LARGE RANDOM PLANAR GRAPHS

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joint work with Omer Gimenez and Marc Noy

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- Degree Distribution
- Generating Functions
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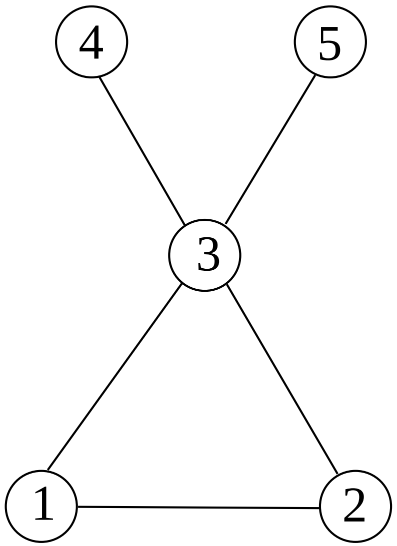
Random Planar Graph

$\mathcal{R}_n$ ... labelled planar graphs with n vertices:



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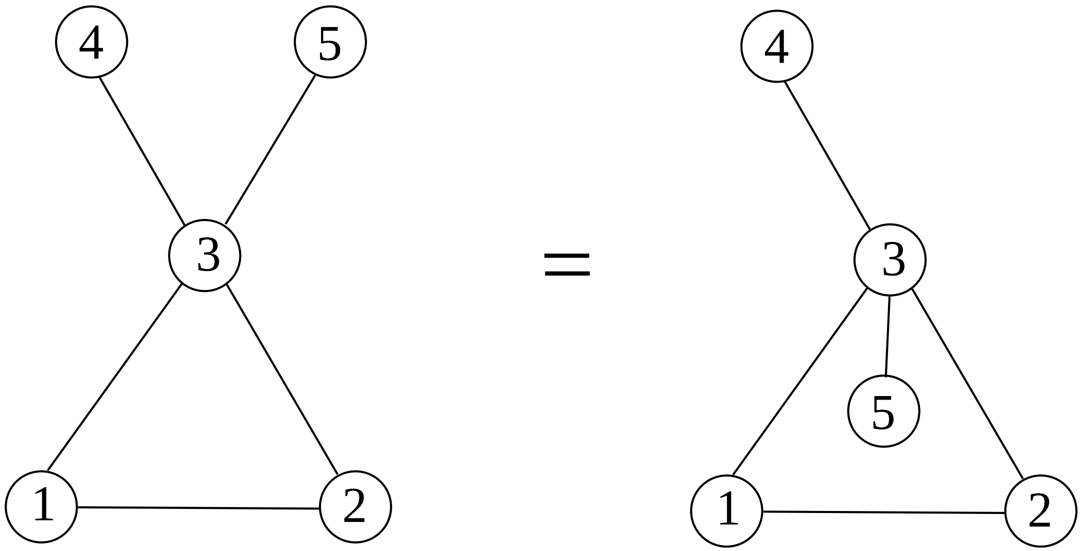

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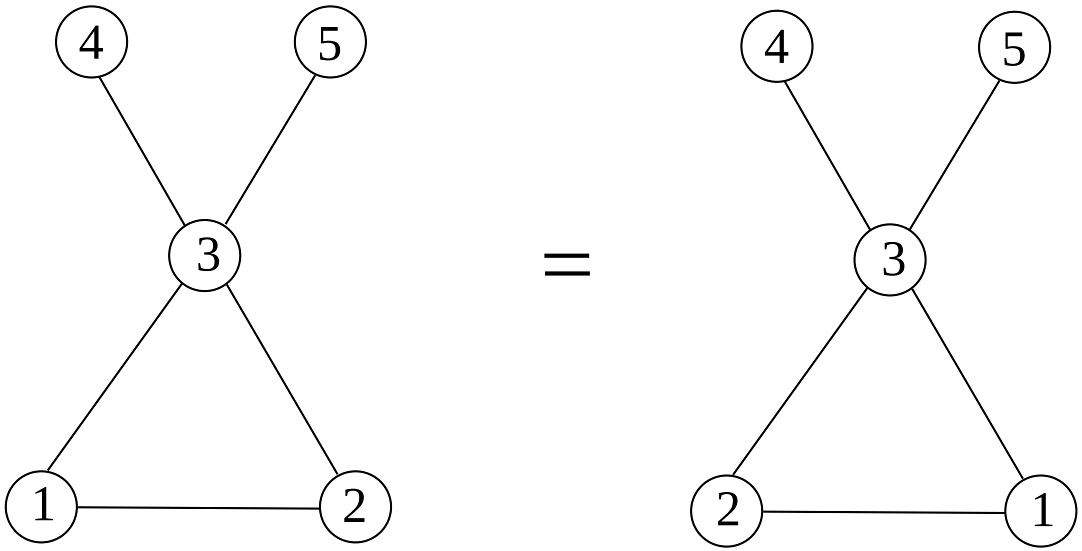

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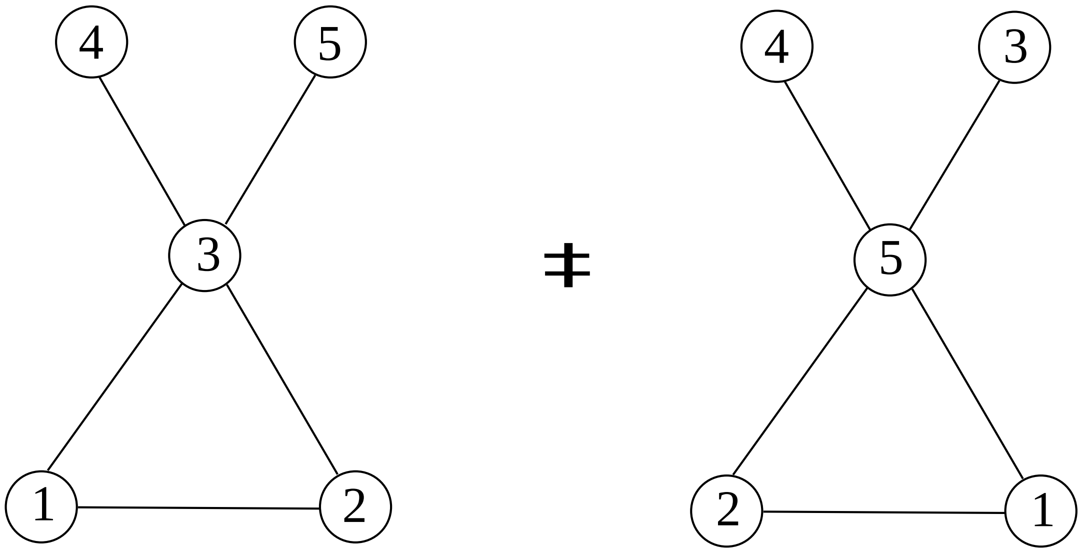
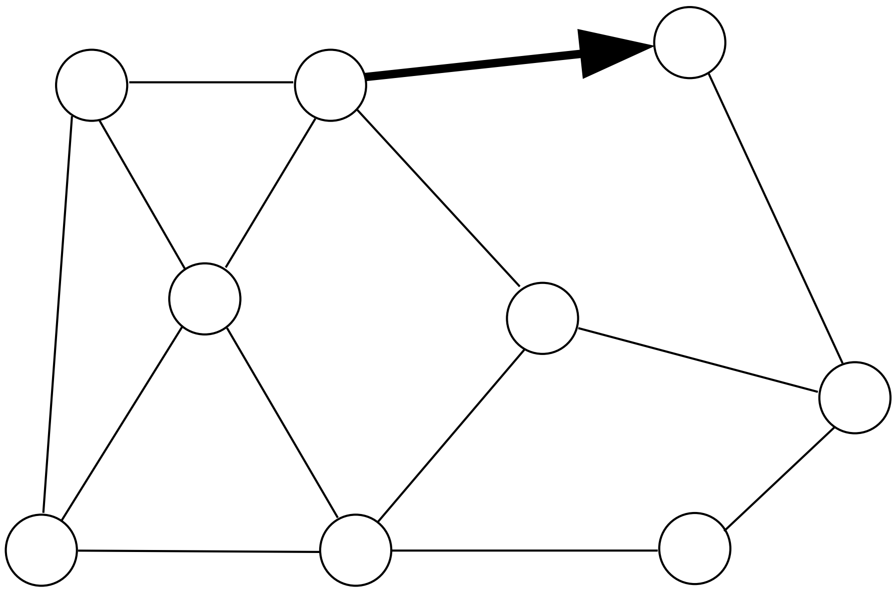


Planar Maps

A **planar map** is a planar graph together with its embedding in the plane
(usually with a rooted edge):



Maps

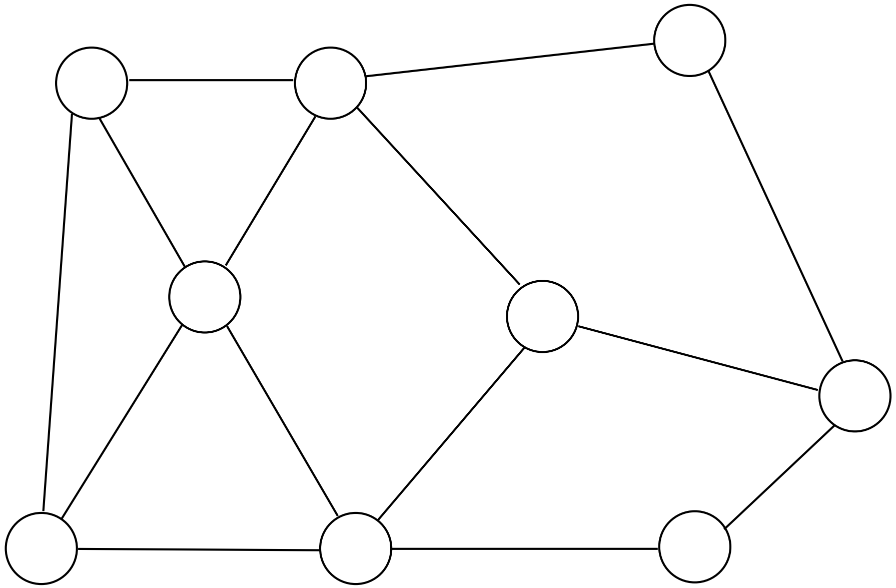
Tutte, Bender, Canfield, Gao, Wormald, Liskovets, Flajolet, Bousquet-Melou, Schaeffer, Bouttier, Guitter, Di Francesco ...

The counting problem for rooted maps is **relatively easy** and many things can be worked out **explicitly** and **asymptotically**.

Several statistics (including maximum degree and diameter) are known. Some of them are very difficult to deal with.

Connectedness

2-connected: one has to remove at least 2 vertices to disconnect



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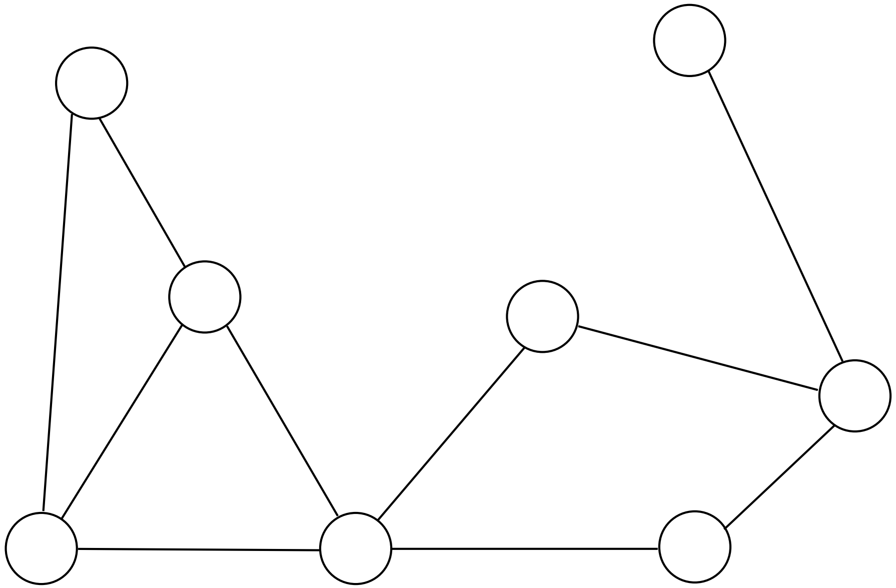

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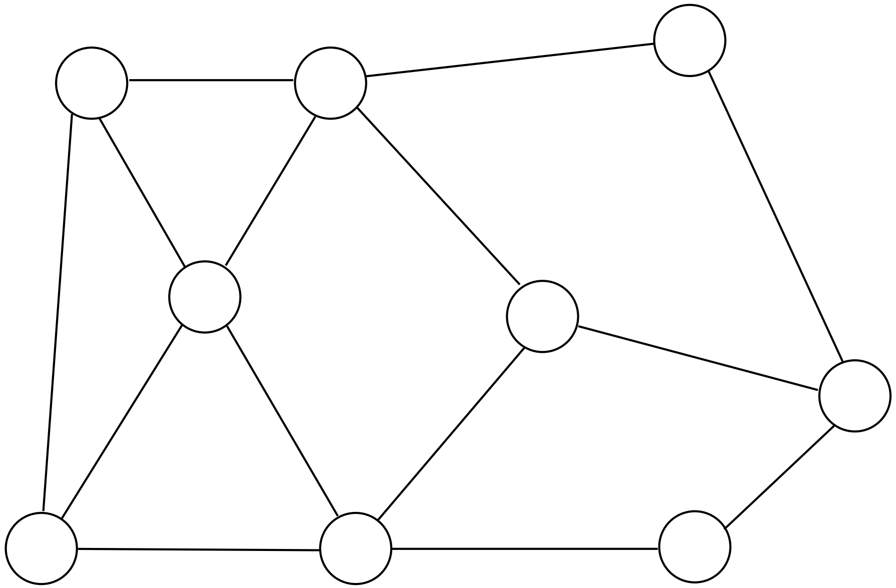

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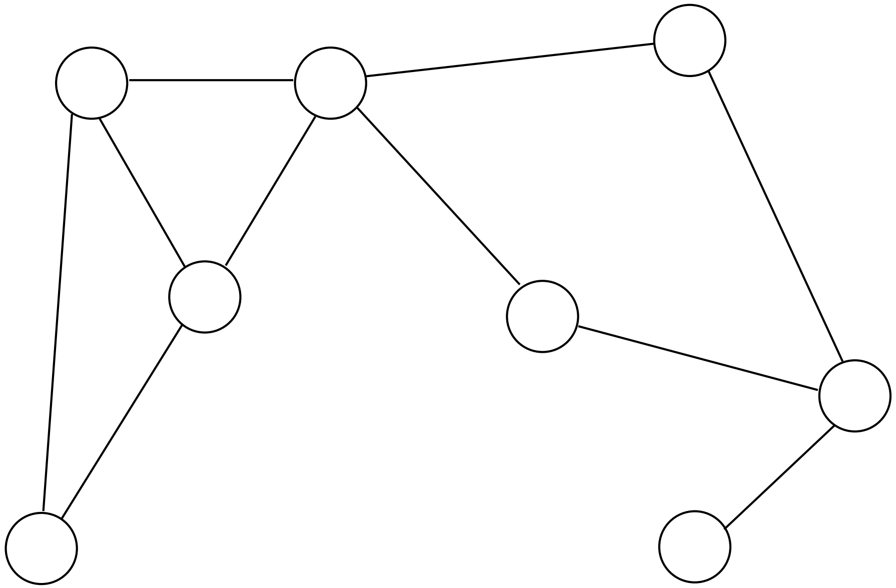

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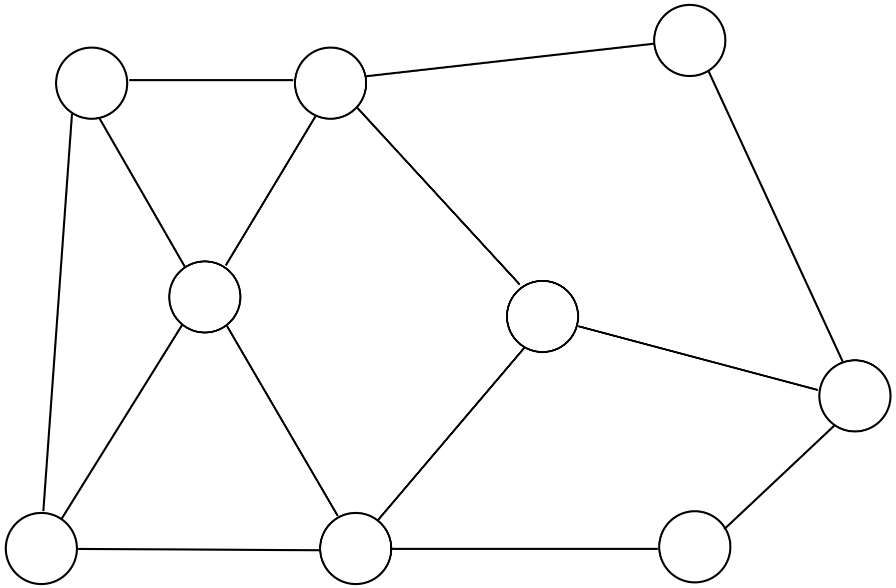

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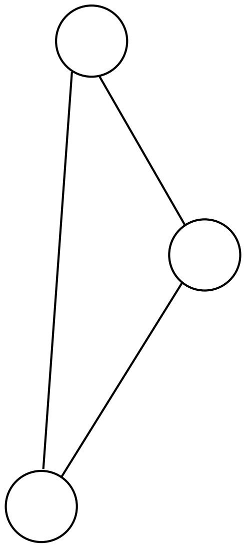

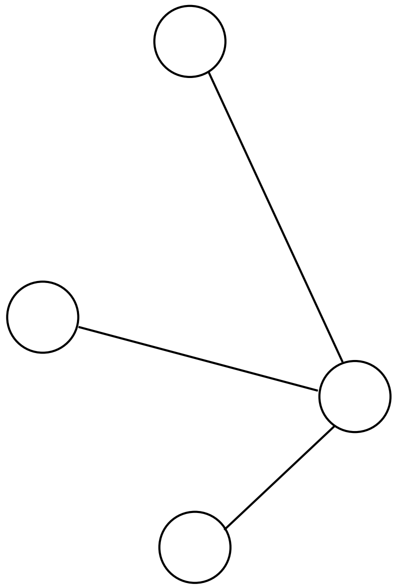

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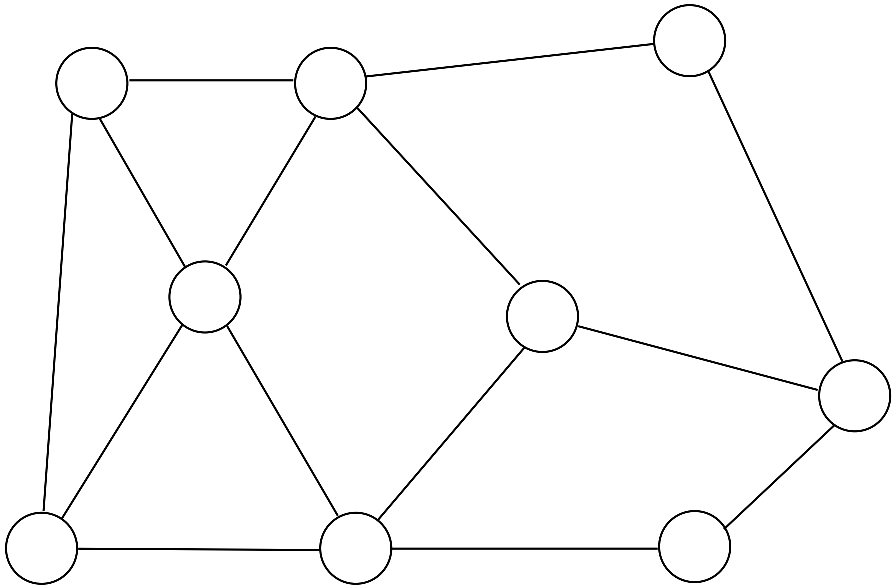
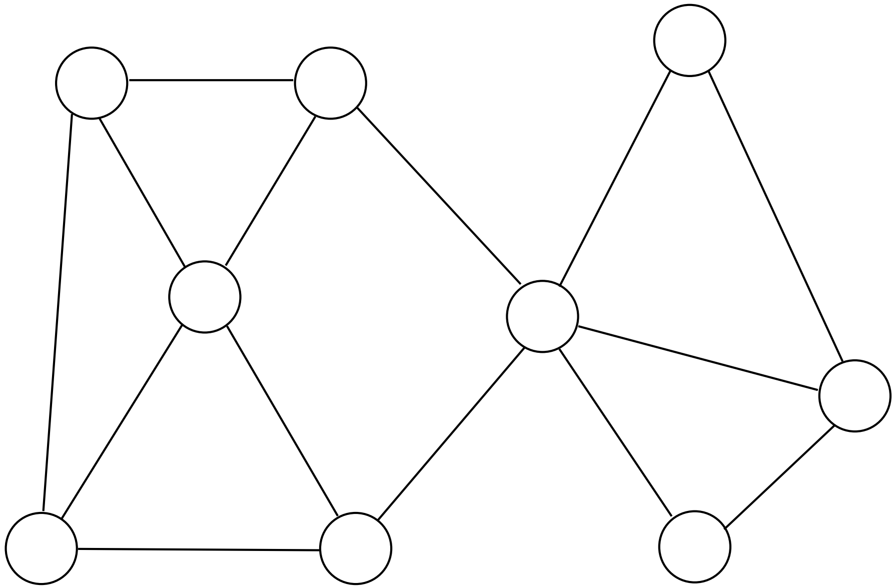


Connectedness

NOT 2-connected:



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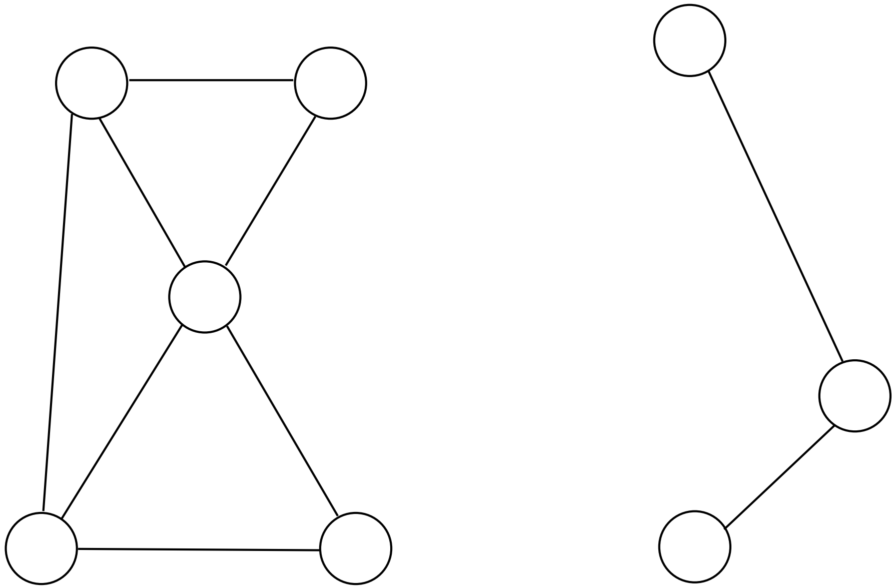

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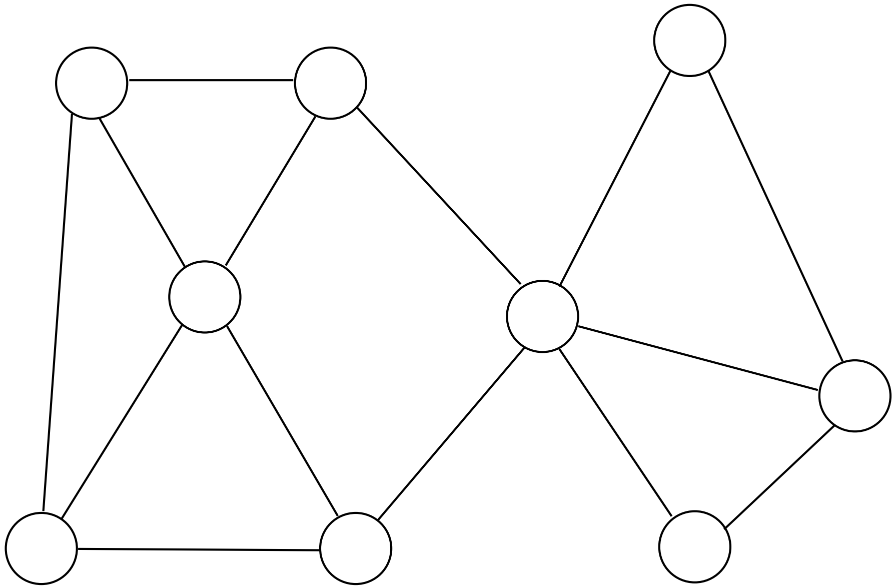
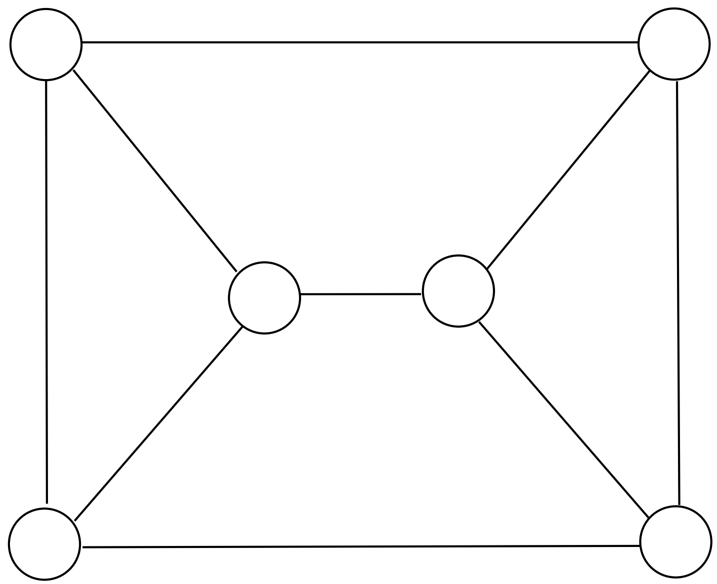


Connectedness

3-connected: one has to remove at least 3 vertices to disconnect



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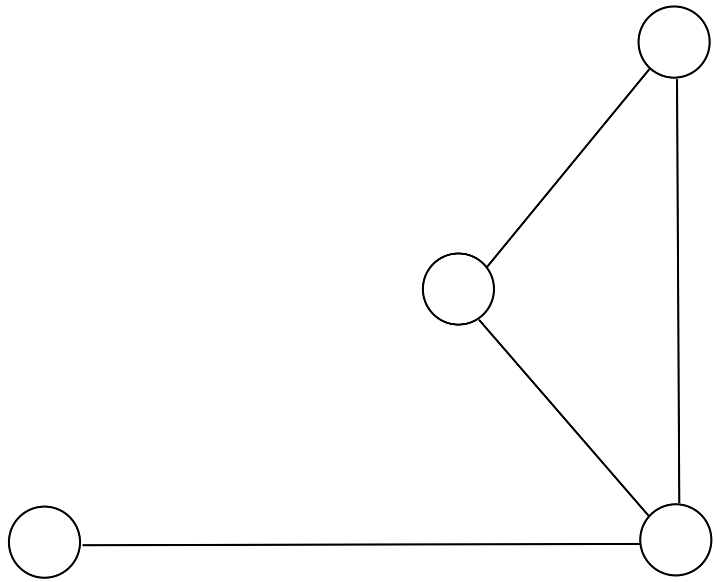

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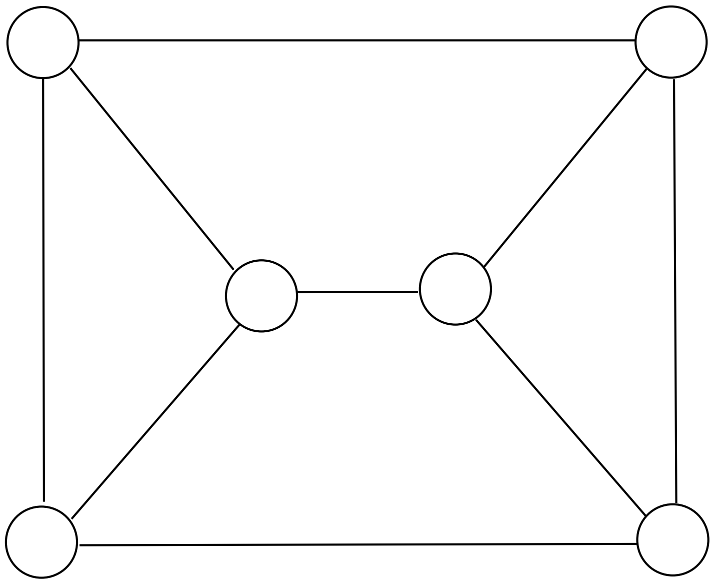


Planar Maps vs. Planar Graphs

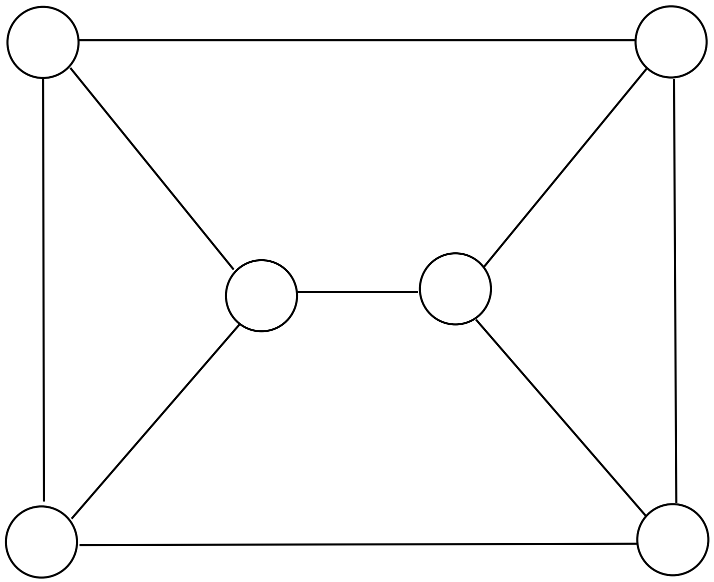
Whitney's Theorem

Every 3-connected planar graph has a unique embedding into the plane.

$\Longrightarrow$ The counting problem of **rooted 3-connected planar maps** is equivalent to the counting problem of **rooted (labelled) 3-connected planar graphs** (despite of a factor $(n-1)!$)

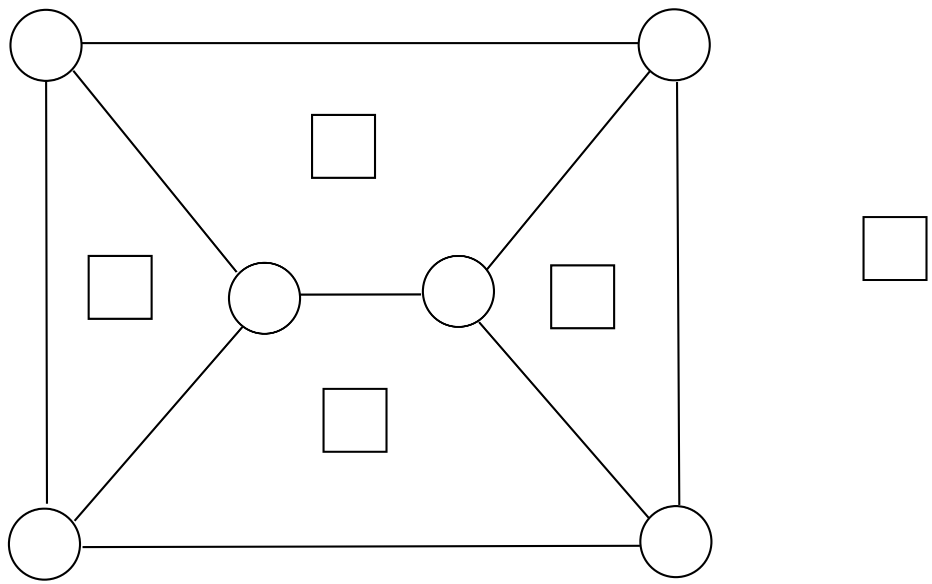
3-Connected Maps

Quadrangulations



3-Connected Maps

Quadrangulations



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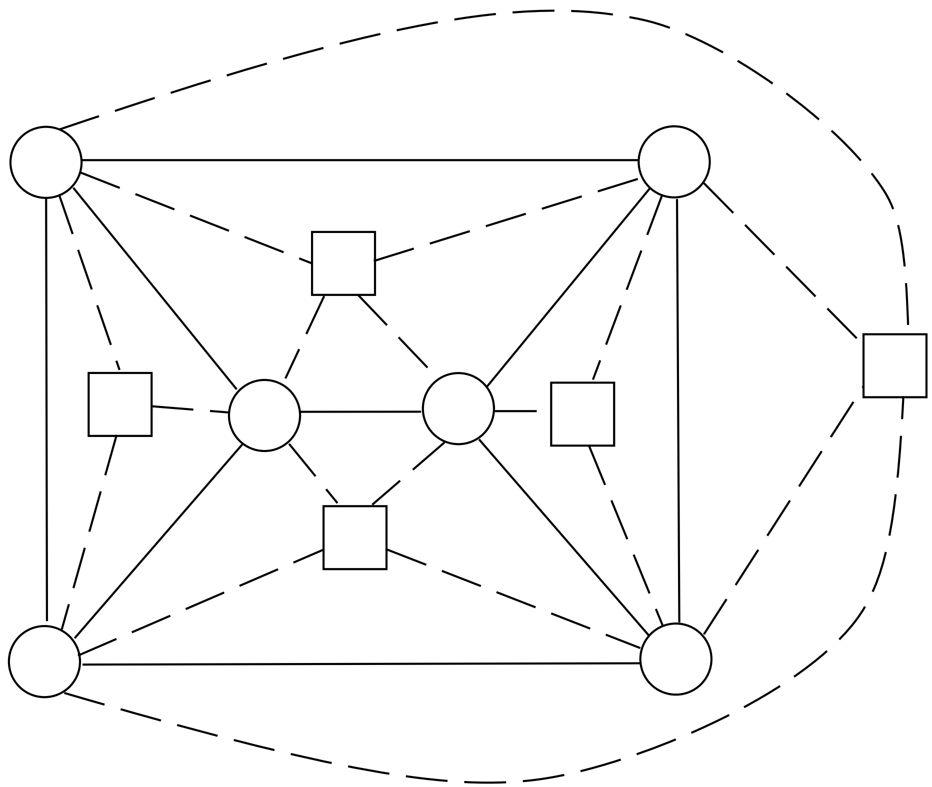

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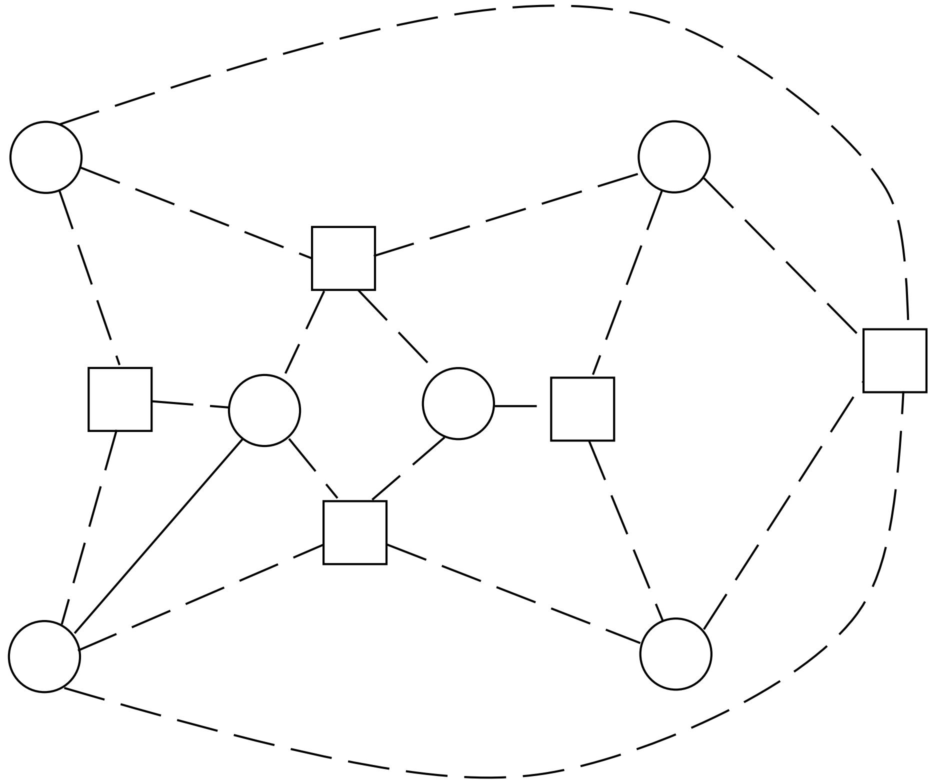

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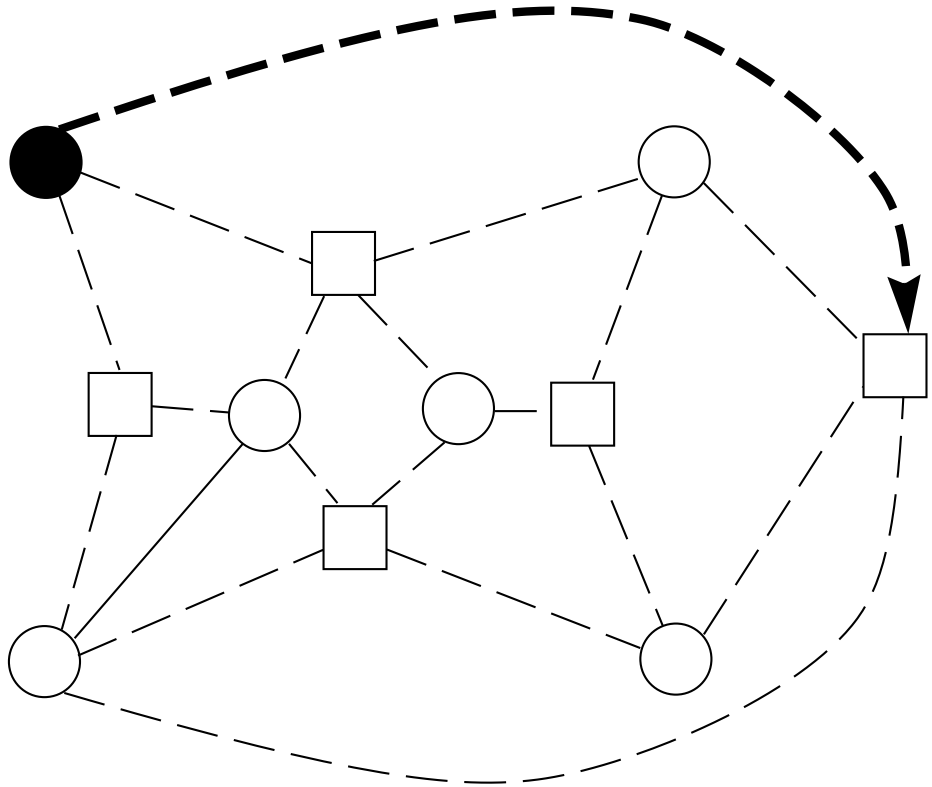

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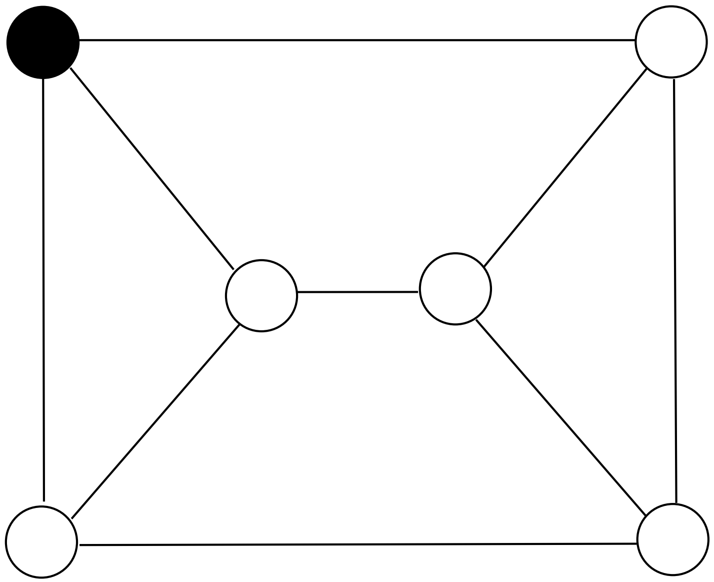


3-Connected Maps

q_{ijk} ... number of edge-rooted 3-connected maps with
$i+1$ vertices of type 1 ($\circ$),
$j+1$ vertices of type 2 ($\Box$), and with
root vertex of degree $k+1$

$$Q(x,y,w) = \sum_{i,j,k} q_{i,j,k} \cdot x^i \, y^j \, w^k$$

Theorem [Mullin+Schellenberg, D+Gimenez+Noy]

$$\boxed{Q(x,y,w) = xyw\left(\frac{1}{1+wy} + \frac{1}{1+x} - 1\right) - \frac{UV}{(1+U+V)^3} \cdot W(R,S,w)}$$

with ...

3-Connected Maps

with **algebraic function** $U = U(x,y)$, $V = V(x,y)$ given by

$$\boxed{U = x(V+1)^2}, \qquad \boxed{V = y(U+1)^2}$$

and

$$\boxed{W(U,V,w) = \frac{-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}}{2(V+1)^2(Vw+U^2+2U+1)}}$$

with polynomials $w_1 = w_1(U,V,w)$ and $w_2 = w_2(U,V,w)$ given by

$$\begin{aligned} w_1 =& -UVw^2 + w(1+4V+3UV^2+5V^2+U^2+2U+2V^3+3U^2V+7UV) \\ &+ (U+1)^2(U+2V+1+V^2), \\ w_2 =& U^2V^2w^2 - 2wUV(2U^2V+6UV+2V^3+3UV^2+5V^2+U^2+2U+4V+1) \\ &+ (U+1)^2(U+2V+1+V^2)^2. \end{aligned}$$

Random Planar Graphs

Denise, Vasconcellos, Welsh (1996)

$$\boxed{\mathbb{P}\{e(\mathcal{R}_n) > \frac{3}{2}n\} \to 1, \quad \mathbb{P}\{e(\mathcal{R}_n) < \frac{5}{2}n\} \to 1}.$$

$e(\mathcal{R}_n)$ ... **number of edges** in random planar graphs $\mathcal{R}_n$
Note that $0 \le e \le 3n$ for all planar graphs.

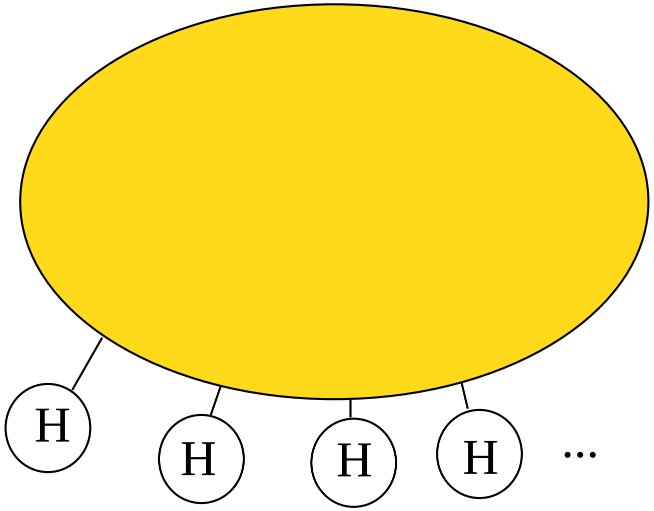
McDiarmid, Steger, Welsh (2005)

$$\boxed{\mathbb{P}\{H \text{ appears in } \mathcal{R}_n \text{ at least } \alpha n \text{ times}\} \to 1}$$

H ... any fixed planar graph, $\alpha > 0$ sufficiently small.

Random Planar Graphs

Appearance of H:



Random Planar Graphs

Consequences:

$$\boxed{\mathbb{P}\{\text{There are} \geq \alpha n \text{ vertices of degree } k\} \to 1}$$

$k > 0$ a given integer, $\alpha > 0$ sufficiently small.

$$\boxed{\mathbb{P}\{\text{There are} \geq C^n \text{ automorphisms}\} \to 1}$$

for some $C > 1$.

Random Planar Graphs

Further Results:

$$\boxed{\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \ge \gamma > 0}$$

[McDiarmid+Reed]

$$\boxed{\mathbb{E}\,\Delta(\mathcal{R}_n) = \Theta(\log n)}$$

$\Delta(\mathcal{R}_n)$ ... maximum degree in $\mathcal{R}_n$

The number of planar graphs

[Bender, Gao, Wormald (2002)]

b_n ... number of **2-connected** labelled planar graphs

$$\boxed{b_n \sim c \cdot n^{-\frac{7}{2}} \gamma_2^n \, n!}, \quad \gamma_2 = 26.18...$$

[Gimenez+Noy (2005)]

g_n .... number of all labelled planar graphs

$$\boxed{\boxed{g_n \sim c \cdot n^{-\frac{7}{2}} \gamma^n \, n!}}, \quad \gamma = 27.22...$$

The number of planar graphs

[Gimenez+Noy (2005)]

- $e(\mathcal{R}_n)$ satisfies a **central limit theorem**:

$$\mathbb{E}\, e(\mathcal{R}_n) \sim 2.21... \cdot n, \quad \mathbb{V}\, e(\mathcal{R}_n) \sim c \cdot n.$$

$$\mathbb{P}\left\{|e(\mathcal{R}_n) - 2.21... \cdot n| > \varepsilon n\right\} \leq e^{-\alpha(\varepsilon)\cdot n}$$

- **Connectedness:**

$$\mathbb{P}\left\{\mathcal{R}_n \text{ is connected}\right\} \to e^{-\nu} = 0.96...$$

$$\text{number of components of } \mathcal{R}_n =: C_n \to 1 + Po(\nu).$$

Degree Distribution

Theorem [D.+Gimenez+Noy]

Let $d_{n,k}$ be the probability that a random node in a random planar graph $\mathcal{R}_n$ has degree k. Then the limit

$$\boxed{d_k := \lim_{n\to\infty} d_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} d_k w^k}$$

can be explicitly computed.

d_1	d_2	d_3	d_4	d_5	d_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

Degree Distribution

More precisely ...

- Implicit equation for $D_0(y,w)$:

$$1+D_0=(1+y\boxed{w})\exp\left(\frac{\sqrt{S}(D_0(t-1)+t)}{4(3t+1)(D_0+1)}-\right.$$
$$\left.-\frac{D_0^2(t^4-12t^2+20t-9)+D_0(2t^4+6t^3-6t^2+10t-12)+t^4+6t^3+9t^2}{4(t+3)(D_0+1)(3t+1)}\right),$$

where $t=t(y)$ satisfies $y+1=\frac{1+2t}{(1+3t)(1-t)}\exp\left(-\frac{1}{2}\frac{t^2(1-t)(18+36t+5t^2)}{(3+t)(1+2t)(1+3t)^2}\right)$.
and $S=(D_0(t-1)+t)(D_0(t-1)^3+t(t+3)^2)$.

- Explicit expressions in terms of $D_0(y,w)$:

$$D_2(y,w),\ D_3(y,w),\ B_0(y,w),\ B_2(y,w),\ B_3(y,w)$$

- Explict expression for $p(w)$:

$$\boxed{p(w)=-e^{B_0(1,w)-B_0(1,1)}B_2(1,w)+e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w)}$$

Degree Distribution

Consequences

- Expected number $X_{n,k}$ of vertices of degree k:

$$\boxed{\mathbb{E}\, X_{n,k} = d_{n,k} \cdot n \sim d_k \cdot n}, \qquad d_k > 0.$$

- Tails of the degree distribution:

$$\boxed{\boxed{d_k \sim c \cdot k^{-\frac{1}{2}}\, q^k}}, \qquad q = 0.79...$$

Degree Distribution

Conjecture for maximum degree $\Delta(\mathcal{R}_n)$:

$$\boxed{\mathbb{E}\,\Delta(\mathcal{R}_n) \sim \frac{\log n}{\log(1/q)}}$$

Remark.

Corresponding results on the **degree distribution** and the **maximum degree** are known for **random planar maps**: [Liskovets, Gao+Wormald]

Degree Distribution

Theorem [D.+Gimenez+Noy]

Let $d_{n,k}^{(2)}$ resp. $d_{n,k}^{(3)}$ be the probability that a random node in a random 2-connectet resp. 3-connected planar graph with n vertices has degree k. Then the limits

$$d_k^{(2)} := \lim_{n\to\infty} d_{n,k}^{(2)} \quad \text{and} \quad d_k^{(3)} := \lim_{n\to\infty} d_{n,k}^{(3)}$$

exists. The probability generating functions

$$p^{(2)}(w) = \sum_{k\geq 1} d_k^{(2)} w^k \quad \text{and} \quad p^{(3)}(w) = \sum_{k\geq 1} d_k^{(2)} w^k$$

can be explicitly computed. Asymptotically we have

$$\boxed{d_k^{(2)} \sim c\cdot k^{\frac{1}{2}}\, q^k}, \quad q = \sqrt{7}-2 \quad \text{and} \quad \boxed{d_k^{(3)} \sim c\cdot k^{-\frac{1}{2}}\, q^k}, \quad q = 0.673...$$

Generating Functions

- g_n ... **all** planar graphs with n vertices:

$$g(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}$$

- c_n ... **connected** planar graphs with n vertices:

$$c(x) = \sum_{n \geq 0} c_n \frac{x^n}{n!}$$

- b_n ... **2-connected** planar graphs with n vertices:

$$b(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$

Generating Functions

- $g_{n,m}$ ... **all** planar graphs with n vertices and m edges:

$$g(x,y) = \sum_{n,m \ge 0} g_{n,m} \frac{x^n}{n!} y^m$$

- $c_{n,m}$ ... **connected** planar graphs with n vertices and m edges:

$$c(x,y) = \sum_{n,m \ge 0} c_{n,m} \frac{x^n}{n!} y^m$$

- $b_{n,m}$ ... **2-connected** planar graphs with n vertices and m edges:

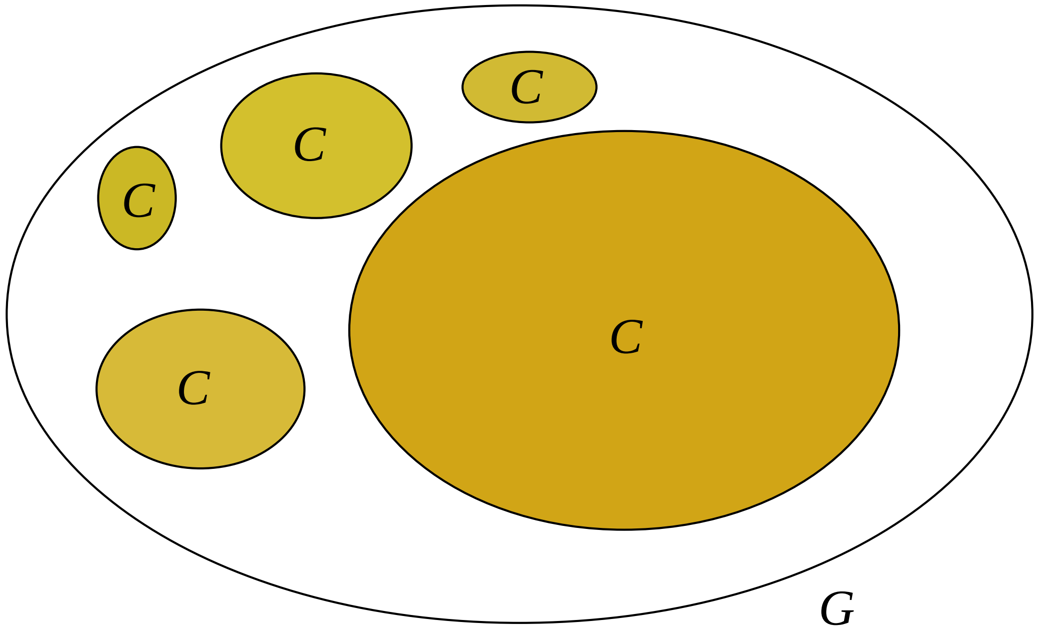
$$b(x,y) = \sum_{n,m \ge 0} b_{n,m} \frac{x^n}{n!} y^m$$

Generating Functions

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right), \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y}, \\
\frac{M(x,D)}{2x^2D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}, \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \\
U &= xy(1+V)^2, \\
V &= y(1+U)^2.
\end{aligned}
$$

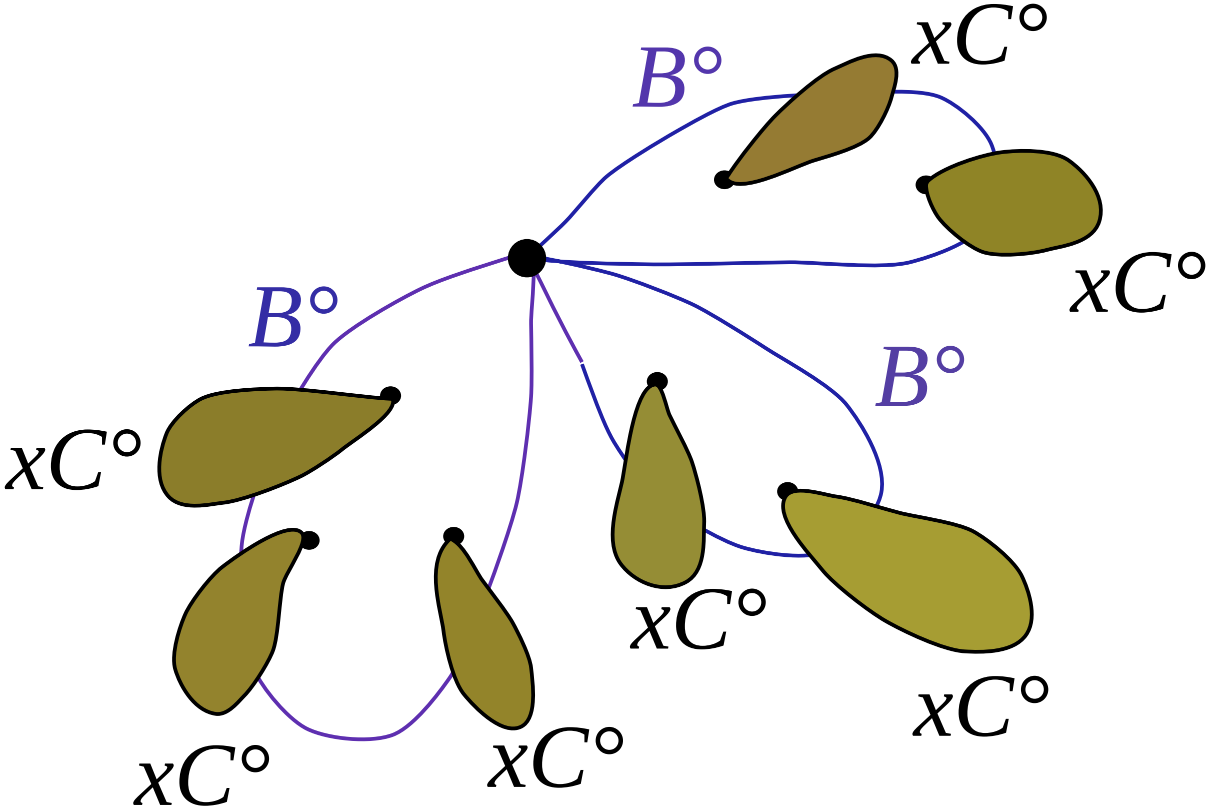
Generating Functions

$$\boxed{G(x,y) = \exp\left(C(x,y)\right)}$$



Generating Functions

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right)$$



Generating Functions

$C^\bullet = \frac{\partial C}{\partial x}$ ... GF, where one vertex is marked but not counted

w ... additional variable that *counts* the **degree of the marked vertex**

Generating functions:

$G^\bullet(x,y,w)$	**all rooted** planar graphs
$C^\bullet(x,y,w)$	**connected rooted** planar graphs
$B^\bullet(x,y,w)$	**2-connected rooted** planar graphs
$T^\bullet(x,y,w)$	**3-connected rooted** planar graphs

Note that $G^\bullet(x,y,1) = \dfrac{\partial G}{\partial x}(x,y)$ etc.

Generating Functions

$$
\begin{aligned}
G^{\bullet}(x,y,w) &= \exp\left(C(x,y,1)\right) C^{\bullet}(x,y,w), \\
C^{\bullet}(x,y,w) &= \exp\left(B^{\bullet}\left(xC^{\bullet}(x,y,1), y, w\right)\right), \\
w\frac{\partial B^{\bullet}(x,y,w)}{\partial w} &= xyw\exp\left(S(x,y,w) + \frac{1}{x^2D(x,y,w)}T^{\bullet}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) \\
D(x,y,w) &= (1+yw)\exp\left(S(x,y,w) + \frac{1}{x^2D(x,y,w)}\times\right. \\
&\qquad \left.\times\, T^{\bullet}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1 \\
S(x,y,w) &= xD(x,y,1)\left(D(x,y,w) - S(x,y,w)\right), \\
T^{\bullet}(x,y,w) &= \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy} + \frac{1}{1+xy} - 1 - \right. \\
&\qquad \left. - \frac{(u+1)^2\left(-w_1(u,v,w) + (u-w+1)\sqrt{w_2(u,v,w)}\right)}{2w(vw+u^2+2u+1)(1+u+v)^3}\right), \\
u(x,y) &= xy(1+v(x,y))^2, \quad v(x,y) = y(1+u(x,y))^2.
\end{aligned}
$$

Asymptotics for Generating Functions

Singularity analysis

Suppose that

$$f(z) = \sum_{n \geq 0} a_n z^n = A_0 + A_2 Z^2 + A_3 Z^3 + O(Z^4),$$

with

$$Z = \sqrt{1 - \frac{z}{\rho}}$$

(plus some technical conditions).

$$\Longrightarrow \quad \boxed{a_n = \frac{3A_3}{4\sqrt{\pi}} \rho^{-n} n^{-5/2} + O(\rho^{-n} n^{-3})}$$

Asymptotics for Generating Functions

3-connected planar graphs

$$\tilde{u}_0(y) = -\frac{1}{3} + \sqrt{\frac{4}{9} + \frac{1}{3y}}$$

$$r(y) = \frac{\tilde{u}_0(y)}{y(1 + y(1 + \tilde{u}_0(y))^2)^2},$$

$$\tilde{X} = \sqrt{1 - \frac{x}{r(y)}}$$

$$\Longrightarrow \quad \boxed{T^\bullet(x,y,w) = \tilde{T}_0(y,w) + \tilde{T}_2(y,w)\tilde{X}^2 + \tilde{T}_3(y,w)\tilde{X}^3 + O(\tilde{X}^4)}$$

Asymptotics for Generating Functions

2-connected planar graphs

$\tau(x)$ ... inverse function of $r(y)$

$$D(R(y), y, 1) = \tau(R(y))$$

$$X = \sqrt{1 - \frac{x}{R(y)}}$$

$$\Longrightarrow \quad \boxed{D(x, y, w) = D_0(y, w) + D_2(y, w)X^2 + D_3(y, w)X^3 + O(X^4)},$$

$$\Longrightarrow \quad \boxed{B^\bullet(x, y, w) = B_0(y, w) + B_2(y, w)X^2 + B_3(y, w)X^3 + O(X^4)}$$

Asymptotics for Generating Functions

Lemma

$$f(x) = \sum_{n \geq 0} \boxed{a_n} \frac{x^n}{n!} = f_0 + f_2 X^2 + f_3 \boxed{X^3} + \mathcal{O}(X^4), \qquad X = \sqrt{1 - \frac{x}{\rho}},$$

$$H(x,z,w) = h_0(x,w) + h_2(x,w) Z^2 + h_3(x,w) \boxed{Z^3} + \mathcal{O}(Z^4),$$

$$Z = \sqrt{1 - \frac{z}{\boxed{f(\rho)}}},$$

$$f_H(x) = H(x, \boxed{f(x)}, w) = \sum_{n \geq 0} \boxed{b_n(w)} \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{\lim_{n \to \infty} \frac{b_n(w)}{a_n} = -\frac{h_2(\rho,w)}{f_0} + \frac{h_3(\rho,w)}{f_3} \left(-\frac{f_2}{f_0} \right)^{3/2}}.$$

Asymptotics for Generating Functions

connected planar graphs

$$C^\bullet(x,1,w) = \exp\left(B^\bullet\left(xC'(x),1,w\right)\right)$$

Application of the lemma with

$$f(x) = xC'(x)$$

and

$$H(x,z,w) = xe^{B^\bullet(z,1,w)}.$$

Thank You