}$ -bounding property for each analytic P-ideal  $\mathcal{I}$ .

Finally, based on a theorem of Fremlin we show that the  $\mathcal{Z}$ -bounding property is equivalent to the Sacks property.

## 2. AROUND THE ALMOST DISJOINTNESS NUMBER OF IDEALS

For any ideal  $\mathcal{I}$  on  $\omega$  denote  $\mathfrak{a}(\mathcal{I})$  the minimum of the cardinalities of infinite  $\mathcal{I}$ -MAD families.

To start the investigation of this cardinal invariant we recall the definition of two special classes of analytic P-ideals: the density ideals and the summable ideals (see [Fa]).

**Definition 2.1.** Let  $h : \omega \rightarrow \mathbb{R}^+$  be a function such that  $\sum_{n \in \omega} h(n) = \infty$ . The *summable ideal corresponding to  $h$*  is

$$\mathcal{I}_h = \left\{ A \subseteq \omega : \sum_{n \in A} h(n) < \infty \right\}.$$

Let  $\langle P_n : n < \omega \rangle$  be a decomposition of  $\omega$  into pairwise disjoint nonempty finite sets and let  $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$  be a sequences of probability measures,  $\mu_n : \mathcal{P}(P_n) \rightarrow [0, 1]$ . The *density ideal generated by  $\vec{\mu}$*  is

$$\mathcal{Z}_{\vec{\mu}} = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \mu_n(A \cap P_n) = 0 \right\}.$$

A summable ideal  $\mathcal{I}_h$  is tall iff  $\lim_{n \rightarrow \infty} h(n) = 0$ ; and a density ideal  $\mathcal{Z}_{\vec{\mu}}$  is tall iff

$$(\dagger) \quad \lim_{n \rightarrow \infty} \max_{i \in P_n} \mu_n(\{i\}) = 0.$$

Clearly the density zero ideal  $\mathcal{Z}$  is a tall density ideal, and the summable and the density ideals are proper ideals.

**Theorem 2.2.** (1)  $\mathfrak{a}(\mathcal{I}_h) > \omega$  for any summable ideal  $\mathcal{I}_h$ .

(2)  $\mathfrak{a}(\mathcal{Z}_{\vec{\mu}}) = \omega$  for any tall density ideal  $\mathcal{Z}_{\vec{\mu}}$ .

*Proof.* (1): We show that if  $\{A_n : n < \omega\} \subseteq \mathcal{I}_h^+$  is  $\mathcal{I}$ -AD then there is  $B \in \mathcal{I}_h^+$  such that  $B \cap A_n \in \mathcal{I}$  for  $n \in \omega$ .

For each  $n \in \omega$  let  $B_n \subseteq A_n \setminus \cup\{A_m : m < n\}$  be finite such that  $\sum_{i \in B_n} h(i) > 1$ , and put

$$B = \cup\{B_n : n \in \omega\}.$$

(2): Write  $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$  and  $\mu_n$  concentrates on  $P_n$ . By  $(\dagger)$  we have  $\lim_{n \rightarrow \infty} |P_n| = \infty$ .

Now for each  $n$  we can choose  $k_n \in \omega$  and a partition  $\{P_{n,k} : k < k_n\}$  of  $P_n$  such that

- (a)  $\lim_{n \rightarrow \infty} k_n = \infty$ ,
- (b) if  $k < k_n$  then  $\mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$ .

Put  $A_k = \cup\{P_{n,k} : k < k_n\}$  for each  $k \in \omega$ . We show that  $\{A_k : k \in \omega\}$  is a  $\mathcal{Z}_{\vec{\mu}}$ -MAD family.

If  $k_n > k$  then  $\mu_n(A_k \cap P_n) = \mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$ . Since for an arbitrary  $k$  for all but finitely many  $n$  we have  $k_n > k$  it follows that

$$\limsup_{n \rightarrow \infty} \mu_n(A_k \cap P_n) = \limsup_{n \rightarrow \infty} \mu_n(P_{n,k}) \geq \limsup_{n \rightarrow \infty} \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}} > 0,$$

thus  $A_k \in \mathcal{Z}_{\vec{\mu}}^+$ .

Assume that  $X \in \mathcal{Z}_{\vec{\mu}}^+$ . Pick  $\varepsilon > 0$  with  $\limsup_{n \rightarrow \infty} \mu_n(X \cap P_n) > \varepsilon$ . For a large enough  $k$  we have  $\frac{1}{2^{k+1}} < \frac{\varepsilon}{2}$  so if  $k < k_n$  then

$$\mu_n(P_n \setminus \cup\{P_{n,i} : i \leq k\}) \leq \frac{1}{2^{k+1}} < \frac{\varepsilon}{2}.$$

So for each large enough  $n$  there is  $i_n \leq k$  such that  $\mu_n(X \cap P_{n,i_n}) > \frac{\varepsilon}{2^{(k+1)}}$ . Then  $i_n = i$  for infinitely many  $n$ , so  $\limsup_{n \rightarrow \infty} \mu_n(X \cap A_i) \geq \frac{\varepsilon}{2^{(k+1)}}$ , and so  $X \cap A_i \in \mathcal{Z}_{\vec{\mu}}^+$ .  $\square$

This Theorem gives new proof of the following well-known fact:

**Corollary 2.3.** *The density zero ideal  $\mathcal{Z}$  is not a summable ideal.*

Given two ideals  $\mathcal{I}$  and  $\mathcal{J}$  on  $\omega$  write  $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$  (see [Ru]) iff there is a function  $f : \omega \rightarrow \omega$  such that

$$\mathcal{I} = \{I \subseteq \omega : f^{-1}I \in \mathcal{J}\},$$

and write  $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$  (see [LaZh]) iff there is a finite-to-one function  $f : \omega \rightarrow \omega$  such that

$$\mathcal{I} = \{I \subseteq \omega : f^{-1}I \in \mathcal{J}\}.$$

The following Observations imply that there are  $\mathcal{I}$ -MAD families of cardinality  $\mathfrak{c}$  for each analytic P-ideal  $\mathcal{I}$ .

**Observation 2.4.** *Assume that  $\mathcal{I}$  and  $\mathcal{J}$  are ideals on  $\omega$ ,  $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$  witnessed by a function  $f : \omega \rightarrow \omega$ . If  $\mathcal{A}$  is an  $\mathcal{I}$ -AD family then  $\{f^{-1}A : A \in \mathcal{A}\}$  is a  $\mathcal{J}$ -AD family.*

**Observation 2.5.**  *$\text{fin} \leq_{\text{RB}} \mathcal{I}$  for any analytic P-ideal  $\mathcal{I}$ .*

*Proof.* Let  $\mathcal{I} = \text{Exh}(\varphi)$  for some lower semicontinuous finite submeasure  $\varphi$  on  $\omega$ . Since  $\omega \notin \mathcal{I}$  we have  $\lim_{n \rightarrow \infty} \varphi(\omega \setminus n) = \varepsilon > 0$ . Hence by the lower semicontinuous property of  $\varphi$  for each  $n > 0$  there is  $m > n$  such that  $\varphi([n, m]) > \varepsilon/2$ .

So there is a partition  $\{I_n : n < \omega\}$  of  $\omega$  into finite pieces such that  $\varphi(I_n) > \varepsilon/2$  for each  $n \in \omega$ . Define the function  $f : \omega \rightarrow \omega$  by the stipulation  $f''I_n = \{n\}$ . Then  $f$  witnesses  $\text{fin} \leq_{\text{RB}} \mathcal{I}$ .  $\square$

For any analytic P-ideal  $\mathcal{I}$  denote  $\bar{\mathfrak{a}}(\mathcal{I})$  the minimum of the cardinalities of uncountable  $\mathcal{I}$ -MAD families.

Clearly  $\mathfrak{a}(\mathcal{I}) > \omega$  implies  $\mathfrak{a}(\mathcal{I}) = \bar{\mathfrak{a}}(\mathcal{I})$ , especially  $\mathfrak{a}(\mathcal{I}_h) = \bar{\mathfrak{a}}(\mathcal{I}_h)$  for summable ideals.

**Theorem 2.6.**  $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \leq \mathfrak{a}$  for each density ideal  $\mathcal{Z}_{\vec{\mu}}$ .

*Proof.* Let  $f : \omega \rightarrow \omega$  be the finite-to-one function defined by  $f^{-1}\{n\} = P_n$  where  $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$  and  $\mu_n : \mathcal{P}(P_n) \rightarrow [0, 1]$ . Specially  $f$  witnesses  $\text{fin} \leq_{\text{RB}} \mathcal{Z}_{\vec{\mu}}$ .

Let  $\mathcal{A}$  be an uncountable (fin-)MAD family. We show that  $f^{-1}[\mathcal{A}] = \{f^{-1}A : A \in \mathcal{A}\}$  is a  $\mathcal{Z}_{\vec{\mu}}$ -MAD family.

By Observation 2.4,  $f^{-1}[\mathcal{A}]$  is a  $\mathcal{Z}_{\vec{\mu}}$ -AD family.

To show the maximality let  $X \in \mathcal{Z}_{\vec{\mu}}^+$  be arbitrary,  $\limsup_{n \rightarrow \infty} \mu_n(X \cap P_n) = \varepsilon > 0$ . Thus

$$J = \{n \in \omega : \mu_n(X \cap P_n) > \varepsilon/2\}$$

is infinite. So there is  $A \in \mathcal{A}$  such that  $A \cap J$  is infinite.

Then  $f^{-1}A \in f^{-1}[\mathcal{A}]$  and  $X \cap f^{-1}A \in \mathcal{Z}_{\vec{\mu}}^+$  because there are infinitely many  $n$  such that we have  $P_n \subseteq f^{-1}A$  and  $\mu_n(X \cap P_n) > \varepsilon/2$ .  $\square$

**Problem 2.7.** Does  $\bar{\mathfrak{a}}(\mathcal{I}) \leq \mathfrak{a}$  hold for each analytic P-ideal  $\mathcal{I}$ ?

**Theorem 2.8.**  $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$  provided that  $\mathcal{I}$  is an analytic P-ideal.

*Remark.* If  $\mathcal{X} \subset [\omega]^\omega$  is an infinite almost disjoint family then there is a tall ideal  $\mathcal{I}$  such that  $\mathcal{X}$  is  $\mathcal{I}$ -MAD. So the Theorem above does not hold for an arbitrary tall ideal on  $\omega$ .

*Proof.*  $\mathcal{I} = \text{Exh}(\varphi)$  for some lower semicontinuous finite submeasure  $\varphi$ .

Let  $\mathcal{A}$  be an uncountable  $\mathcal{I}$ -AD family of cardinality smaller than  $\mathfrak{b}$ . We show that  $\mathcal{A}$  is not maximal.

There exists an  $\varepsilon > 0$  such that the set

$$\mathcal{A}_\varepsilon = \{A \in \mathcal{A} : \lim_{n \rightarrow \infty} \varphi(A \setminus n) > \varepsilon\}$$

is uncountable. Let  $\mathcal{A}' = \{A_n : n \in \omega\} \subseteq \mathcal{A}_\varepsilon$  be a set of pairwise distinct elements of  $\mathcal{A}_\varepsilon$ . We can assume that these sets are pairwise disjoint. For each  $A \in \mathcal{A} \setminus \mathcal{A}'$  choose a function  $f_A \in \omega^\omega$  such that

$$(*_A) \varphi((A \cap A_n) \setminus f_A(n)) < 2^{-n} \text{ for each } n \in \omega.$$

Using the assumption  $|\mathcal{A}| < \mathfrak{b}$  there exists a strictly increasing function  $f \in \omega^\omega$  such that  $f_A \leq^* f$  for each  $A \in \mathcal{A} \setminus \mathcal{A}'$ . For each  $n$  pick  $g(n) > f(n)$  such that  $\varphi(A_n \cap [f(n), g(n))) > \varepsilon$ , and let

$$X = \bigcup_{n \in \omega} (A_n \cap [f(n), g(n))).$$

Clearly  $X \in \mathcal{Z}_{\bar{\mu}}^+$  because for each  $n < \omega$  there is  $m$  such that  $A_m \cap [f(m), g(m)) \subseteq X \setminus n$  and so  $\varphi(X \setminus n) \geq \varphi(A_m \cap [f(m), g(m))) > \varepsilon$ , i.e.  $\lim_{n \rightarrow \infty} \varphi(X \setminus n) \geq \varepsilon$ .

We have to show that  $X \cap A \in \mathcal{Z}_{\bar{\mu}}$  for each  $A \in \mathcal{A}$ . If  $A = A_n$  for some  $n$  then  $X \cap A = X \cap A_n = A_n \cap [f(n), g(n))$ , i.e. the intersection is finite.

Assume now that  $A \in \mathcal{A} \setminus \mathcal{A}'$ . Let  $\delta > 0$ . We show that if  $k$  is large enough then  $\varphi((X \cap A) \setminus k) < \delta$ .

There is  $N \in \omega$  such that  $2^{-N+1} < \delta$  and  $f_A(n) \leq f(n)$  for each  $n \geq N$ .

Let  $k$  be so large that  $k$  contains the finite set  $\bigcup_{n < N} [f(n), g(n))$ .

Now  $(X \cap A) \setminus k = \bigcup_{n \in \omega} (A_n \cap A \cap [f(n), g(n))) \setminus k$  and  $(A_n \cap A \cap [f(n), g(n))) \setminus k = \emptyset$  if  $n < N$  so

$$\begin{aligned} (X \cap A) \setminus k &= \bigcup_{n \geq N} (A_n \cap A \cap [f(n), g(n))) \setminus k \subseteq \\ &\bigcup_{n \geq N} ((A_n \cap A) \setminus f(n)) \subseteq \bigcup_{n \geq N} ((A_n \cap A) \setminus f_A(n)). \end{aligned}$$

Thus by  $(*_A)$  we have

$$\varphi((X \cap A) \setminus k) \leq \sum_{n \geq N} \varphi(A_n \cap A \setminus f_A(n)) \leq \sum_{n \geq N} \frac{1}{2^n} = 2^{-N+1} < \delta.$$

□

### 3. COHEN-INDESTRUCTIBLE $\mathcal{I}$ -MAD FAMILIES

If  $\varphi$  is a lower semicontinuous finite submeasure on  $\omega$  then clearly  $\varphi$  is determined by  $\varphi \upharpoonright [\omega]^{<\omega}$ . Using this observation one can define forcing indestructibility of  $\mathcal{I}$ -MAD families for an analytic P-ideal  $\mathcal{I}$ . The following Theorem is a modification of Kunen's proof for existence of Cohen-indestructible MAD family from CH (see [Ku] Ch. VIII Th. 2.3.).

**Theorem 3.1.** *Assume CH. For each analytic P-ideal  $\mathcal{I}$  then there is an uncountable Cohen-indestructible  $\mathcal{I}$ -MAD family.*

*Proof.* We will define the uncountable Cohen-indestructible  $\mathcal{I}$ -MAD family  $\{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+$  by recursion on  $\xi \in \omega_1$ . The family  $\{A_\xi : \xi < \omega_1\}$  will be fin-AD as well. Our main concern is that we do have  $\mathfrak{a}(\mathcal{I}) > \omega$  so it is not automatic that  $\{A_\eta : \eta < \xi\}$  is not maximal for  $\xi < \omega_1$ .

Denote  $\mathbb{C}$  the Cohen forcing. Let  $\mathcal{I} = \text{Exh}(\varphi)$  be an analytic P-ideal. Let  $\{\langle p_\xi, \dot{X}_\xi, \delta_\xi \rangle : \omega \leq \xi < \omega_1\}$  be an enumeration of all triples  $\langle p, \dot{X}, \delta \rangle$  such

that  $p \in \mathbb{C}$ ,  $\dot{X}$  is a nice name for a subset of  $\omega$ , and  $\delta$  is a positive rational number.

Write  $\varepsilon = \lim_{n \rightarrow \infty} \varphi(\omega \setminus n) > 0$ . Partition  $\omega$  into infinite sets  $\{A_m : m < \omega\}$  such that  $\lim_{n \rightarrow \infty} \varphi(A_m \setminus n) = \varepsilon$  for each  $m < \omega$ .

Assume  $\xi \geq \omega$  and we have  $A_\eta \in \mathcal{I}^+$  for  $\eta < \xi$  such that  $\{A_\eta : \eta < \xi\}$  is a fin-AD so especially an  $\mathcal{I}$ -AD family.

**Claim:** There is  $X \in \mathcal{I}^+$  such that  $|X \cap A_\zeta| < \omega$  for  $\zeta < \xi$ .

*Proof of the Claim.* Write  $\xi = \{\zeta_i : i < \omega\}$ . Recursion on  $j \in \omega$  we can choose  $x_j \in [A_{\ell_j}]^{<\omega}$  for some  $\ell_j \in \omega$  such that

- (i)  $\varphi(x_j) \geq \varepsilon/2$ ,
- (ii)  $x_j \cap (\cup_{i \leq j} A_{\zeta_i}) = \emptyset$ .

Assume that  $\{x_i : i < j\}$  is chosen. Pick  $\ell_j \in \omega \setminus \{\zeta_i : i < j\}$ . Let  $m \in \omega$  such that  $A_{\ell_j} \cap \cup\{A_{\zeta_i} : i \leq j\} \subseteq m$ . Since  $\varphi(A_{\ell_j} \setminus m) \geq \varepsilon$  there is  $x_j \in [A_{\ell_j} \setminus m]^{<\omega}$  with  $\varphi(x_j) \geq \varepsilon/2$ .

Let  $X = \cup\{x_j : j < \omega\}$ . Then  $|A_\zeta \cap X| < \omega$  for  $\zeta < \xi$  and  $\lim_{n \rightarrow \infty} \varphi(X \setminus n) \geq \varepsilon/2$ .  $\square$

If  $p_\xi$  does not force (a) and (b) below then let  $A_\xi$  be  $X$  from the Claim.

- (a)  $\lim_{n \rightarrow \infty} \check{\varphi}(\dot{X}_\xi \setminus n) > \check{\delta}_\xi$ ,
- (b)  $\forall \eta < \xi \ \dot{X}_\xi \cap \check{A}_\eta \in \mathcal{I}$ .

Assume  $p_\xi \Vdash (a) \wedge (b)$ . Let  $\{B_k^\xi : k \in \omega\} = \{A_\eta : \eta < \xi\}$  and  $\{p_k^\xi : k \in \omega\} = \{p' \in \mathbb{C} : p' \leq p_\xi\}$  be enumerations. Clearly for each  $k \in \omega$  we have

$$p_k^\xi \Vdash \lim_{n \rightarrow \infty} \check{\varphi}((\dot{X}_\xi \setminus \cup\{\check{B}_l^\xi : l \leq \check{k}\}) \setminus n) > \check{\delta}_\xi,$$

so we can choose a  $q_k^\xi \leq p_k^\xi$  and a finite  $a_k^\xi \subseteq \omega$  such that  $\varphi(a_k^\xi) > \delta_\xi$  and  $q_k^\xi \Vdash \check{a}_k^\xi \subseteq (\dot{X}_\xi \setminus \cup\{\check{B}_l^\xi : l \leq \check{k}\}) \setminus \check{k}$ . Let  $A_\xi = \cup\{a_k^\xi : k \in \omega\}$ . Clearly  $A_\xi \in \mathcal{I}^+$  and  $\{A_\eta : \eta \leq \xi\}$  is a fin-AD family.

Thus  $\mathcal{A} = \{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+$  is a fin-AD family.

We show that  $\mathcal{A}$  is a Cohen-indestructible  $\mathcal{I}$ -MAD. Assume otherwise there is a  $\xi$  such that  $p_\xi \Vdash \lim_{n \rightarrow \infty} \check{\varphi}(\dot{X}_\xi \setminus n) > \check{\delta}_\xi \wedge \forall \eta < \omega_1 \ \dot{X}_\xi \cap \check{A}_\eta \in \mathcal{I}$ , specially  $p_\xi \Vdash (a) \wedge (b)$ . There is a  $p_k^\xi \leq p_\xi$  and an  $N$  such that  $p_k^\xi \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{N}) < \check{\delta}_\xi$ . We can assume  $k \geq N$ , so  $p_k^\xi \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{k}) < \check{\delta}_\xi$ . By the choice of  $q_k^\xi$  and  $a_k^\xi$  we have  $q_k^\xi \Vdash \check{a}_k^\xi \subseteq (\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{k}$ , so  $q_k^\xi \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{k}) > \check{\delta}_\xi$ , contradiction.  $\square$

#### 4. TOWERS IN $\mathcal{I}^*$

Let  $\mathcal{I}$  be an ideal on  $\omega$ . A  $\subseteq^*$ -decreasing sequence  $\langle A_\alpha : \alpha < \kappa \rangle$  is a *tower* in  $\mathcal{I}^*$  if (a) it is a tower (i.e. there is no  $X \in [\omega]^\omega$  with  $X \subseteq^* A_\alpha$  for  $\alpha < \kappa$ ), and (b)  $A_\alpha \in \mathcal{I}^*$  for  $\alpha < \kappa$ . Under CH it is straightforward to construct towers in  $\mathcal{I}^*$  for each tall analytic P-ideal  $\mathcal{I}$ . The existence of such towers is consistent with  $2^\omega > \omega_1$  as well by the Theorem 4.2 below. Denote  $\mathbb{C}_\alpha$  the standard forcing adding  $\alpha$  Cohen reals by finite conditions.

**Lemma 4.1.** *Let  $\mathcal{I} = \text{Exh}(\varphi)$  be a tall analytic P-ideal in the ground model  $V$ . Then there is a set  $X \in V^{\mathbb{C}_1} \cap \mathcal{I}$  such that  $|X \cap S| = \omega$  for each  $S \in [\omega]^\omega \cap V$ .*

*Proof.* Since  $\mathcal{I}$  is tall we have  $\lim_{n \rightarrow \infty} \varphi(\{n\}) = 0$ . Fix a partition  $\langle I_n : n \in \omega \rangle$  of  $\omega$  into finite intervals such that  $\varphi(\{x\}) < \frac{1}{2^n}$  for  $x \in I_{n+1}$  (we can not say anything about  $\varphi(\{x\})$  for  $x \in I_0$ ). Then  $X' \in \mathcal{I}$  whenever  $|X' \cap I_n| \leq 1$  for each  $n$ .

Let  $\{i_k^n : k < k_n\}$  be the increasing enumeration of  $I_n$ . Our forcing  $\mathbb{C}$  adds a Cohen real  $c \in \omega^\omega$  over  $V$ . Let

$$X_\alpha = \{i_k^n : c(n) \equiv k \pmod{k_n}\} \in V^{\mathbb{C}} \cap \mathcal{I}.$$

A trivial density argument shows that  $|X_\alpha \cap S| = \omega$  for each  $S \in V \cap [\omega]^\omega$ .  $\square$

**Theorem 4.2.**  $\Vdash_{\mathbb{C}_{\omega_1}}$  "There exists a tower in  $\mathcal{I}^*$  for each tall analytic P-ideal  $\mathcal{I}$ ."

*Proof.* Let  $V$  be a countable transitive model and  $G$  be a  $\mathbb{C}_{\omega_1}$ -generic filter over  $V$ . Let  $\mathcal{I} = \text{Exh}(\varphi)$  be a tall analytic P-ideal in  $V[G]$  with some lower semicontinuous finite submeasure  $\varphi$  on  $\omega$ . There is a  $\delta < \omega_1$  such that  $\varphi \upharpoonright [\omega]^{<\omega} \in V[G_\delta]$  where  $G_\delta = G \cap \mathbb{C}_\delta$ , so we can assume  $\varphi \upharpoonright [\omega]^{<\omega} \in V$ .

Work in  $V[G]$  recursion on  $\omega_1$  we construct the tower  $\bar{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$  in  $\mathcal{I}^*$  such that  $\bar{A} \upharpoonright \alpha \in V[G_\alpha]$ .

Because  $\mathcal{I}$  contains infinite elements we can construct in  $V$  a sequence  $\langle A_n : n \in \omega \rangle$  in  $\mathcal{I}^*$  which is strictly  $\subseteq^*$ -descending, i.e.  $|A_n \setminus A_{n+1}| = \omega$  for  $n \in \omega$ . Assume  $\langle A_\xi : \xi < \alpha \rangle$  are done.

Since  $\mathcal{I}$  is a P-ideal there is  $A'_\alpha \in \mathcal{I}^*$  with  $A'_\alpha \subseteq^* A_\beta$  for  $\beta < \alpha$ .

By lemma 4.1 there is a set  $X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}$  such that  $X_\alpha \cap S \neq \emptyset$  for each  $S \in [\omega]^\omega \cap V[G_\alpha]$ .

Let  $A_\alpha = A'_\alpha \setminus X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}^*$  so  $S \not\subseteq^* A_\alpha$  for any  $S \in V[G_\alpha] \cap [\omega]^\omega$ . Hence  $V[G] \models \langle A_\alpha : \alpha < \omega_1 \rangle$  is a tower in  $\mathcal{I}^*$ .  $\square$

**Problem 4.3.** Do there exist towers in  $\mathcal{I}^*$  for some tall analytic P-ideal  $\mathcal{I}$  in ZFC?

## 5. UNBOUNDING AND DOMINATING NUMBERS OF IDEALS

A *supported relation* (see [Vo]) is a triple  $\mathcal{R} = (A, R, B)$  where  $R \subseteq A \times B$ ,  $\text{dom}(R) = A$ ,  $\text{ran}(R) = B$ , and we always assume that for each  $b \in B$  there is an  $a \in A$  such that  $\langle a, b \rangle \notin R$ .

The *unbounding and dominating numbers of  $\mathcal{R}$* :

$$\mathfrak{b}(\mathcal{R}) = \min\{|A'| : A' \subseteq A \wedge \forall b \in B \ A' \not\subseteq R^{-1}\{b\}\},$$

$$\mathfrak{d}(\mathcal{R}) = \min\{|B'| : B' \subseteq B \wedge A = R^{-1}B'\}.$$

For example  $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$  and  $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$ . Note that  $\mathfrak{b}(\mathcal{R})$  and  $\mathfrak{d}(\mathcal{R})$  are defined for each  $\mathcal{R}$ , but in general  $\mathfrak{b}(\mathcal{R}) \leq \mathfrak{d}(\mathcal{R})$  does not hold.

We recall the definition of Galois-Tukey connection of relations.



**Definition 5.1.** ([Vo]) Let  $\mathcal{R}_1 = (A_1, R_1, B_1)$  and  $\mathcal{R}_2 = (A_2, R_2, B_2)$  be supported relations. A pair of functions  $\phi : A_1 \rightarrow A_2$ ,  $\psi : B_2 \rightarrow B_1$  is a *Galois-Tukey connection from  $\mathcal{R}_1$  to  $\mathcal{R}_2$* , in notation  $(\phi, \psi) : \mathcal{R}_1 \preceq \mathcal{R}_2$  if  $a_1 R_1 \psi(b_2)$  whenever  $\phi(a_1) R_2 b_2$ . In a diagram:

$$\begin{array}{ccc} \psi(b_2) \in B_1 & \xleftarrow{\psi} & B_2 \ni b_2 \\ R_1 & \longleftarrow & R_2 \\ a_1 \in A_1 & \xrightarrow{\phi} & A_2 \ni \phi(a_1) \end{array}$$

We write  $\mathcal{R}_1 \preceq \mathcal{R}_2$  if there is a Galois-Tukey connection from  $\mathcal{R}_1$  to  $\mathcal{R}_2$ . If  $\mathcal{R}_1 \preceq \mathcal{R}_2$  and  $\mathcal{R}_2 \preceq \mathcal{R}_1$  also hold then we say  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are *Galois-Tukey equivalent*, in notation  $\mathcal{R}_1 \equiv \mathcal{R}_2$ .

**Fact 5.2.** *If  $\mathcal{R}_1 \preceq \mathcal{R}_2$  then  $\mathfrak{b}(\mathcal{R}_1) \geq \mathfrak{b}(\mathcal{R}_2)$  and  $\mathfrak{d}(\mathcal{R}_1) \leq \mathfrak{d}(\mathcal{R}_2)$ .*

**Theorem 5.3.** *If  $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$  then  $(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$ .*

*Proof.* Fix a finite-to-one function  $f : \omega \rightarrow \omega$  witnessing  $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$ .

Define  $\phi, \psi : \omega^\omega \rightarrow \omega^\omega$  as follows:

$$\begin{aligned} \phi(x)(i) &= \max(x'' f^{-1}\{i\}), \\ \psi(y)(j) &= y(f(j)). \end{aligned}$$

We prove two claims.

**Claim 5.3.1.**  $(\phi, \psi) : (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$ .

*Proof of the claim.* We show that if  $\phi(x) \leq_{\mathcal{I}} y$  then  $x \leq_{\mathcal{J}} \psi(y)$ . Indeed,  $I = \{i : \phi(x)(i) > y(i)\} \in \mathcal{I}$ . Assume that  $f(j) = i \notin I$ . Then  $\phi(x)(i) = \max(x'' f^{-1}\{i\}) \leq y(i)$ . Since  $y(i) = \psi(y)(j)$ , so

$$x(j) \leq \max(x'' f^{-1}\{f(j)\}) \leq y(f(j)) = \psi(y)(j)$$

Since  $f^{-1}I \in \mathcal{J}$  this yields  $x \leq_{\mathcal{J}} \psi(y)$ .  $\square$

**Claim 5.3.2.**  $(\psi, \phi) : (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$ .

*Proof of the claim.* We show that if  $\psi(y) \leq_{\mathcal{J}} x$  then  $y \leq_{\mathcal{I}} \phi(x)$ . Assume on the contrary that  $y \not\leq_{\mathcal{I}} \phi(x)$ . Then  $A = \{i \in \omega : y(i) > \phi(x)(i)\} \in \mathcal{I}^+$ . By definition of  $\phi$ , we have  $A = \{i : y(i) > \max(x'' f^{-1}\{i\})\}$ .

Let  $B = f^{-1}A \in \mathcal{J}^+$ . For  $j \in B$  we have  $f(j) \in A$  and so

$$\psi(y)(j) = y(f(j)) > \phi(x)(f(j)) = \max(x'' f^{-1}\{f(j)\}) \geq x(j).$$

Hence  $\psi(y) \not\leq_{\mathcal{J}} x$ , contradiction.  $\square$

These claims prove the statement of the Theorem, so we are done.  $\square$

By Fact 5.2 we have:

**Corollary 5.4.** *If  $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$  holds then  $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}_{\mathcal{J}}$  and  $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}_{\mathcal{J}}$ .*

By Observation 2.5 this yields:

**Corollary 5.5.** *If  $\mathcal{I}$  is an analytic  $P$ -ideal then  $(\omega^\omega, \leq^*, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$ , and  $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$  and  $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$ .*

## 6. $\mathcal{I}$ -BOUNDING AND $\mathcal{I}$ -DOMINATING FORCING NOTIONS

**Definition 6.1.** Let  $\mathcal{I}$  be a Borel ideal on  $\omega$ . A forcing notion  $\mathbb{P}$  is  $\mathcal{I}$ -*bounding* if

$$\Vdash_{\mathbb{P}} \forall A \in \mathcal{I} \exists B \in \mathcal{I} \cap V \ A \subseteq B;$$

$\mathbb{P}$  is  $\mathcal{I}$ -*dominating* if

$$\Vdash_{\mathbb{P}} \exists B \in \mathcal{I} \forall A \in \mathcal{I} \cap V \ A \subseteq^* B.$$

**Theorem 6.2.** *Let  $\mathcal{I}$  be a tall analytic  $P$ -ideal. If  $\mathbb{P}$  is  $\mathcal{I}$ -bounding then  $\mathbb{P}$  is  $\omega^\omega$ -bounding as well; if  $\mathbb{P}$  is  $\mathcal{I}$ -dominating then  $\mathbb{P}$  adds dominating reals.*

*Proof.* Assume that  $\mathcal{I} = \text{Exh}(\varphi)$  for some lower semicontinuous finite submeasure  $\varphi$ . For  $A \in \mathcal{I}$  let

$$d_A(n) = \min\{k \in \omega : \varphi(A \setminus k) < 2^{-n}\}.$$

Clearly if  $A \subseteq B \in \mathcal{I}$  then  $d_A \leq d_B$ .

It is enough to show that  $\{d_A : A \in \mathcal{I}\}$  is cofinal in  $\langle \omega^\omega, \leq^* \rangle$ . Let  $f \in \omega^\omega$ . Since  $\mathcal{I}$  is a tall ideal we have  $\lim_{k \rightarrow \infty} \varphi(\{k\}) = 0$  but  $\lim_{m \rightarrow \infty} (\omega \setminus m) = \varepsilon > 0$ . Thus for all but finite  $n \in \omega$  we can choose a finite set  $A_n \subseteq \omega \setminus f(n)$  such that  $2^{-n} \leq \varphi(A_n) < 2^{-n+1}$  so  $A = \cup\{A_n : n \in \omega\} \in \mathcal{I}$  and  $f \leq^* d_A$ .

Why? We can assume if  $k \geq f(n)$  then  $\varphi(\{k\}) < 2^{-n}$ . Let  $n$  be so large such that  $2^{-n} < \varepsilon$ . Now if there is no a suitable  $A_n$  then  $\varphi(\omega \setminus f(n)) \leq 2^{-n} < \varepsilon$ , contradiction.  $\square$

The converse of the first implication of Theorem 6.2 is not true by the following Proposition.

**Proposition 6.3.** *The random forcing is not  $\mathcal{I}$ -bounding for any tall summable and tall density ideal  $\mathcal{I}$ .*

*Proof.* Denote  $\mathbb{B}$  the random forcing and  $\lambda$  the Lebesgue-measure.

If  $\mathcal{I} = \mathcal{I}_h$  is a tall summable ideal then we can chose pairwise disjoint sets  $H(n) \in [\omega]^\omega$  such that  $\sum_{l \in H(n)} h(l) = 1$  and  $\max\{h(l) : l \in H(n)\} < 2^{-n}$  for each  $n \in \omega$ . Let  $H(n) = \{l_k^n : k \in \omega\}$ . For each  $n$  fix a partition  $\{[B_k^n] : k \in \omega\}$  of  $\mathbb{B}$  such that  $\lambda(B_k^n) = h(l_k^n)$  for each  $k \in \omega$ . Let  $\dot{X}$  be a  $\mathbb{B}$ -name such that  $\Vdash_{\mathbb{B}} \dot{X} = \{\check{l}_k^n : [B_k^n] \in \dot{G}\}$ . Clearly  $\Vdash_{\mathbb{B}} \dot{X} \in \mathcal{I}_h$ .  $\dot{X}$  shows that  $\mathbb{B}$  is not  $\mathcal{I}_h$ -bounding.

Assume on the contrary that there is a  $[B] \in \mathbb{B}$  and an  $A \in \mathcal{I}_h$  such that  $[B] \Vdash \dot{X} \subseteq \check{A}$ . There is an  $n \in \omega$  such that

$$\sum_{l_k^n \in A} \lambda(B_k^n) = \sum_{l_k^n \in A} h(l_k^n) < \lambda(B).$$

Choose a  $k$  such that  $l_k^n \notin A$  and  $[B_k^n] \wedge [B] \neq [\emptyset]$ . We have  $[B_k^n] \wedge [B] \Vdash \check{l}_k^n \in \dot{X} \setminus \check{A}$ , contradiction.

If  $\mathcal{I} = \mathcal{Z}_{\bar{\mu}}$  is a tall density ideal then for each  $n$  fix a partition  $\{[B_k^n] : k \in P_n\}$  of  $\mathbb{B}$  such that  $\lambda(B_k^n) = \mu_n(\{k\})$  for each  $k$ . Let  $\dot{X}$  be a  $\mathbb{B}$ -name such that  $\Vdash_{\mathbb{B}} \dot{X} = \{\check{k} : [\check{B}_k^n] \in \dot{G}\}$ . Clearly  $\Vdash_{\mathbb{B}} \dot{X} \in \mathcal{Z}_{\bar{\mu}}$ .  $\dot{X}$  shows that  $\mathbb{B}$  is not  $\mathcal{Z}_{\bar{\mu}}$ -bounding.

Assume on the contrary that there is a  $[B] \in \mathbb{B}$  and an  $A \in \mathcal{Z}_{\bar{\mu}}$  such that  $[B] \Vdash \dot{X} \subseteq \check{A}$ . There is an  $n \in \omega$  such that

$$\sum_{k \in A \cap P_n} \lambda(B_k^n) = \mu_n(A \cap P_n) < \lambda(B).$$

Choose a  $k \in P_n \setminus A$  such that  $[B_k^n] \wedge [B] \neq [\emptyset]$ . We have  $[B_k^n] \wedge [B] \Vdash \check{k} \in \dot{X} \setminus \check{A}$ , contradiction.  $\square$

The converse of the second implication of Theorem 6.2 is not true as well: the Hechler forcing is a counterexample according to the following Theorem.

**Theorem 6.4.** *If  $\mathbb{P}$  is  $\sigma$ -centered then  $\mathbb{P}$  is not  $\mathcal{I}$ -dominating for any tall analytic P-ideal  $\mathcal{I}$ .*

*Proof.* Assume that  $\mathcal{I} = \text{Exh}(\varphi)$  for some lower semicontinuous finite submeasure  $\varphi$ . Let  $\varepsilon = \lim_{n \rightarrow \infty} \varphi(\omega \setminus n) > 0$ .

Let  $\mathbb{P} = \cup\{C_n : n \in \omega\}$  where  $C_n$  is centered for each  $n$ . Assume on the contrary that  $\Vdash_{\mathbb{P}} \dot{X} \in \mathcal{I} \wedge \forall A \in \mathcal{I} \cap V \ A \subseteq^* \dot{X}$  for some  $\mathbb{P}$ -name  $\dot{X}$ .

For each  $A \in \mathcal{I}$  choose a  $p_A \in \mathbb{P}$  and a  $k_A \in \omega$  such that

$$(o) \quad p_A \Vdash \check{A} \setminus \check{k}_A \subseteq \dot{X} \wedge \varphi(\dot{X} \setminus \check{k}_A) < \varepsilon/2.$$

For each  $n, k \in \omega$  let  $\mathcal{C}_{n,k} = \{A \in \mathcal{I} : p_A \in C_n \wedge k_A = k\}$ , and let  $B_{n,k} = \bigcup \mathcal{C}_{n,k}$ . We show that for each  $n$  and  $k$

$$\varphi(B_{n,k} \setminus k) \leq \varepsilon/2.$$

Assume indirectly  $\varphi(B_{n,k} \setminus k) > \varepsilon/2$  for some  $n$  and  $k$ . There is a  $k'$  such that  $\varphi(B_{n,k} \cap [k, k']) > \varepsilon/2$  and there is a finite  $\mathcal{D} \subseteq \mathcal{C}_{n,k}$  such that  $B_{n,k} \cap [k, k'] = (\cup \mathcal{D}) \cap [k, k']$ . Choose a common extension  $q$  of  $\{p_A : A \in \mathcal{D}\}$ . Now we have  $q \Vdash \cup\{A \setminus \check{k} : A \in \check{\mathcal{D}}\} \subseteq \dot{X}$  and so

$$q \Vdash \varepsilon/2 < \varphi(\check{B}_{n,k} \cap [\check{k}, \check{k}']) = \varphi((\cup \check{\mathcal{D}}) \cap [\check{k}, \check{k}']) \leq \varphi(\dot{X} \cap [\check{k}, \check{k}']) \leq \varphi(\dot{X} \setminus \check{k}),$$

which contradicts (o).

So for each  $n$  and  $k$  the set  $\omega \setminus B_{n,k}$  is infinite, so  $\omega \setminus B_{n,k}$  contains an infinite  $D_{n,k} \in \mathcal{I}$ . Let  $D \in \mathcal{I}$  such that  $D_{n,k} \subseteq^* D$  for each  $n, k \in \omega$ .

Then there is no  $n, k$  such that  $D \subseteq^* B_{n,k}$ . Contradiction.  $\square$

By this Theorem and by Lemma 4.1 the Cohen forcing is neither  $\mathcal{I}$ -dominating nor  $\mathcal{I}$ -bounding for any tall analytic P-ideal  $\mathcal{I}$ .

Finally in the rest of the paper we compare the Sacks property and the  $\mathcal{I}$ -bounding property.

**Theorem 6.5.** *If  $\mathbb{P}$  has the Sacks property then  $\mathbb{P}$  is  $\mathcal{I}$ -bounding for each analytic P-ideal  $\mathcal{I}$ .*

*Proof.* Let  $\mathcal{I} = \text{Exh}(\varphi)$ . Assume  $\Vdash_{\mathbb{P}} \dot{X} \in \mathcal{I}$ . Let  $d_{\dot{X}}$  be a  $\mathbb{P}$ -name for an element of  $\omega^\omega$  such that  $\Vdash_{\mathbb{P}} d_{\dot{X}}(\check{n}) = \min\{k \in \omega : \varphi(\dot{X} \setminus k) < 2^{-\check{n}}\}$ . We know that  $\mathbb{P}$  is  $\omega^\omega$ -bounding. If  $p \Vdash d_{\dot{X}} \leq \check{f}$  for some strictly increasing  $f \in \omega^\omega$  then by the Sacks property there is a  $q \leq p$  and a slalom  $S : \omega \rightarrow [[\omega]^{<\omega}]^{<\omega}$ ,  $|S(n)| \leq n$  such that

$$q \Vdash \forall^\infty n \dot{X} \cap [f(n), f(n+1)) \in S(n).$$

Now let

$$A = \bigcup_{n \in \omega} \{D \in S(n) : \varphi(D) < 2^{-n}\}.$$

$A \in \mathcal{I}$  because  $\varphi(A \setminus f(n)) \leq \sum_{k \geq n} \varphi(A \cap [f(k), f(k+1))) \leq \sum_{k \geq n} \frac{k}{2^k}$ . Clearly  $q \Vdash \dot{X} \subseteq^* \dot{A}$ .  $\square$

A supported relation  $\mathcal{R} = (A, R, B)$  is called *Borel-relation* iff there is a Polish space  $X$  such that  $A, B \subseteq X$  and  $R \subseteq X^2$  are Borel sets. Similarly a Galois-Tukey connection  $(\phi, \psi) : \mathcal{R}_1 \preceq \mathcal{R}_2$  between Borel-relations is called *Borel GT-connection* iff  $\phi$  and  $\psi$  are Borel functions. To be Borel-relation and Borel GT-connection is absolute for transitive models containing all relevant codes.

Some important Borel-relation:

(A):  $(\mathcal{I}, \subseteq, \mathcal{I})$  and  $(\mathcal{I}, \subseteq^*, \mathcal{I})$  for a Borel ideal  $\mathcal{I}$ .

(B): Denote  $\text{Slm}$  the set of *slaloms on  $\omega$* , i.e.  $S \in \text{Slm}$  iff  $S : \omega \rightarrow [\omega]^{<\omega}$  and  $|S(n)| = 2^n$  for each  $n$ . Let  $\sqsubseteq$  and  $\sqsubseteq^*$  be the following relations on  $\omega^\omega \times \text{Slm}$ :

$$f \sqsubseteq^{(*)} S \iff \forall^{(\infty)} n \in \omega f(n) \in S(n).$$

The supported relations  $(\omega^\omega, \sqsubseteq, \text{Slm})$  and  $(\omega^\omega, \sqsubseteq^*, \text{Slm})$  are Borel-relations.

(C): Denote  $\ell_1^+$  the set of positive summable series. Let  $\leq$  be the coordinate-wise and  $\leq^*$  the almost everywhere coordinate-wise ordering on  $\ell_1^+$ .  $(\ell_1^+, \leq, \ell_1^+)$  and  $(\ell_1^+, \leq^*, \ell_1^+)$  are Borel-relations.

**Definition 6.6.** Let  $\mathcal{R} = (A, R, B)$  be a Borel-relation. A forcing notion  $\mathbb{P}$  is  $\mathcal{R}$ -*bounding* if

$$\Vdash_{\mathbb{P}} \forall a \in A \exists b \in B \cap V a R b;$$

$\mathcal{R}$ -*dominating* if

$$\Vdash_{\mathbb{P}} \exists b \in B \forall a \in A \cap V a R b.$$

For example the property  $\mathcal{I}$ -bounding/dominating is the same as  $(\mathcal{I}, \subseteq^*$ ,  $\mathcal{I}$ )-bounding/dominating.

We can reformulate some classical properties of forcing notions:

$\omega^\omega$ -bounding	$\equiv$	$(\omega^\omega, \leq^{(*)}, \omega^\omega)$ -bounding
adding dominating reals	$\equiv$	$(\omega^\omega, \leq^*, \omega^\omega)$ -dominating
Sacks property	$\equiv$	$(\omega^\omega, \sqsubseteq^{(*)}, \text{Slm})$ -bounding
adding a slalom capturing all ground model reals	$\equiv$	$(\omega^\omega, \sqsubseteq^*, \text{Slm})$ -dominating

If  $\mathcal{R} = (A, R, B)$  is a supported relation then let  $\mathcal{R}^\perp = (B, \neg R^{-1}, A)$  where  $b(\neg R^{-1})a$  iff not  $aRb$ . Clearly  $(\mathcal{R}^\perp)^\perp = \mathcal{R}$  and  $\mathfrak{b}(\mathcal{R}) = \mathfrak{d}(\mathcal{R}^\perp)$ . Now if  $\mathcal{R}$  is a Borel-relation then  $\mathcal{R}^\perp$  is a Borel-relation too, and a forcing notion is  $\mathcal{R}$ -bounding iff it is not  $\mathcal{R}^\perp$ -dominating.

**Fact 6.7.** *Assume  $\mathcal{R}_1 \preceq \mathcal{R}_2$  are Borel-relations with Borel GT-connection and  $\mathbb{P}$  is a forcing notion. If  $\mathbb{P}$  is  $\mathcal{R}_2$ -bounding/dominating then  $\mathbb{P}$  is  $\mathcal{R}_1$ -bounding/dominating.*

By Corollary 5.5 this yields

**Corollary 6.8.** *For each analytic P-ideal  $\mathcal{I}$  (1) a poset  $\mathbb{P}$  is  $\leq_{\mathcal{I}}$ -bounding iff it is  $\omega^\omega$ -bounding, (2) forcing with a poset  $\mathbb{P}$  adds  $\leq_{\mathcal{I}}$ -dominating reals iff this forcing adds dominating reals.*

Fremlin's two results have interesting corollaries for us.

**Lemma 6.9.** ([Fr] 524C) *Let  $(P, \leq)$  be a partially ordered set such that  $\sup\{p, q\}$  is defined for all  $p, q \in P$ . Suppose  $d$  is a metric on  $P$  such that  $(P, d)$  is a polish space and  $\sup\{\cdot, \cdot\} : P^2 \rightarrow P$  is uniformly continuous. Then  $(P, \leq, P) \preceq (\ell_1^+, \leq, \ell_1^+)$  with a Borel GT-connection.*

**Corollary 6.10.** *There is a Borel GT-connection  $(\mathcal{I}, \subseteq, \mathcal{I}) \preceq (\ell_1^+, \leq, \ell_1^+)$  for each analytic P-ideal  $\mathcal{I}$ .*

*Proof.* Let  $\mathcal{I} = \text{Exh}(\varphi)$  where  $\varphi$  is a finite lower semicontinuous submeasure on  $\omega$  with  $\varphi(A) = 0$  iff  $A = \emptyset$ . One can show that  $(\mathcal{I}, \subseteq)$  with  $d(A, B) = \varphi(A \Delta B)$  satisfy the requirements of Lemma 6.9.  $\square$

Note that there is no any Galois-Tukey connection from  $(\ell_1^+, \leq, \ell_1^+)$  to  $(\mathcal{Z}, \subseteq, \mathcal{Z})$  so they are not GT-equivalent (see [LoVe] Th. 7.).

**Theorem 6.11.** ([Fr] 524I) *There is a Borel GT-connection  $(\ell_1^+, \leq^*, \ell_1^+) \equiv (\omega^\omega, \sqsubseteq^*, \text{Sln})$ .*

**Corollary 6.12.** *If  $\mathbb{P}$  adds a slalom capturing all ground model reals then  $\mathbb{P}$  is  $\mathcal{I}$ -dominating for each analytic P-ideal  $\mathcal{I}$ .*

*Proof.* By Fact 6.7 and Theorem 6.11 adding slalom is the same as  $(\ell_1^+, \leq^*, \ell_1^+)$ -dominating. Let  $\dot{x}$  be a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \dot{x} \in \ell_1^+ \wedge \forall y \in \ell_1^+ \cap V y \leq^* \dot{x}$ . Moreover let  $\dot{X}$  be a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \dot{X} = \{z \in \ell_1^+ : |z \setminus \dot{x}| < \omega, \forall n (z(n) \neq \dot{x}(n) \Rightarrow z(n) \in \omega)\}$ . Let  $(\phi, \psi) : (\mathcal{I}, \subseteq, \mathcal{I}) \preceq (\ell_1^+, \leq, \ell_1^+)$  be a Borel GT-connection. Now if  $\dot{A}$  is a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \forall z \in \dot{X} \psi(z) \subseteq^* \dot{A}$  then  $\dot{A}$  shows that  $\mathbb{P}$  is  $\mathcal{I}$ -dominating.  $\square$

Denote  $\mathbb{D}$  the dominating forcing and  $\text{LOC}$  the Localization forcing. Because of the complicated GT-connections it seems hard to construct an  $\mathcal{I}$ -dominating set in  $V^{\text{LOC}}$ . Despite  $\text{LOC}$  it is much more simple to show directly that the two-step iteration  $\mathbb{D} * \text{LOC}$  is  $\mathcal{I}$ -dominating.

**Observation 6.13.** *If  $\mathcal{I}$  is an arbitrary analytic P-ideal then two step iteration  $\mathbb{D} * \text{LOC}$  is  $\mathcal{I}$ -dominating.*

Indeed, let  $\mathcal{I} \in V \subseteq M \subseteq N$  be transitive models,  $d \in M \cap \omega^\omega$  be strictly increasing and dominating over  $V$ , and  $S \in N$ ,  $S : \omega \rightarrow [[\omega]^{<\omega}]^{<\omega}$ ,  $|S(n)| \leq n$  a slalom which captures all reals from  $M$ . Now if

$$X_n = \cup\{A \in S(n) \cap \mathcal{P}([d(n), d(n+1)) : \varphi(A) < 2^{-n}\}$$

then it is easy to see that  $Y \subseteq^* \cup\{X_n : n \in \omega\} \in \mathcal{I} \cap N$  for each  $Y \in V \cap \mathcal{I}$ .

**Problem 6.14.** Does  $\mathcal{Z}$ -dominating (or  $\mathcal{I}$ -dominating) imply adding slaloms?

We will use the following deep result of Fremlin to prove Theorem 6.16.

**Theorem 6.15.** ([Fr] 526G) *There is a family  $\{P_f : f \in \omega^\omega\}$  of Borel subsets of  $\ell_1^+$  such that the following hold:*

- (i)  $\ell_1^+ = \cup\{P_f : f \in \omega^\omega\}$ ,
- (ii) if  $f \leq g$  then  $P_f \subseteq P_g$ ,
- (iii)  $(P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z})$  with a Borel GT-connection for each  $f$ .

**Theorem 6.16.**  $\mathbb{P}$  is  $\mathcal{Z}$ -bounding iff  $\mathbb{P}$  has the Sacks property.

*Proof.* Let  $\{P_f : f \in \omega^\omega\}$  be a family satisfying (i), (ii), and (iii) in Theorem 6.15, and fix Borel GT-connections  $(\phi_f, \psi_f) : (P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z})$  for each  $f \in \omega^\omega$ . Assume  $\mathbb{P}$  is  $\mathcal{Z}$ -bounding and  $\Vdash_{\mathbb{P}} \dot{x} \in \ell_1^+$ .  $\mathbb{P}$  is  $\omega^\omega$ -bounding by Theorem 6.2 so using (ii) we have  $\Vdash_{\mathbb{P}} \ell_1^+ = \cup\{P_f : f \in \omega^\omega \cap V\}$ . We can choose a  $\mathbb{P}$ -name  $\dot{f}$  for an element of  $\omega^\omega \cap V$  such that  $\Vdash_{\mathbb{P}} \dot{x} \in P_{\dot{f}}$ . By  $\mathcal{Z}$ -bounding property of  $\mathbb{P}$  there is a  $\mathbb{P}$ -name  $\dot{A}$  for an element of  $\mathcal{Z} \cap V$  such that  $\Vdash_{\mathbb{P}} \phi_{\dot{f}}(\dot{x}) \subseteq \dot{A}$ , so  $\Vdash_{\mathbb{P}} \dot{x} \leq \psi_{\dot{f}}(\dot{A}) \in \ell_1^+ \cap V$ . So we have  $\mathbb{P}$  is  $(\ell_1^+, \leq^{(*)}, \ell_1^+)$ -bounding. By Theorem 6.11 and Fact 6.7  $\mathbb{P}$  has the Sacks property.

The converse implication was proved in Theorem 6.5. □

**Problem 6.17.** Does the  $\mathcal{I}$ -bounding property imply the Sacks property for each tall analytic P-ideal  $\mathcal{I}$ ?

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