LARGE RANDOM PLANAR GRAPHS

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joint work with Omer Gimenez and Marc Noy

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* supported by the Austrian Science Foundation FWF, grant S9600.
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Random Planar Graph

\( R_n \) ... labelled planar graphs with \( n \) vertices:
Random Planar Graph

$\mathcal{R}_n$ ... labelled planar graphs with $n$ vertices:
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$\mathcal{R}_n$ ... labelled planar graphs with $n$ vertices:
Planar Maps

A **planar map** is a planar graph together with its embedding in the plane (usually with a rooted edge):
Maps

Tutte, Bender, Canfield, Gao, Wormald, Liskovets, Flajolet, Bousquet-Melou, Schaeffer, Bouttier, Guitter, Di Francesco ...

The counting problem for rooted maps is relatively easy and many things can be worked out explicitly and asymptotically.

Several statistics (including maximum degree and diameter) are known. Some of them are very difficult to deal with.
**Connectedness**

2-connected: one has to remove at least 2 vertices to disconnect
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3-connected: one has to remove at least 3 vertices to disconnect
Connectedness

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Planar Maps vs. Planar Graphs

Whitney’s Theorem

Every 3-connected planar graph has a unique embedding into the plane.

⇒ The counting problem of rooted 3-connected planar maps is equivalent to the counting problem of rooted (labelled) 3-connected planar graphs (despite of a factor $(n - 1)!$)
3-Connected Maps

Quadrangulations
3-Connected Maps

Quadrangulations
3-Connected Maps

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3-Connected Maps

$q_{ijk} \ldots$ number of edge-rooted 3-connected maps with
$i + 1$ vertices of type 1 ($\circ$),
$j + 1$ vertices of type 2 ($\square$), and with
root vertex of degree $k + 1$

$$Q(x, y, w) = \sum_{i,j,k} q_{i,j,k} \cdot x^i y^j w^k$$

**Theorem** [Mullin+Schellenberg, D+Gimenez+Noy]

$$Q(x, y, w) = xyw \left( \frac{1}{1 + wy} + \frac{1}{1 + x} - 1 \right) - \frac{UV}{(1 + U + V)^3} \cdot W(R, S, w)$$

with ...
3-Connected Maps

with algebraic function \( U = U(x, y), \ V = V(x, y) \) given by

\[
U = x(V + 1)^2, \quad V = y(U + 1)^2
\]

and

\[
W(U, V, w) = \frac{-w_1(U, V, w) + (U - w + 1)\sqrt{w_2(U, V, w)}}{2(V + 1)^2(Vw + U^2 + 2U + 1)}
\]

with polynomials \( w_1 = w_1(U, V, w) \) and \( w_2 = w_2(U, V, w) \) given by

\[
w_1 = -UVw^2 + w(1 + 4V + 3UV^2 + 5V^2 + U^2 + 2U + 2V^3 + 3U^2V + 7UV) \\
+ (U + 1)^2(U + 2V + 1 + V^2),
\]

\[
w_2 = U^2V^2w^2 - 2wUV(2U^2V + 6UV + 2V^3 + 3UV^2 + 5V^2 + U^2 + 2U + 4V + 1) \\
+ (U + 1)^2(U + 2V + 1 + V^2)^2.
\]
Random Planar Graphs

Denise, Vasconcellos, Welsh (1996)

\[ \mathbb{P}\{e(R_n) > \frac{3}{2}n\} \to 1, \quad \mathbb{P}\{e(R_n) < \frac{5}{2}n\} \to 1. \]

\( e(R_n) \) ... number of edges in random planar graphs \( R_n \)
Note that \( 0 \leq e \leq 3n \) for all planar graphs.

McDiarmid, Steger, Welsh (2005)

\[ \mathbb{P}\{H \text{ appears in } R_n \text{ at least } \alpha n \text{ times}\} \to 1 \]

\( H \) ... any fixed planar graph, \( \alpha > 0 \) sufficiently small.
Random Planar Graphs

Appearance of $H$: 

![Diagram showing the appearance of $H$]
Random Planar Graphs

Consequences:

\[ \Pr \{ \text{There are } \geq \alpha n \text{ vertices of degree } k \} \rightarrow 1 \]

$k > 0$ a given integer, $\alpha > 0$ sufficiently small.

\[ \Pr \{ \text{There are } \geq C^n \text{ automorphisms} \} \rightarrow 1 \]

for some $C' > 1$. 

Random Planar Graphs

Further Results:

\[ \mathbb{P} \{ \mathcal{R}_n \text{ is connected} \} \geq \gamma > 0 \]

[McDiarmid+Reed]

\[ \mathbb{E} \Delta(\mathcal{R}_n) = \Theta(\log n) \]

\( \Delta(\mathcal{R}_n) \) ... maximum degree in \( \mathcal{R}_n \)
The number of planar graphs

[Bender, Gao, Wormald (2002)]

\( b_n \) ... number of \textbf{2-connected} labelled planar graphs

\[
 b_n \sim c \cdot n^{-\frac{7}{2}} \gamma_2^n n!, \quad \gamma_2 = 26.18... 
\]

[Gimenez+Noy (2005)]

\( g_n \) ... number of all labelled planar graphs

\[
 g_n \sim c \cdot n^{-\frac{7}{2}} \gamma^n n!, \quad \gamma = 27.22... 
\]
The number of planar graphs

[Gimenez+Noy (2005)]

- $e(\mathcal{R}_n)$ satisfies a central limit theorem:
  
  $\mathbb{E} e(\mathcal{R}_n) \sim 2.21\ldots \cdot n$, \quad $\forall e(\mathcal{R}_n) \sim c \cdot n$.

  $\mathbb{P} \{|e(\mathcal{R}_n) - 2.21\ldots \cdot n| > \varepsilon n\} \leq e^{-\alpha(\varepsilon) \cdot n}$

- Connectedness:

  $\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \to e^{-\nu} = 0.96\ldots$

  number of components of $\mathcal{R}_n := C_n \to 1 + Po(\nu)$. 
Degree Distribution

**Theorem [D. + Gimenez + Noy]**

Let \( d_{n,k} \) be the probability that a random node in a random planar graph \( \mathcal{R}_n \) has degree \( k \). Then the limit

\[
   d_k := \lim_{n \to \infty} d_{n,k}
\]

exists. The probability generating function

\[
p(w) = \sum_{k \geq 1} d_k w^k
\]

can be explicitly computed.

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<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( d_3 )</th>
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Degree Distribution

More precisely ...

• Implicit equation for $D_0(y, w)$:

$$
1 + D_0 = (1 + y[w]) \exp \left( \frac{\sqrt{S} (D_0 (t - 1) + t)}{4(3t + 1)(D_0 + 1)} - \frac{D_0^2 (t^4 - 12t^2 + 20t - 9) + D_0 (2t^4 + 6t^3 - 6t^2 + 10t - 12) + t^4 + 6t^3 + 9t^2)}{4(t + 3)(D_0 + 1)(3t + 1)} \right),
$$

where $t = t(y)$ satisfies $y + 1 = \frac{1 + 2t}{(1 + 3t)(1 - t)} \exp \left( -\frac{1}{2} \frac{t^2(1 - t)(18 + 36t + 5t^2)}{(3 + t)(1 + 2t)(1 + 3t)^2} \right)$.

and $S = (D_0(t - 1) + t)(D_0(t - 1)^3 + t(t + 3)^2)$.

• Explicit expressions in terms of $D_0(y, w)$:

$$D_2(y, w), \ D_3(y, w), \ B_0(y, w), \ B_2(y, w), \ B_3(y, w)$$

• Explicit expression for $p(w)$:

$$p(w) = -e^{B_0(1,w) - B_0(1,1)} B_2(1, w) + e^{B_0(1,w) - B_0(1,1)} \frac{1 + B_2(1, 1)}{B_3(1, 1)} B_3(1, w)$$
Degree Distribution

Consequences

• Expected number $X_{n,k}$ of vertices of degree $k$:

$$\mathbb{E} X_{n,k} = d_{n,k} \cdot n \sim d_k \cdot n, \quad d_k > 0.$$  

• Tails of the degree distribution:

$$d_k \sim c \cdot k^{-\frac{1}{2}} q^k, \quad q = 0.79...$$
Degree Distribution

Conjecture for maximum degree $\Delta(R_n)$:

$$\mathbb{E} \Delta(R_n) \sim \frac{\log n}{\log(1/q)}$$

Remark.

Corresponding results on the degree distribution and the maximum degree are known for random planar maps: [Liskovets, Gao+Wormald]
Degree Distribution

**Theorem [D.‡Gimenez‡Noy]**

Let \( d_{n,k}^{(2)} \) resp. \( d_{n,k}^{(3)} \) be the probability that a random node in a random 2-connectet resp. 3-connected planar graph with \( n \) vertices has degree \( k \). Then the limits

\[
\lim_{n \to \infty} d_{n,k}^{(2)} \quad \text{and} \quad \lim_{n \to \infty} d_{n,k}^{(3)}
\]

exists. The probability generating functions

\[
p^{(2)}(w) = \sum_{k \geq 1} d_{k}^{(2)} w^k \quad \text{and} \quad p^{(3)}(w) = \sum_{k \geq 1} d_{k}^{(3)} w^k
\]

can be explicitly computed. Asymptotically we have

\[
d_{k}^{(2)} \sim c \cdot k^{1/2} q^k, \quad q = \sqrt{7} - 2 \quad \text{and} \quad d_{k}^{(3)} \sim c \cdot k^{-1/2} q^k, \quad q = 0.673\ldots
\]
Generating Functions

• $g_n$ ... all planar graphs with $n$ vertices:

$$g(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!}$$

• $c_n$ ... connected planar graphs with $n$ vertices:

$$c(x) = \sum_{n \geq 0} c_n \frac{x^n}{n!}$$

• $b_n$ ... 2-connected planar graphs with $n$ vertices:

$$b(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$$
Generating Functions

- $g_{n,m}$ ... all planar graphs with $n$ vertices and $m$ edges:
  \[ g(x, y) = \sum_{n,m \geq 0} g_{n,m} \frac{x^n}{n!} y^m \]

- $c_{n,m}$ ... connected planar graphs with $n$ vertices and $m$ edges:
  \[ c(x, y) = \sum_{n,m \geq 0} c_{n,m} \frac{x^n}{n!} y^m \]

- $b_{n,m}$ ... 2-connected planar graphs with $n$ vertices and $m$ edges:
  \[ b(x, y) = \sum_{n,m \geq 0} b_{n,m} \frac{x^n}{n!} y^m \]
Generating Functions

\[ G(x, y) = \exp \left( C(x, y) \right), \]
\[ \frac{\partial C(x, y)}{\partial x} = \exp \left( \frac{\partial B}{\partial x} \left( x \frac{\partial C(x, y)}{\partial x}, y \right) \right), \]
\[ \frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y}, \]
\[ \frac{M(x, D)}{2x^2D} = \log \left( \frac{1 + D}{1 + y} \right) - \frac{xD^2}{1 + xD}, \]
\[ M(x, y) = x^2y^2 \left( \frac{1}{1 + xy} + \frac{1}{1 + y} - 1 - \frac{(1 + U)^2(1 + V)^2}{(1 + U + V)^3} \right), \]
\[ U = xy(1 + V)^2, \]
\[ V = y(1 + U)^2. \]
Generating Functions

\[ G(x, y) = \exp(C(x, y)) \]
\[
\frac{\partial C(x, y)}{\partial x} = \exp \left( \frac{\partial B}{\partial x} \left( x \frac{\partial C(x, y)}{\partial x}, y \right) \right)
\]
Generating Functions

\[ C^\bullet = \frac{\partial C}{\partial x} \quad \text{GF, where one vertex is marked but not counted} \]

\[ w \quad \text{additional variable that counts the degree of the marked vertex} \]

Generating functions:

\[
\begin{align*}
    G^\bullet(x, y, w) & \quad \text{all rooted planar graphs} \\
    C^\bullet(x, y, w) & \quad \text{connected rooted planar graphs} \\
    B^\bullet(x, y, w) & \quad \text{2-connected rooted planar graphs} \\
    T^\bullet(x, y, w) & \quad \text{3-connected rooted planar graphs}
\end{align*}
\]

Note that \( G^\bullet(x, y, 1) = \frac{\partial G}{\partial x}(x, y) \) etc.
Generating Functions

\[ G^\bullet(x, y, w) = \exp(C(x, y, 1)) C^\bullet(x, y, w), \]

\[ C^\bullet(x, y, w) = \exp(B^\bullet(xC^\bullet(x, y, 1), y, w)), \]

\[ w \frac{\partial B^\bullet(x, y, w)}{\partial w} = xyw \exp \left( S(x, y, w) + \frac{1}{x^2 D(x, y, w)} T^\bullet \left( x, D(x, y, 1), \frac{D(x, y, w)}{D(x, y, 1)} \right) \right) - 1 \]

\[ D(x, y, w) = (1 + yw) \exp \left( S(x, y, w) + \frac{1}{x^2 D(x, y, w)} \times \right. \]

\[ \times T^\bullet \left( x, D(x, y, 1), \frac{D(x, y, w)}{D(x, y, 1)} \right) \left. \right) - 1 \]

\[ S(x, y, w) = xD(x, y, 1) (D(x, y, w) - S(x, y, w)), \]

\[ T^\bullet(x, y, w) = \frac{x^2 y^2 w^2}{2} \left( \frac{1}{1 + wy} + \frac{1}{1 + xy} - 1 - \frac{(u + 1)^2 \left( -w_1(u, v, w) + (u - w + 1) \sqrt{w_2(u, v, w)} \right)}{2w(vw + u^2 + 2u + 1)(1 + u + v)^3} \right), \]

\[ u(x, y) = xy(1 + v(x, y))^2, \quad v(x, y) = y(1 + u(x, y))^2. \]
Asymptotics for Generating Functions

Singularity analysis

Suppose that

\[ f(z) = \sum_{n \geq 0} a_n z^n = A_0 + A_2 Z^2 + A_3 Z^3 + O(Z^4), \]

with

\[ Z = \sqrt{1 - \frac{z}{\rho}} \]

(plus some technical conditions).

\[ \implies a_n = \frac{3 A_3}{4 \sqrt{\pi}} \rho^{-n} n^{-5/2} + O(\rho^{-n} n^{-3}) \]
3-connected planar graphs

\[ \tilde{u}_0(y) = -\frac{1}{3} + \sqrt{\frac{4}{9} + \frac{1}{3y}} \]

\[ r(y) = \frac{\tilde{u}_0(y)}{y(1 + y(1 + \tilde{u}_0(y))^2)^2} \]

\[ \tilde{X} = \sqrt{1 - \frac{x}{r(y)}} \]

\[ \implies T^\bullet(x, y, w) = \tilde{T}_0(y, w) + \tilde{T}_2(y, w) \tilde{X}^2 + \tilde{T}_3(y, w) \tilde{X}^3 + O(\tilde{X}^4) \]
Asymptotics for Generating Functions

2-connected planar graphs

\[ \tau(x) \quad \text{... inverse function of } r(y) \]

\[ D(R(y), y, 1) = \tau(R(y)) \]

\[ X = \sqrt{1 - \frac{x}{R(y)}} \]

\[ \Rightarrow \]

\[ D(x, y, w) = D_0(y, w) + D_2(y, w)X^2 + D_3(y, w)X^3 + O(X^4), \]

\[ \Rightarrow \]

\[ B^\bullet(x, y, w) = B_0(y, w) + B_2(y, w)X^2 + B_3(y, w)X^3 + O(X^4) \]
Asymptotics for Generating Functions

Lemma

\[ f(x) = \sum_{n \geq 0} \frac{a_n x^n}{n!} = f_0 + f_2 X^2 + f_3 X^3 + O(X^4), \quad X = \sqrt{1 - \frac{x}{\rho}}, \]

\[ H(x, z, w) = h_0(x, w) + h_2(x, w) Z^2 + h_3(x, w) Z^3 + O(Z^4), \quad Z = \sqrt{1 - \frac{z}{f(\rho)}}, \]

\[ f_H(x) = H(x, f(x), w) = \sum_{n \geq 0} b_n(w) \frac{x^n}{n!} \]

\[ \lim_{n \to \infty} \frac{b_n(w)}{a_n} = \frac{h_2(\rho, w)}{f_0} + \frac{h_3(\rho, w)}{f_3} \left( -\frac{f_2}{f_0} \right)^{3/2}. \]
Asymptotics for Generating Functions

connected planar graphs

\[ C^\bullet(x, 1, w) = \exp \left( B^\bullet \left( xC'(x), 1, w \right) \right) \]

Application of the lemma with 

\[ f(x) = xC'(x) \]

and 

\[ H(x, z, w) = xe^{B^\bullet(z, 1, w)}. \]
Thank You