RECURSIVE TREES

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- Cutting down Recursive Trees
- Plane Oriented Recursive Trees

Methodology

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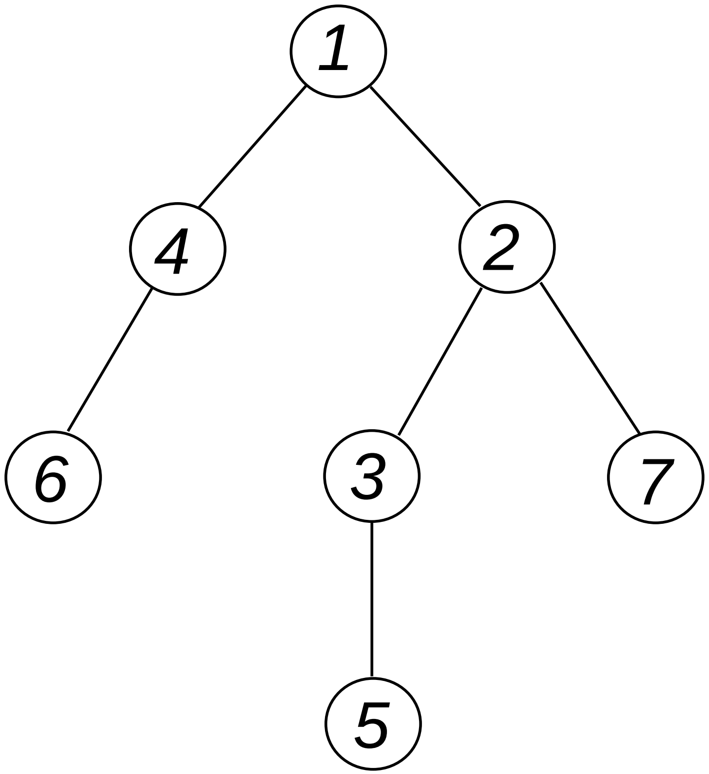
- Heavy use of generating function

Recursive Trees



Recursive Trees

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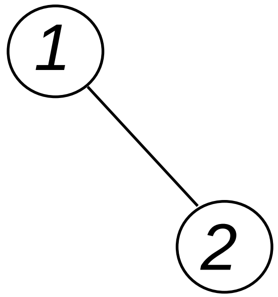
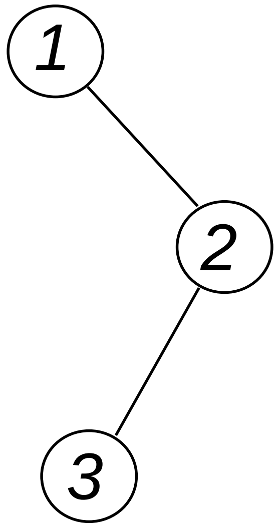


Recursive Trees



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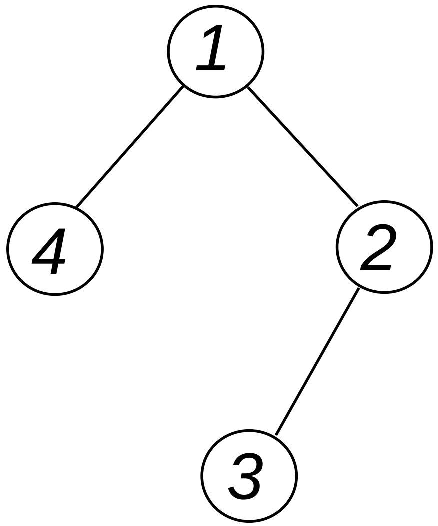

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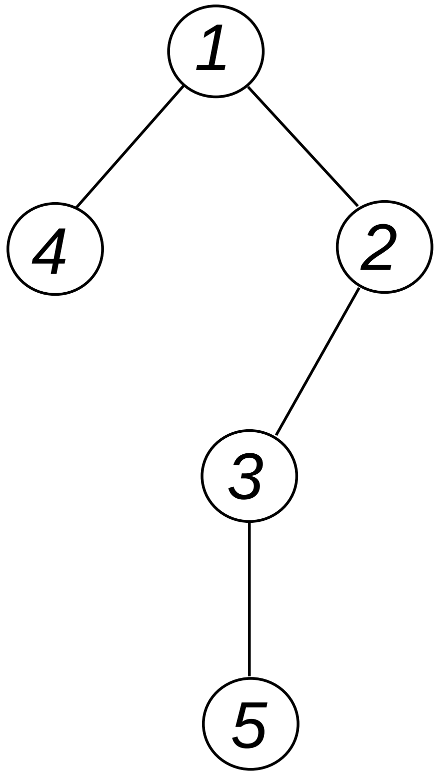
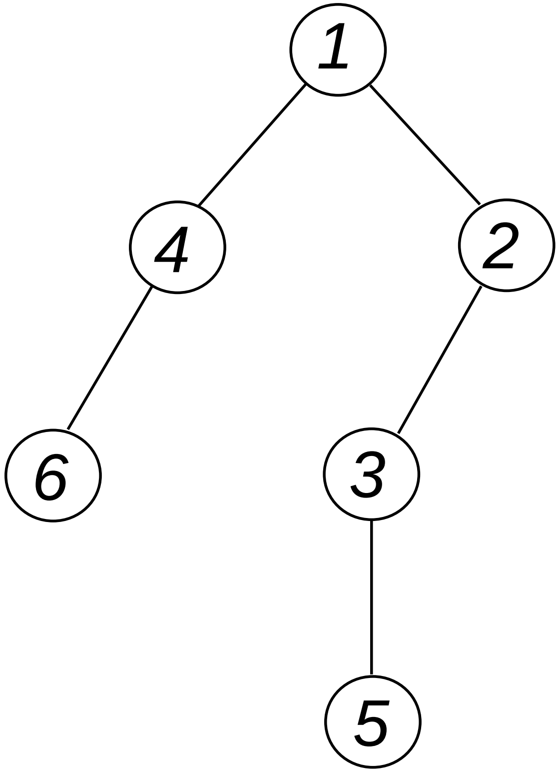
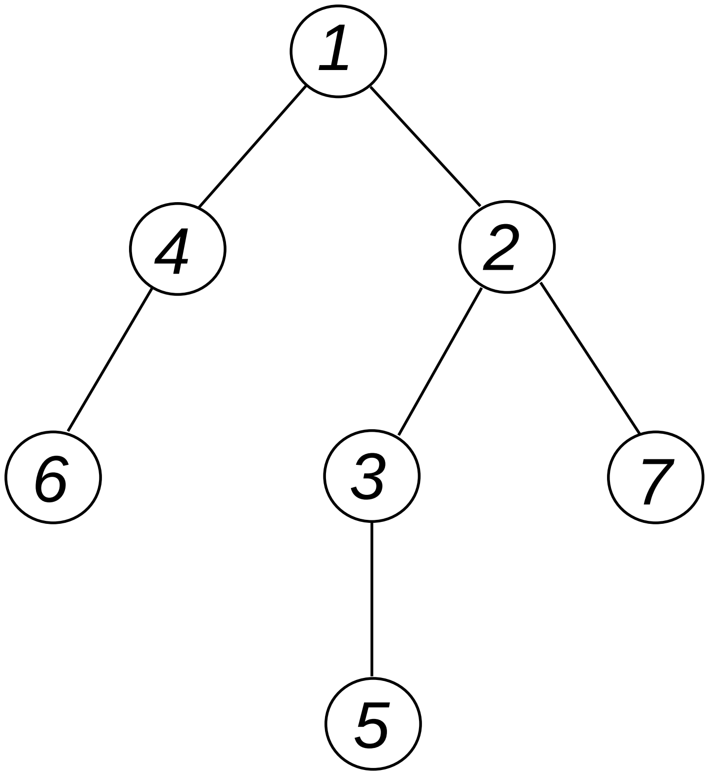


Recursive Trees



Recursive Trees



Recursive Trees

Combinatorial Description

- labelled rooted tree
- labels are strictly increasing
- no left-to-right order (non-planar)

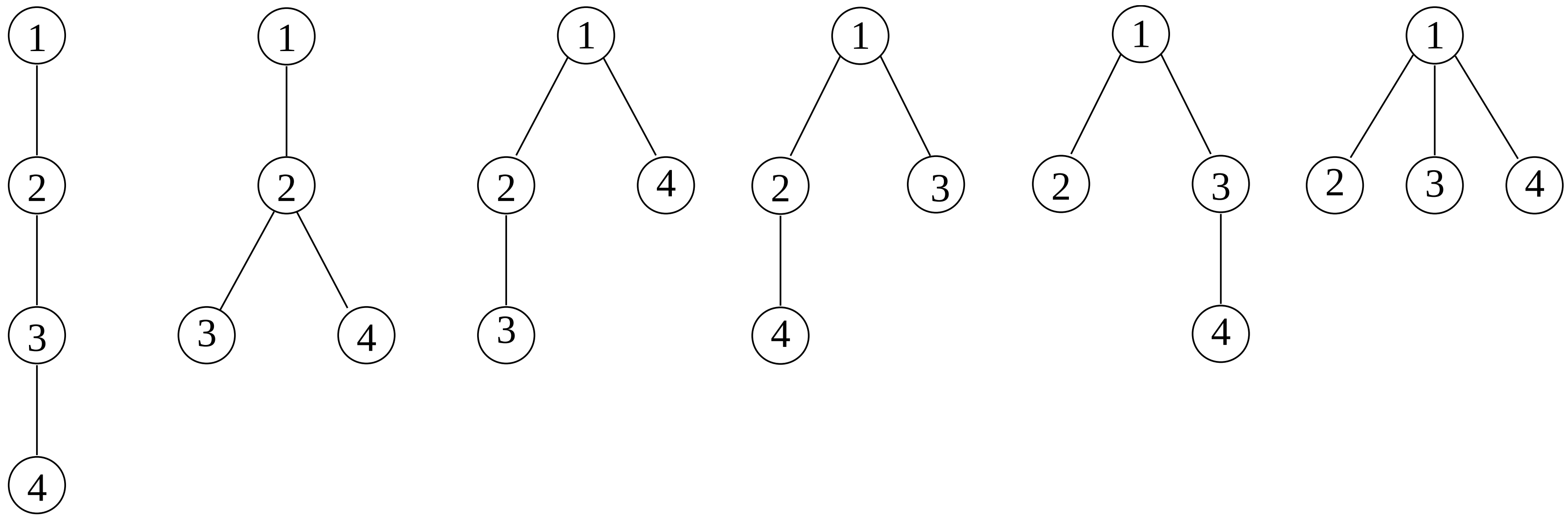
Recursive Trees

Motivations

- spread of epidemics
- pyramid schemes
- familiy trees of preserved copies of ancient texts
- convex hull algorithms
- ...

Enumeration of Recursive Trees

All recursive trees of size 4:



Enumeration of Recursive Trees

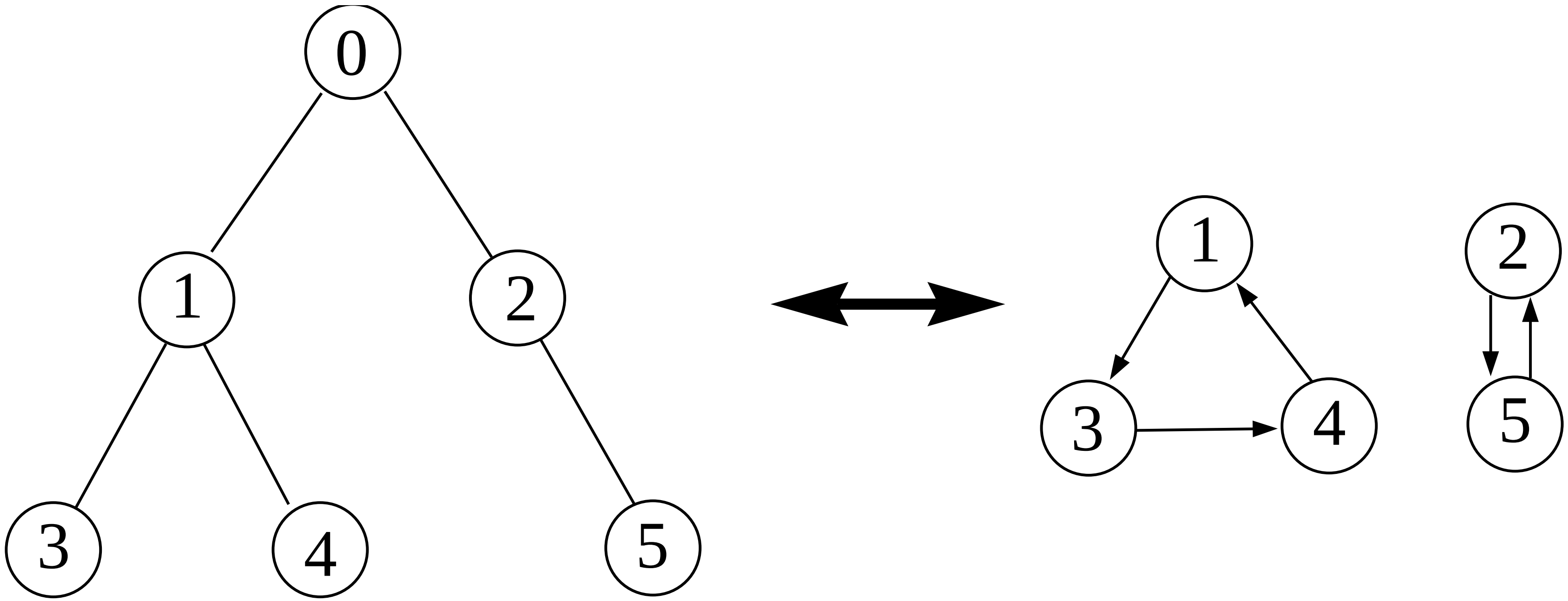
Number of recursive trees

$$\begin{aligned} y_n &= \text{number of recursive trees of size } n \\ &= (n-1)! \end{aligned}$$

The node with label j has exactly $j-1$ possibilities to be inserted $\Longrightarrow y_n = 1 \cdot 2 \cdots (n-1)$.

Enumeration of Recursive Trees

Bijection to permutations



root degree = number of cycles

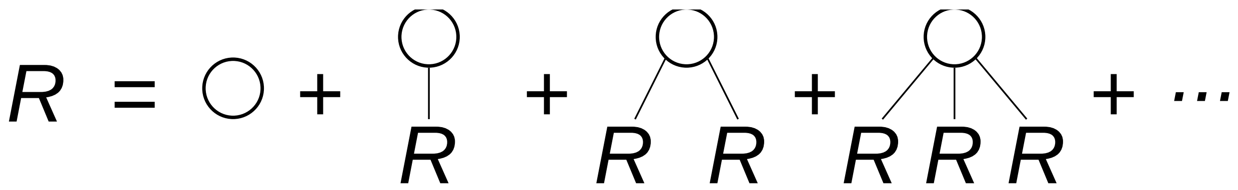
subtree sizes = cycle lengths

Enumeration of Recursive Trees

Generating Functions:

$$y(x) = \sum_{n\ge 1} y_n \frac{x^n}{n!} = \sum_{n\ge 1} \frac{x^n}{n} = \log \frac{1}{1-x}$$

$$\boxed{y'(x) = 1 + y(x) + \frac{y(x)^2}{2!} + \frac{y(x)^3}{3!} + \cdots = e^{y(x)}}$$



A recursive tree can be interpreted as a root followed by an unordered sequence of recursive trees. $(y'(x) = \sum_{n\ge 0} y_{n+1} x^n/n!)$

Random Recursive Trees

Probability Model:

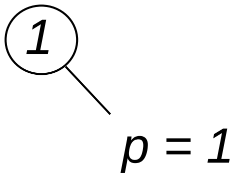
Process of growing trees:

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node with probability $1/(j-1)$.

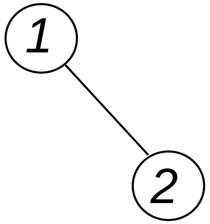
After n steps every tree (of size n) has equal probability $1/(n-1)!$.

Random Recursive Trees

Random Recursive Trees

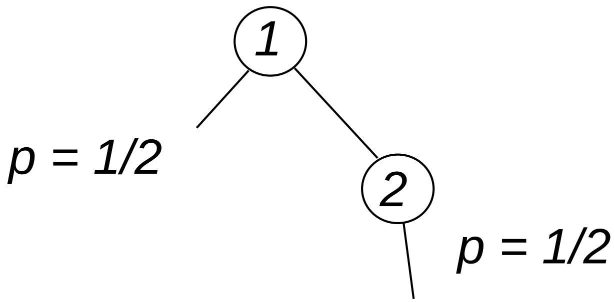


Random Recursive Trees

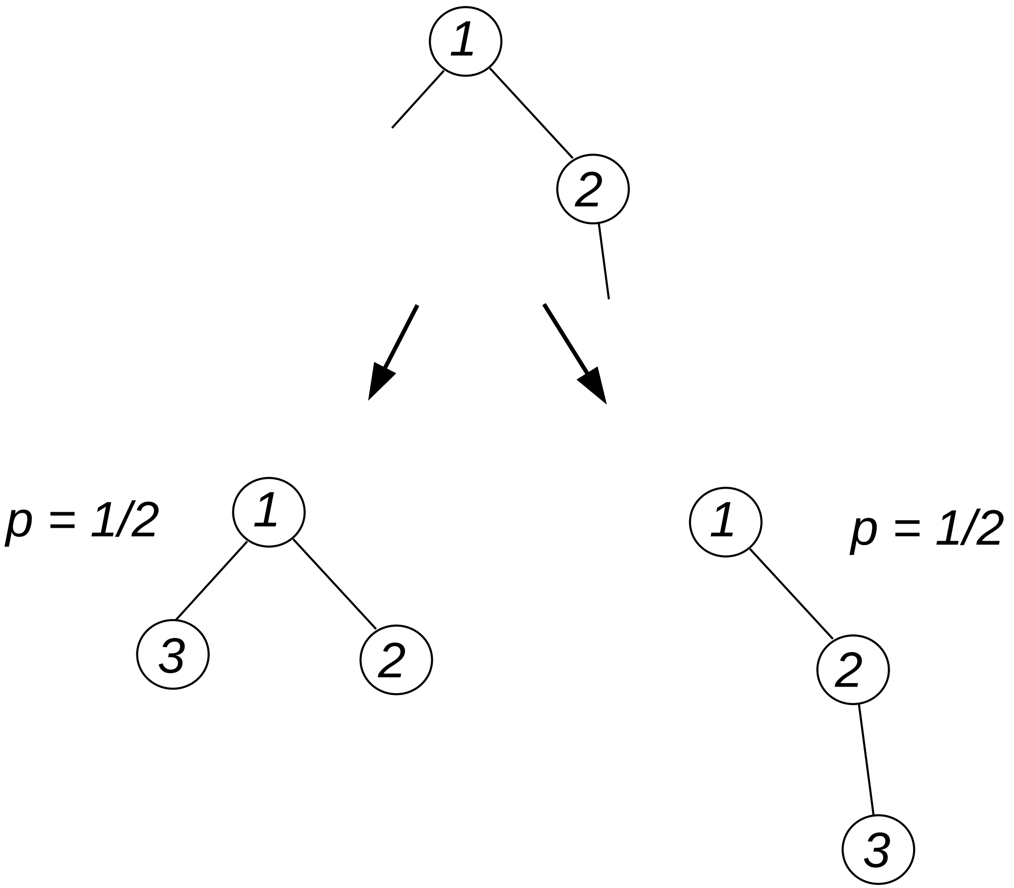


p = 1

Random Recursive Trees

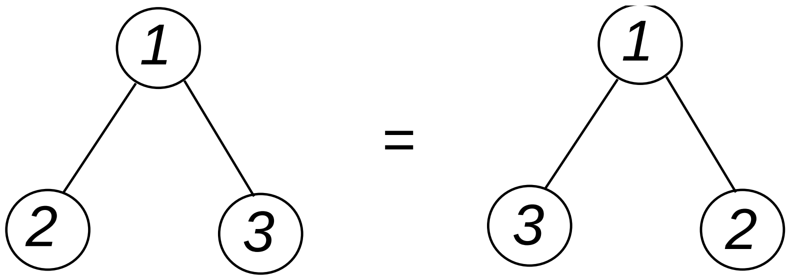


Random Recursive Trees



Random Recursive Trees

Remark: left-to-right order is irrelevant



Profile of Recursive Trees

First sample of shape parameters:

- **insertion depth** of the n-th node: D_n
- **path length**: I_n (sum of alle distances to the root)
- **height** H_n (maximal distance to the root)
- **degree distribution**
- **profile** $X_{n,k}$ (number of nodes at level k)

Profile of Recursive Trees

First sample of shape parameters:

- **insertion depth** of the n-th node: D_n
- **path length**: I_n (sum of alle distances to the root)
- **height** H_n (maximal distance to the root)
- **degree distribution**
- **PROFILE** $X_{n,k}$ (number of nodes at level k)

Profile of Recursive Trees

Relevance of the profile $X_{n,k}$:

- $\mathbb{P}\{D_n = k\} = \dfrac{1}{n-1}\,\mathbb{E}\,X_{n-1,k-1}$

- $I_n = \sum_{k \geq 0} k X_{n,k}$

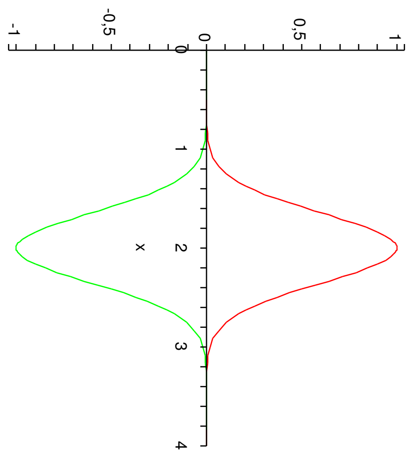
- $H_n = \max\{k \geq 0 : X_{n,k} > 0\}$

- The **profile** descibes the **shape** of the tree.

Profile of Recursive Trees

Average profile:

$$\mathbb{E}\,X_{n,k} = \frac{n}{\sqrt{2\pi \log n}} \left(e^{-\frac{(k-\log n)^2}{2\log n}} + \mathcal{O}\left(\frac{1}{\sqrt{\log n}} \right) \right).$$



Profile of Recursive Trees

Central limit theorem for the insertion depth [Devroye, Mahmoud]

$$\boxed{\frac{D_n - \log n}{\sqrt{\log n}} \to N(0,1)}$$

$$\mathbb{E}\, D_n = \log n + O(1), \qquad \mathbb{V}\, D_n = \log n + O(1).$$

Profile of Recursive Trees

Lemma [Dondajewski+Szymánski]

$$\mathbb{E}\, X_{n,k} = [u^k]\binom{n+u-1}{n-1} = \frac{|s_{n,k}|}{(n-1)!}$$

$s_{n,k}$ ... **Stirling numbers** of the first kind
$|s_{n,k}|$ ... number of permutations of $\{1,\ldots n\}$ with k cycles:

$$\sum_{k=0}^{n} s_{n,k}u^k = u(u-1)\cdots(u-n+1)$$
$$\sum_{k=0}^{n} |s_{n,k}|u^k = u(u+1)\cdots(u+n-1)$$

Stirling Numbers

$$s_{n+1,k} = s_{n,k-1} - n s_{n,k}, \qquad |s_{n+1,k}| = |s_{n,k-1}| + n|s_{n,k}|$$

$s_{n,k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=0$	1								
$n=1$	0	1							
$n=2$	0	-1	1						
$n=3$	0	2	-3	1					
$n=4$	0	-6	11	-6	1				
$n=5$	0	24	-50	35	-10	1			
$n=6$	0	-120	274	-225	85	-15	1		
$n=7$	0	720	-1764	1624	-735	175	-21	1	
$n=8$	0	-5040	13068	-13132	6769	-1960	322	-28	1

Profile of Recursive Trees

Remark

$$\binom{n+\alpha-1}{n} = (-1)^n \binom{-\alpha}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right)$$

Corollary

$$|s_{n,k}| = \frac{(n-1)!(\log n)^k}{k!\Gamma(\frac{k}{\log n}+1)} \left(1 + O\left(\frac{1}{n}\right)\right)$$
$$\sim \frac{n!}{\sqrt{2\pi \log n}} e^{-\frac{(k-\log n)^2}{2\log n}}$$

Proof:

$$\binom{n+u-1}{n-1} \sim \frac{n^u}{\Gamma(u+1)} \quad \Longrightarrow \quad [u^k]\binom{n+u-1}{n-1} \sim \frac{(\log n)^k}{k!\Gamma(\frac{k}{\log n}+1)}.$$

Profile of Recursive Trees

Corollary

$$\mathbb{P}\{D_n = k\} = \frac{\mathbb{E}\, X_{n-1,k-1}}{n-1} \sim \frac{1}{\sqrt{2\pi \log n}} e^{-\frac{(k-\log n)^2}{2\log n}}$$

This implies the **central limit theorem** for D_n.

Cycles in Permutations

$|s_{n,k}|$ = number of permutations of $\{1, \ldots n\}$ with k cycles

Corollary

C_n ... random number of **cycles in permutations**

$$\boxed{\frac{C_n - \log n}{\sqrt{\log n}} \to N(0,1)}$$

Corollary

R_n ... **root degree** of random recursive trees

$$\boxed{\frac{R_n - \log n}{\sqrt{\log n}} \to N(0,1)}$$

Profile of Recursive Trees

Profile polynomial

$$\boxed{W_n(u) = \sum_{k \geq 0} X_{n,k}\, u^k}$$

Lemma. The normalized profile polynomial

$$M_n(u) = \frac{W_n(u)}{\mathbb{E}\, W_n(u)}$$

is a **martingale** (with respect to the natural filtration related to the tree evolution process).

Profile of Recursive Trees

Theorem [Chauvin+Drmota+Jabbour for binary search trees]

$$\left(\frac{W_n(u)}{\mathbb{E}\,W_n(u)}, u \in B\right) \to (M(u), u \in B)$$

for a suitable domain $B \subseteq \mathbb{C}$.

Remarks

- $(M(u), u \in B)$ stochastic process of **random analytic functions**.

- Fixed point equation:

$$\boxed{M(u) \equiv uU^u M^{(1)}(u) + (1-U)^u M^{(2)}(u)},$$

  where $M^{(1)}(u)$ and $M^{(2)}(u)$ are independent copies of $M(u)$, U is uniform in $[0,1]$ and $(U, M^{(1)}(u), M^{(2)}(u))$ are independent.

Profile of Recursive Trees

Theorem [Chauvin+Drmota+Jabbour for binary search trees]

$$\left(\frac{X_{n,\lfloor \alpha \log n \rfloor}}{\mathbb{E}\, X_{n,\lfloor \alpha \log n \rfloor}}, \alpha \in I\right) \to (M(\alpha), \alpha \in I)\,.$$

Idea

$$\begin{aligned} X_{n,k} &= [u^k] W_n(u) \\ &= [u^k] M_n(u) \cdot \mathbb{E}\, W_n(u) \\ &\sim [u^k] M(u) \cdot \mathbb{E}\, W_n(u) \cdot \\ &\sim M(\alpha)[u^k] \mathbb{E}\, W_n(u) = M(\alpha) \mathbb{E}\, X_{n,k}. \end{aligned}$$

$\alpha = k/\log n$ ... saddle point the function $n^u u^{-k}$.

Path Length

Remark

$$M_n'(1) = \frac{I_n - \mathbb{E}\, I_n}{n}$$

Corollary

$$\boxed{\frac{I_n - \mathbb{E}\, I_n}{n} \longrightarrow M'(1)}$$

The random variable $M'(1)$ is not normal. Note also that $\mathbb{E}\, I_n \sim n \log n$.

Leaves in Recursive Trees

Theorem [Najock+Heyde]

L_n ... number of leaves in a random recursive tree of size n

$$\mathbb{P}\{L_n = k\} = \frac{1}{(n-1)!} \left\langle {n-1 \atop k-1} \right\rangle .$$

$\left\langle {n \atop k} \right\rangle$ ... Eulerian numbers

Remark

$\left\langle {n-1 \atop k-1} \right\rangle =$ number or recursive trees of size n with k leaves.

Eulerian Numbers

$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle$$

$\left\langle {n \atop k} \right\rangle$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=0$	1								
$n=1$	1	1							
$n=2$	1	4	1						
$n=3$	1	11	11	1					
$n=4$	1	26	66	26	1				
$n=5$	1	57	302	302	57	1			
$n=6$	1	120	1191	4216	1191	120	1		
$n=7$	1	247	4293	15619	15619	4293	247	1	
$n=8$	1	502	14608	88234	156190	88234	14608	502	1

Leaves in Recursive Trees

Corollary

$$\boxed{\frac{L_n - \frac{n}{2}}{\sqrt{\frac{7}{12}n}} \to N(0,1)}$$

$$\mathbb{E}\, L_n = \frac{n}{2}, \qquad \mathbb{E}\,(L_n)^2 = \frac{1}{12}(3(n-1)^2 + 13(n-1) + 14),$$

Leaves in Recursive Trees

Generating functions

$\ell_{n,k}$ ... number or recursive trees of size n with k leaves.

$$y(x,u) = \sum_{n,k} \ell_{n,k} \cdot u^k \cdot \frac{x^n}{n!} = \sum_{n,k} \mathbb{P}\{L_n = k\} \cdot u^k \cdot \frac{x^n}{n}$$

$$\frac{\partial y(x,u)}{\partial x} = u + e^{y(x,u)} - 1$$

$$\boxed{y(x,u) = (x-1)(u-1) + \log\left(\frac{u-1}{1-e^{(x-1)(u-1)}}\right)}$$

Degree Distribution

$\ell_{n,k}^{(d)}$ ... number or r.t.'s of size n with k nodes of outdegree d.

$L_n^{(d)}$ ... number of nodes of outdegree d in a random r.t. of size n:

$$\mathbb{P}\{L_n^{(d)} = k\} = \frac{\ell_{n,k}^{(d)}}{(n-1)!}$$

$\overline{D}_n$ ... **degree of a random node** in a random r.t. of size n

$$\mathbb{P}\{\overline{D}_n = d\} = \frac{1}{n}\,\mathbb{E}\,L_n^{(d)}$$

Theorem

$$\boxed{\mathbb{P}\{\overline{D}_n = d\} = \frac{1}{2^{d+1}} + O\left(\frac{1}{n^2}\frac{(2\log n)^d}{d!}\right)}$$

Degree Distribution

Generating functions

$$y(x,u) = \sum_{n,k} \ell_{n,k}^{(d)} \cdot u^k \cdot \frac{x^n}{n!} = \sum_{n,k} \mathbb{P}\{L_n^{(d)} = k\} \cdot u^k \cdot \frac{x^n}{n}$$

$$\boxed{\frac{\partial y(x,u)}{\partial x} = e^{y(x,u)} + (u-1)\frac{y(x,u)^d}{d!}}$$

$$Y(x) = \left.\frac{\partial y(x,u)}{\partial u}\right|_{u=1} = \sum_{n\ge 0} \mathbb{E}\, L_n^{(d)} \cdot \frac{x^n}{n} = \sum_{n\ge 0} \mathbb{P}\{\overline{D}_n = d\} \cdot x^n$$

$$\boxed{Y'(x) = \frac{1}{1-x}Y(x) + \frac{1}{d!}\left(\log\frac{1}{1-x}\right)^d}$$

Degree Distribution

$$Y'(x) = \frac{1}{1-x} Y(x) + \frac{1}{d!}\left(\log \frac{1}{1-x}\right)^d$$

$$\Longrightarrow \quad Y(x) = \frac{1}{2^{d+1}} \frac{1}{1-x} + (x-1)\sum_{j=0}^{d} \frac{1}{j!\,2^{d+1-j}}\left(\log \frac{1}{1-x}\right)^j$$

$$\Longrightarrow \quad \mathbb{P}\{\overline{D}_n = d\} = \frac{1}{2^{d+1}} + O\left(\frac{1}{n^2}\frac{(2\log n)^d}{d!}\right)$$

uniformly for $d \le (2-\varepsilon)\log n$.

Corollary

$$\mathbb{P}\{\overline{D}_n > d\} = \frac{1}{2^{d+1}} + O\left(\frac{1}{n^2}\frac{(2\log n)^d}{d!}\right)$$

Maximum Degree

Theorem [Szymánski, Pittel]

Δ_n ... **maximum node degree** in random r.t.'s.

$$\boxed{\mathbb{E}\,\Delta_n \sim \log_2 n}$$

Remark. This degree is much larger than the expected root degree with is about $\log n$.

First Moment Method

X ... discrete random variable on **non-negative integers**.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{X > 0\} \leq \min\{1, \mathbb{E}\, X\}}.$$

Proof

$$\mathbb{E}\, X = \sum_{k \geq 0} k\, \mathbb{P}\{X = k\} \geq \sum_{k \geq 1} \mathbb{P}\{X = k\} = \mathbb{P}\{X > 0\}.$$

Maximum Degree

Upper bound: first moment method

X_d ... number of nodes of degree $> d$:

$$\boxed{\mathbb{E}\, X_d = n\mathbb{P}\{\overline{D}_n > d\}} \qquad \boxed{\Delta_n > d \iff X_d > 0}$$

$$\begin{aligned}
\Longrightarrow \qquad \mathbb{E}\,\Delta_n &= \sum_{d\ge 0} \mathbb{P}\{\Delta_n > d\} \\
&= \sum_{d\ge 0} \mathbb{P}\{X_d > 0\} \\
&\le \sum_{d\ge 0} \min\{1, \mathbb{E}\, X_d\} \\
&\le \sum_{d\le \log_2 n} 1 + n \sum_{d>\log_2 n} \mathbb{P}\{\overline{D}_n > d\} \\
&= \log_2 n + O(1).
\end{aligned}$$

Second Moment Method

Theorem

X ... non-negative random variable with bounded second moment

$$\implies \boxed{\mathbb{P}\{X > 0\} \geq \frac{(\mathbb{E}\, X)^2}{\mathbb{E}\,(X^2)}}.$$

Proof

$$\mathbb{E}\, X = \mathbb{E}\left(X \cdot \mathbf{1}_{[X>0]}\right) \leq \sqrt{\mathbb{E}\,(X^2)}\sqrt{\mathbb{E}\,(\mathbf{1}_{[X>0]}^2)} = \sqrt{\mathbb{E}\,(X^2)}\sqrt{\mathbb{P}\{X > 0\}}$$

Remark In order to apply the second moment method to obtain a lower bound for $\mathbb{E}\,\Delta_n$ one needs estimates for $\mathbb{E}\,(X_d)^2$ which can be derived in a similar fashion as above.

Maximum Degree

Theorem [Goh+Schmutz]

$$\mathbf{P}\{\Delta_n \le d\} = \exp\left(-2^{-(d-\log_2 n+1)}\right) + o(1)$$

Remark. The limiting behaviour of Δ_n is related the the exteme value (= Gumbel) distribution ($F(t) = e^{-e^{-t}}$).

The distribution of Δ_n is extremely concentrated around $d \approx \log_2 n$.

The proof is an analytic "tour de force".

Height of Recursive Trees

Height H_n

Theorem [Devroye 1987, Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \longrightarrow e \qquad (a.s.)}$$

Height Distribution

$F(z)$ solution of

$$\boxed{y\,F(y/e^{1/e}) = \int_0^y F(z/e^{1/e})F(y-z)\,dz}$$

Recursive sequence of generating functions:

$$\boxed{y'_{k+1}(x) = e^{y_k(x)}},\ \ y_0(x) = 0,\ \ y_k(0) = 0.$$

Theorem [Drmota]

$$\mathbb{E}\,H_n = e\log n + O\Big(\sqrt{\log n}\,(\log\log n)\Big).$$

$$\boxed{\mathbf{P}\{H_n \le k\} = F(n/y'_k(1)) + o(1)}$$

$$\mathbf{P}\{|H_n - \mathbb{E}\,H_n| \ge \eta\} \ll e^{-c\eta} \quad (c > 0)$$

Height Distribution

$y_{n,k}$ ... number of r.t.'s of size n and height $\le k$:

$\mathbb{P}\{H_n \le k\} = y_{n,k}/(n-1)!$

$$y_k(x) = \sum_{n\ge 0} \mathbb{P}\{H_n \le k\} \frac{x^n}{n} = \sum_{n\ge 0} y_{n,k} \frac{x^n}{n!}$$

$$\boxed{y'_{k+1}(x) = e^{y_k(x)}}$$

$$Y_k(x) = y'_k(x) = \sum_{n\ge 0} \mathbb{P}\{H_{n+1} \le k\} x^n$$

$$\boxed{Y'_{k+1}(x) = Y_{k+1}(x) Y_k(x)}$$

$(Y_{k+1}(0) = 1)$

Height Distribution

$$y\, F(y/e^{1/e}) = \int_0^y F(z/e^{1/e}) F(y-z)\, dz$$

$$\Psi(u) = \int_0^\infty F(y) e^{-yu}\, dy$$

$$\boxed{\overline{Y}_k(x) = e^{k/e} \cdot \Psi\left(e^{k/e}(1-x)\right)}$$

Height Distribution

- $1 - \overline{Y}_k(0) \sim Ck \left(\frac{2}{e}\right)^k, \qquad \overline{Y}_k(1) = e^{k/e}.$

- $$\boxed{\overline{Y}'_{k+1}(x) = \overline{Y}_{k+1}(x)\overline{Y}_k(x)}$$

- For every positive integer ℓ and for every real number $k > 0$ the difference

$$Y_\ell(x) - \overline{Y}_k(x)$$

has exactly one zero (**"Intersection Property"**).

Height Distribution

- $\overline{Y}_k(x) = \sum_{n\geq 0} \overline{Y}_{n,k} x^n$ is an entire function with coefficients

$$\overline{Y}_{n,k} = \int_0^\infty F\left(ve^{-k/e}\right) v^n e^{-v}\, dv$$

and asymptotically we have

$$\boxed{\overline{Y}_{n,k} = F\left(ne^{-k/e}\right) + o(1)}$$

Height Distribution

Remark:

The functions

$$\overline{y}_k(x) = \int_0^x \overline{Y}_k(t)\,dt = \log \overline{Y}_{k+1}(x)$$

satisfy the recurrence

$$\boxed{\overline{y}_{k+1}(x) = e^{\overline{y}_k(x)}}$$

Height Distribution

Proof idea

- $Y_k(x)$ is approximated by the *auxiliary function* $\overline{Y}_{e_k}(x)$:

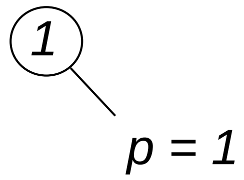
$$Y_k(1) = \overline{Y}_{e_k}(1) \quad \Longleftrightarrow \quad e_k = e \cdot \log Y_k(\rho) \sim k.$$

- $Y_k(x) \approx \overline{Y}_{e_k}(x)$ in a neighbourhood of $x = 1$

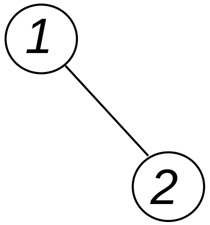
$$\Longrightarrow \quad \boxed{\mathbf{P}\{H_n \le k\} \approx \overline{Y}_{n,e_k} = F\Big(n/Y_k(1)\Big) + o(1)}$$

Plane Oriented Trees

Plane Oriented Trees

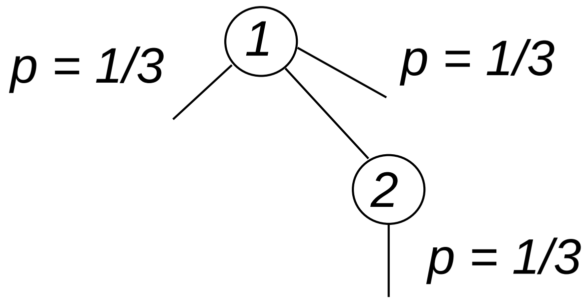


Plane Oriented Trees

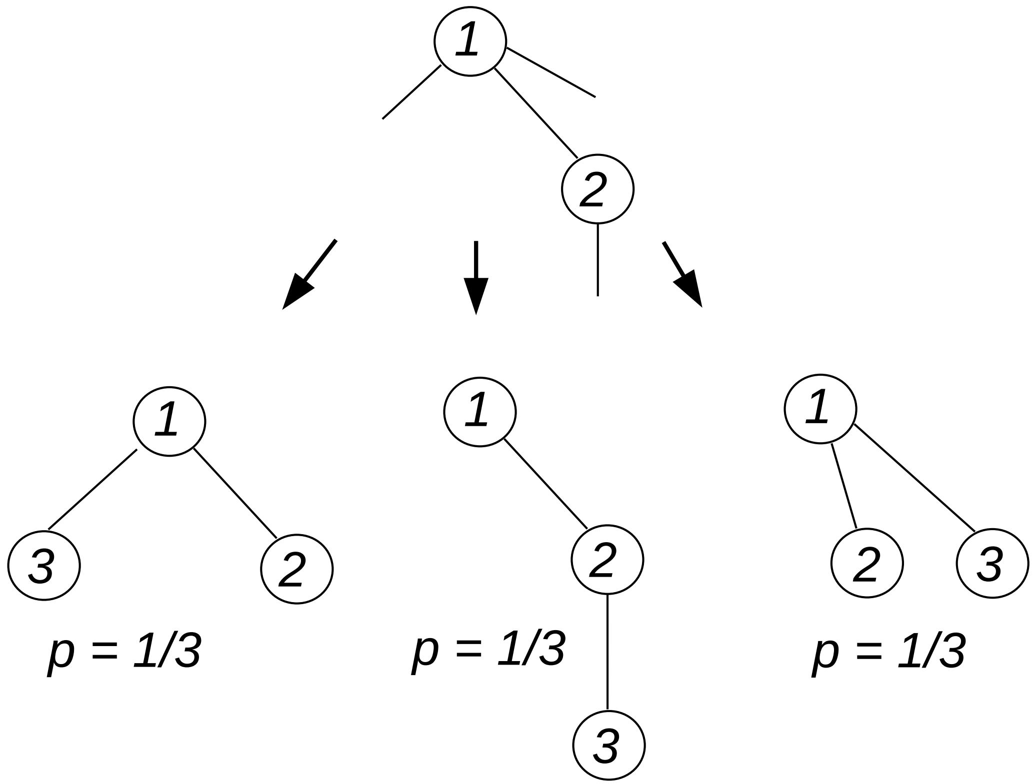


$p = 1$

Plane Oriented Trees



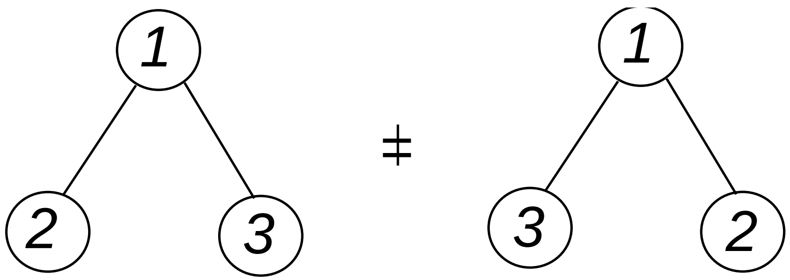
Plane Oriented Trees



p = 1/3 p = 1/3 p = 1/3

Plane Oriented Trees

Remark: left-to-right order is relevant



Plane Oriented Trees

Number of Plane Oriented Trees:

$$\begin{aligned} y_n &= \text{number of plane oriented trees of size } n \\ &= 1 \cdot 3 \cdot 5 \cdots (2n-3) = (2n-3)!! \\ &= \frac{(2n-2)!}{2^{n-1}(n-1)!} \end{aligned}$$

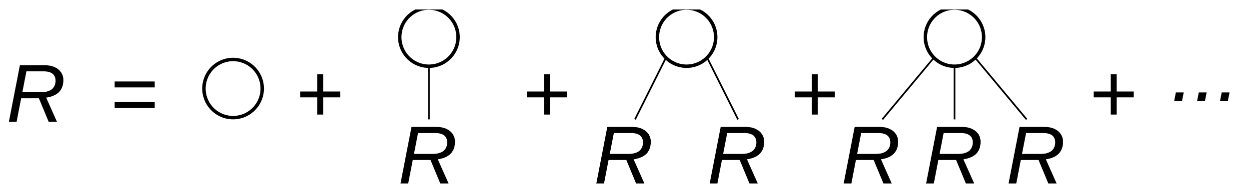
The node with label j has exactly $2j-3$ possibilities to be inserted $\Longrightarrow y_n = 1 \cdot 3 \cdots (2n-3)$.

Plane Oriented Trees

Generating Functions:

$$y(x) = \sum_{n \geq 1} y_n \frac{x^n}{n!} = \sum_{n \geq 1} \frac{1}{2^{n-1}} \binom{2(n-1)}{n-1} \frac{x^n}{n} = 1 - \sqrt{1 - 2x}$$

$$\boxed{y'(x) = 1 + y(x) + y(x)^2 + y(x)^3 + \cdots = \frac{1}{1 - y(x)}}$$



A plane oriented tree can be interpreted as a root followed by an **ordered** sequence of plane oriented trees. ($y'(x) = \sum_{n \geq 0} y_{n+1} x^n / n!$)

Plane Oriented Trees

Probability Model:

Process of growing trees

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node of outdegree d with probability $(d+1)/(2j-3)$.

After n steps every tree (of size n) has equal probability $1/(2n-3)!!$.

Plane Oriented Trees

Depth D_n of the n-th node

$$\mathbb{E}\, D_n = H_{2n-1} - \frac{1}{2} H_{n-1} = \frac{1}{2} \log n + O(1)$$

$$\begin{aligned} \mathbb{V}\, D_n &= H_{2n-1} - \frac{1}{2} H_{n-1} - H^{(2)}_{2n-1} + \frac{1}{4} H^{(2)}_{n-1} \\ &= \frac{1}{2} \log n + O(1) \end{aligned}$$

Central limit theorem:

$$\boxed{\frac{D_n - \frac{1}{2}\log n}{\sqrt{\frac{1}{2}\log n}} \to N(0,1)}$$

Plane Oriented Trees

Number L_n of leaves

$$\mathbb{E}\, L_n = \frac{2n-1}{3}$$

$$\mathbb{V}\, L_n = \frac{n}{9} - \frac{1}{18} - \frac{1}{6(2n-1)}$$

Central limit theorem:

$$\boxed{\frac{L_n - \frac{2}{3}n}{\sqrt{\frac{n}{9}}} \to N(0,1)}$$

Plane Oriented Trees

Distribution of out-degrees

$\overline{D}_n$ ... **degree of a random node** in a random p.o.r.t. of size n

$$\mathbb{P}\{\overline{D}_n = d\} = \frac{4}{(d+1)(d+2)(d+3)} + o(1)$$

Remark. $\frac{4}{(d+1)(d+2)(d+3)} \sim 4\, d^{-3}$ as $d \to \infty$.

Plane Oriented Trees

Root degree R_n

$$\mathbb{P}\{R_n = k\} = \frac{(2n-3-k)!}{2^{n-1-k}(n-1-k)!} \sim \sqrt{\frac{2}{\pi n}} e^{-k^2/(4n)}$$

$$\mathbb{E}\, R_n = \sqrt{\pi n} + O(1)$$

Plane Oriented Trees

Height H_n

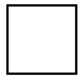
[Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \to \frac{1}{2s} = 1.79556\ldots \qquad (a.s.)}$$

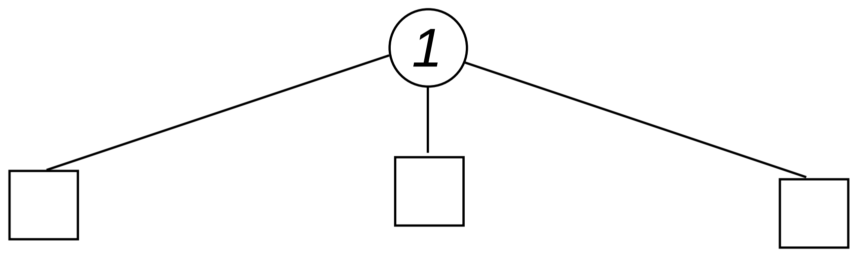
where $s = 0.27846\ldots$ is the positive solution of $se^{s+1} = 1$.

Precise results (as above) are also available ([Drmota]).

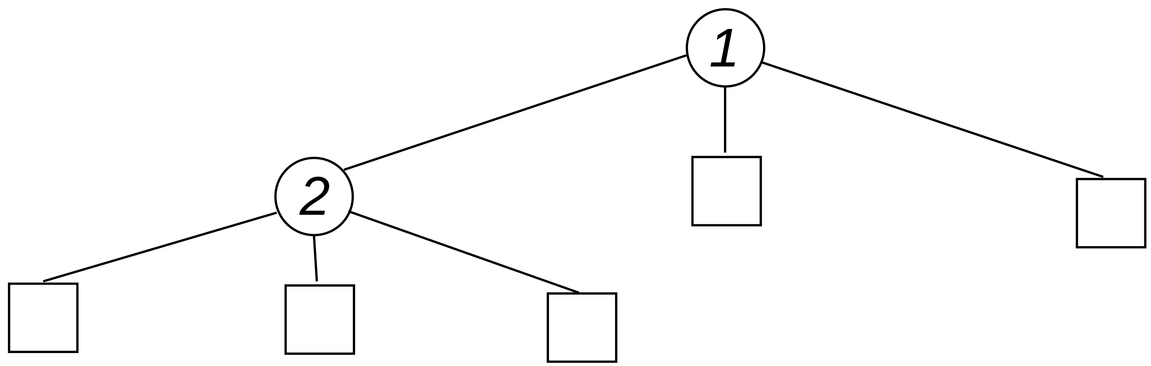
D-ary Recursive Trees



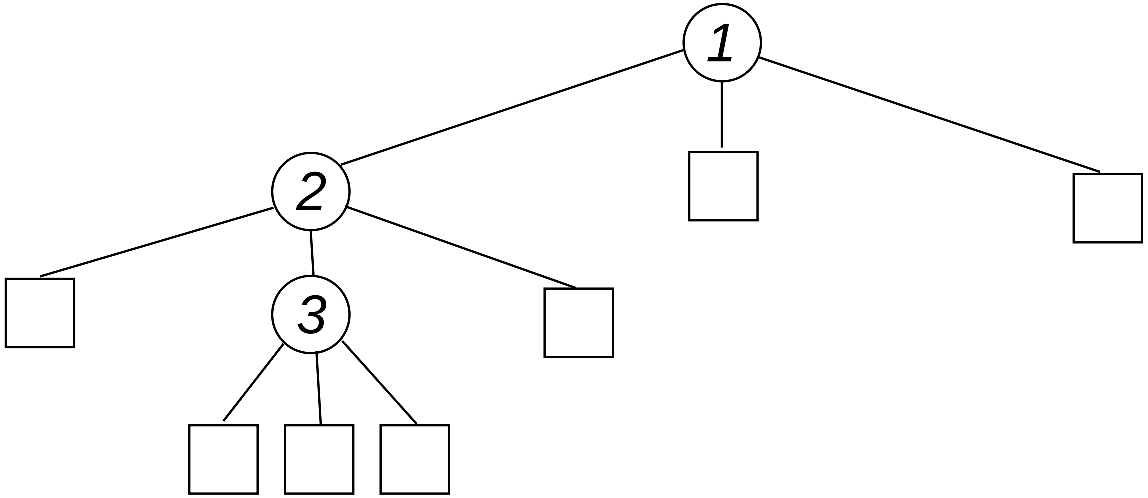
D-ary Recursive Trees



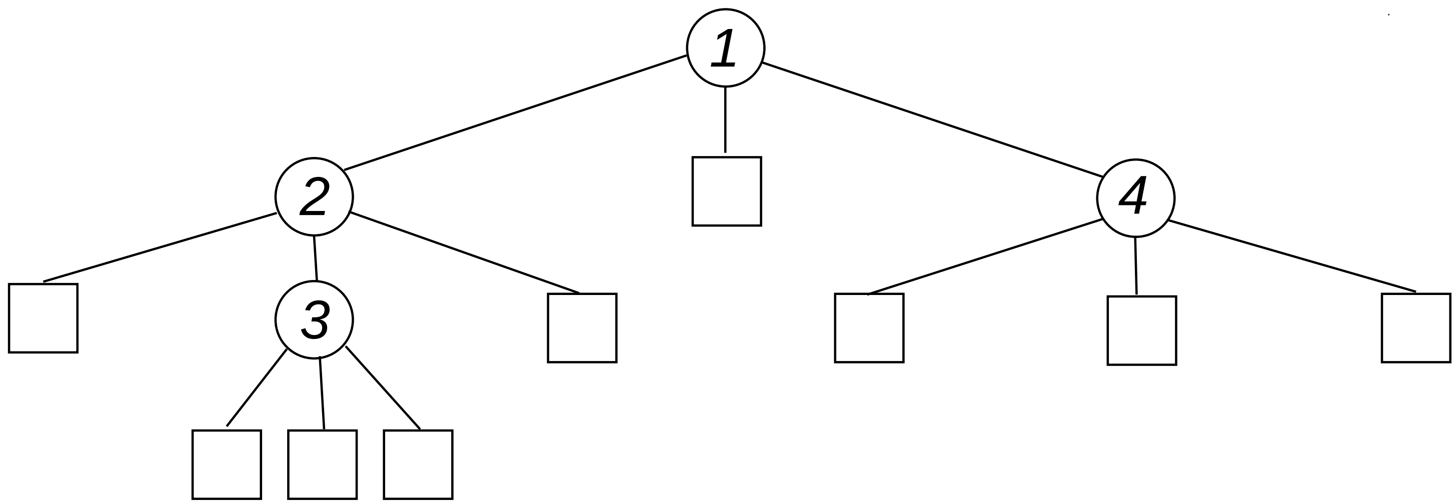
D-ary Recursive Trees



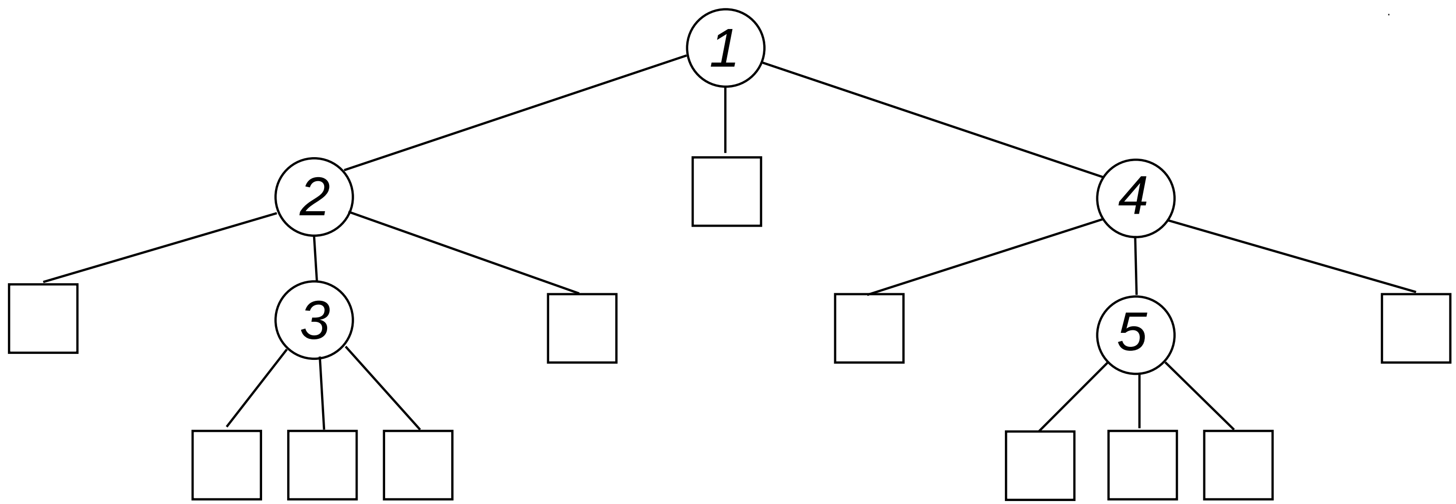
D-ary Recursive Trees



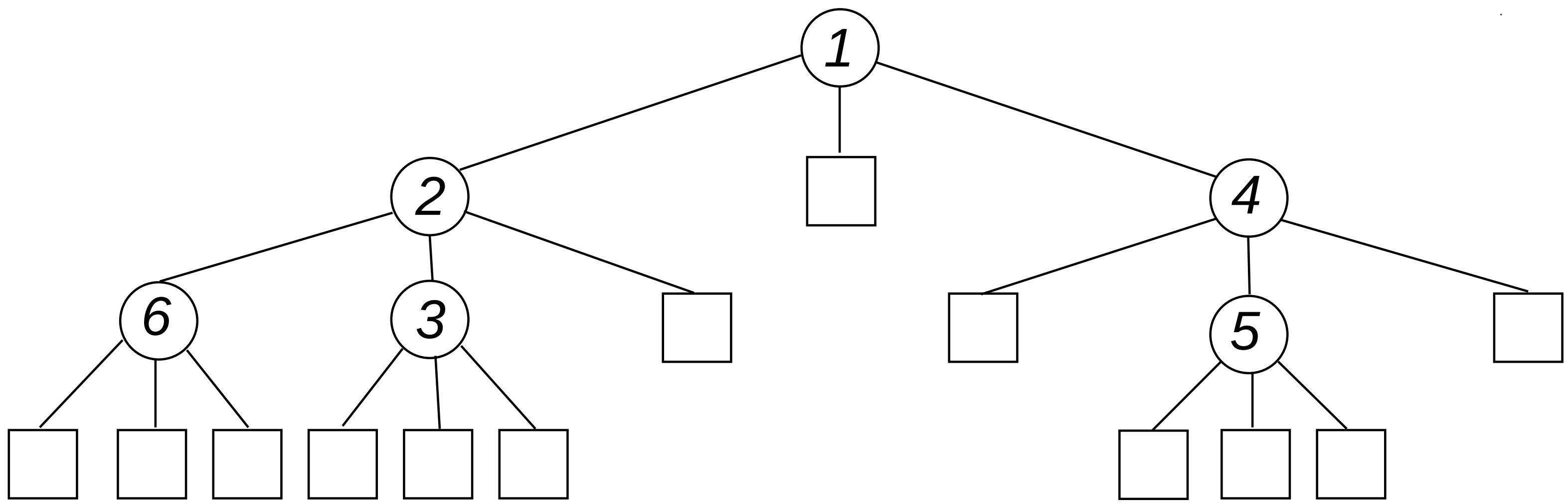
D-ary Recursive Trees



D-ary Recursive Trees



D-ary Recursive Trees



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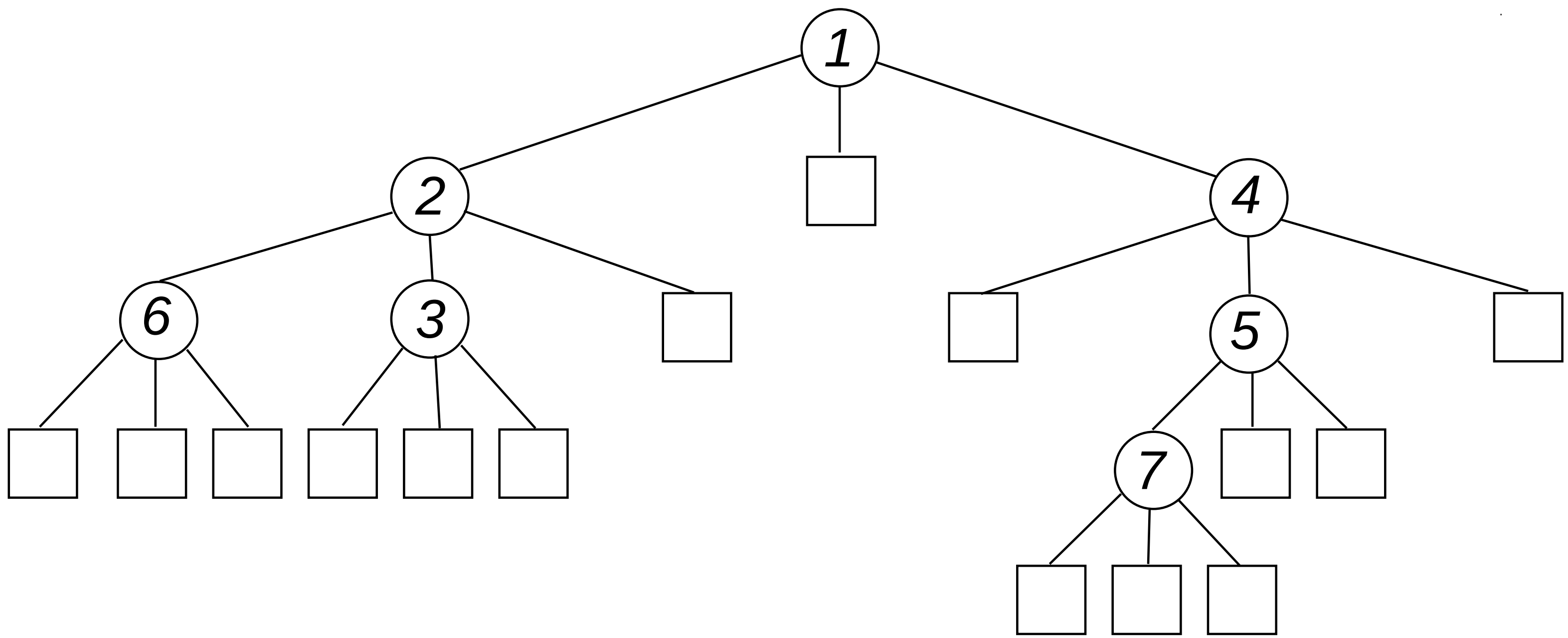

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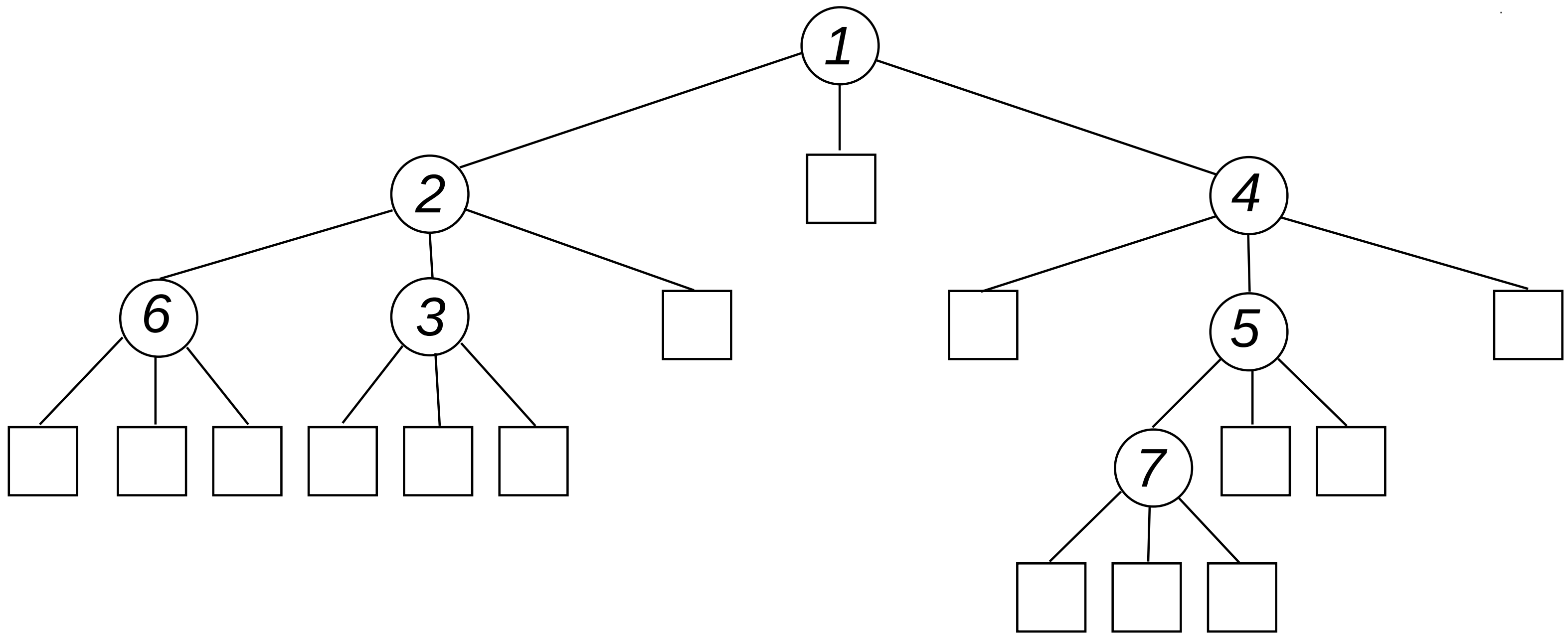
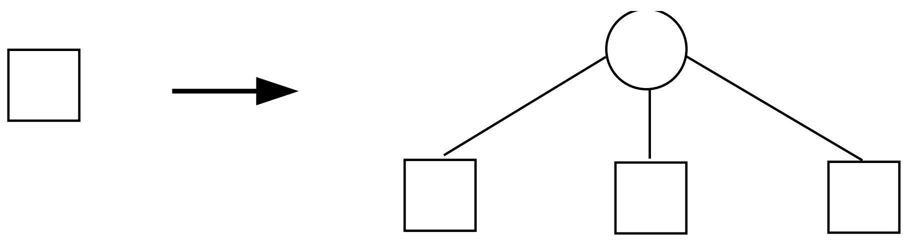
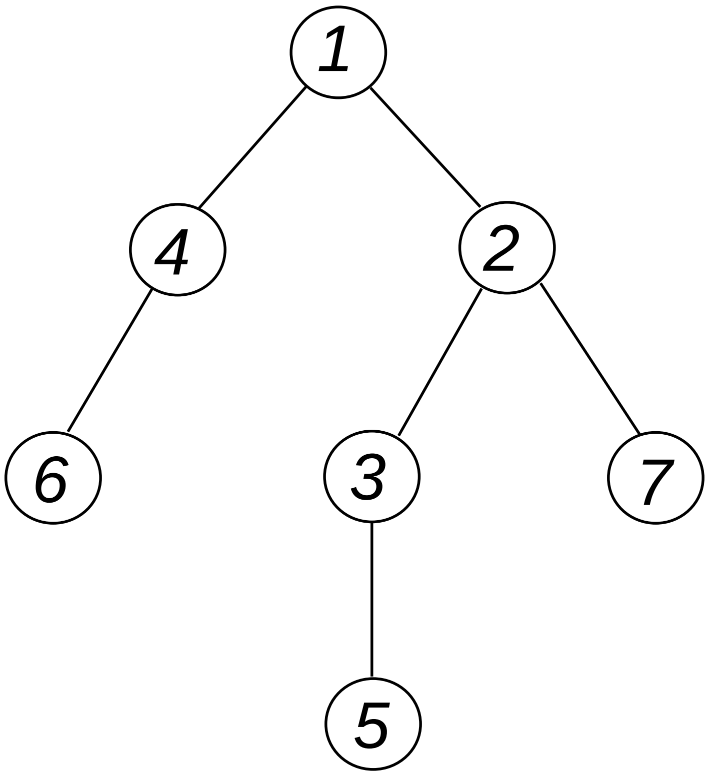




Cutting down Recursive Trees



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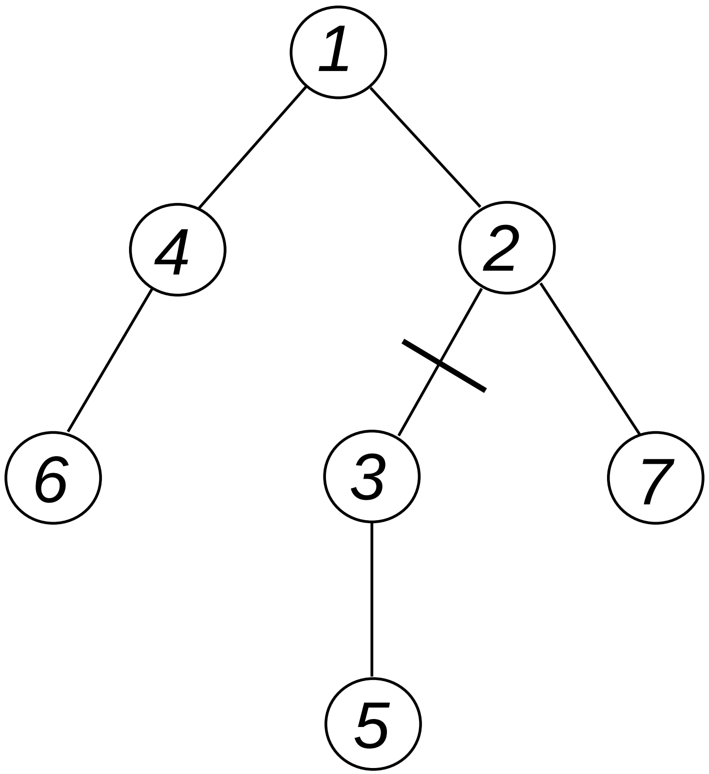

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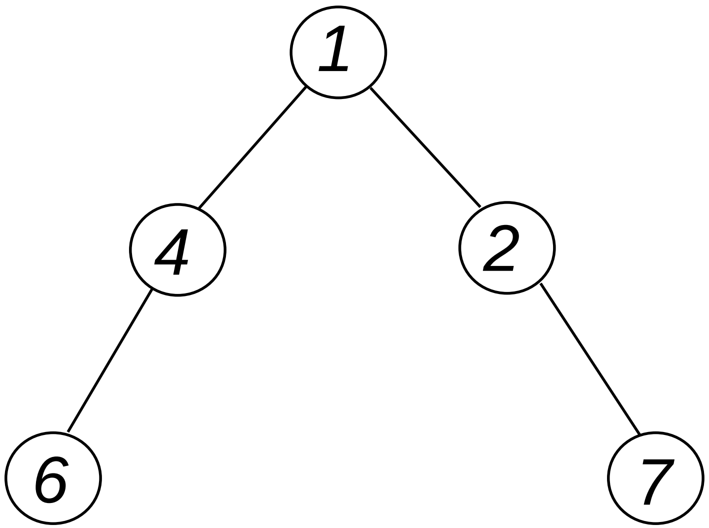

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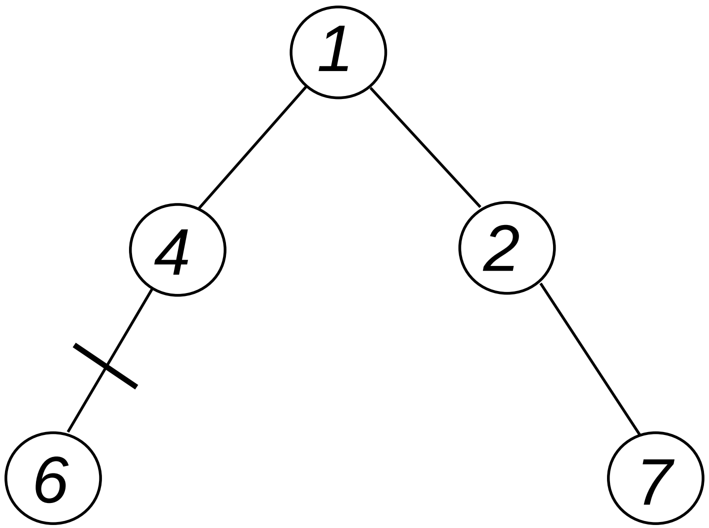

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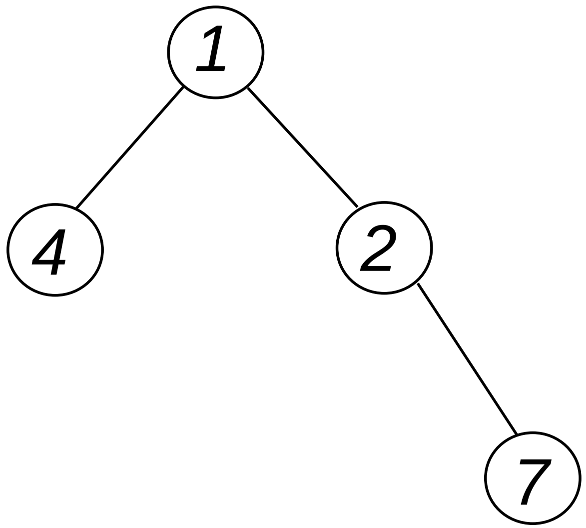

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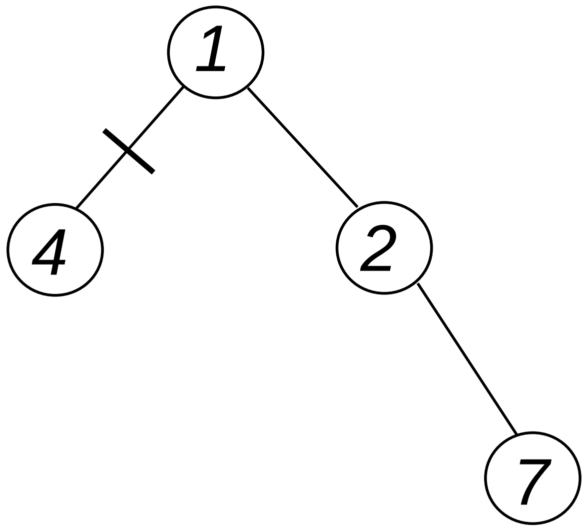

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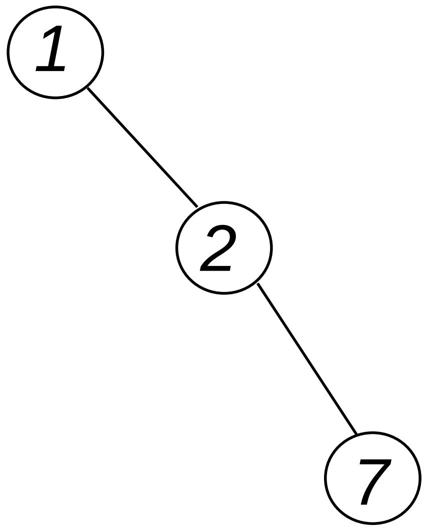

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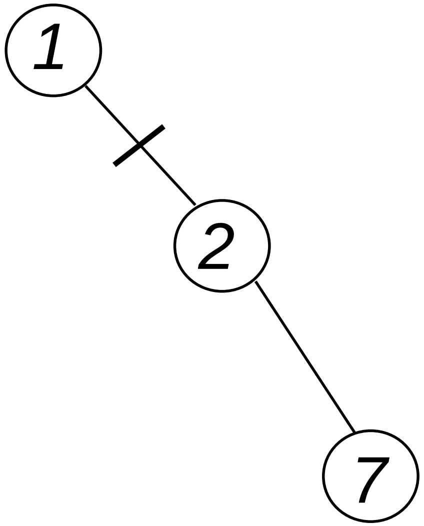


Cutting down Recursive Trees

Cutting down Recursive Trees

X_n ... number of random cuts to cut down a random r.t. of size n.

$X_0 = X_1 = 0$,

$$\boxed{X_n \equiv X_{I_n} + 1} \qquad (n \ge 2),$$

where I_n is a discrete random variable with

$$\mathbb{P}\{I_n = k\} = \frac{1}{(n-k)(n-k+1)} \frac{n}{n-1} \qquad (0 \le k < n)$$

that is independent of $(X_0, X_1, \ldots, X_{n-1})$.

Cutting down Recursive Trees

Lemma

The probability to that the remaining tree has size $= k$ if we cut a random edge in a random recursive tree of size n equals

$$\frac{1}{(n-k)(n-k+1)}\,\frac{n}{n-1}$$

Proof

$$\begin{aligned}\frac{(k-1)!(n-k-1)!}{(n-1)(n-1)!}\sum_{j=1}^{k}\binom{n-j}{n-k} &= \frac{(k-1)!(n-k-1)!}{(n-1)(n-1)!}\binom{n}{n-k+1} \\ &= \frac{1}{(n-k)(n-k+1)}\,\frac{n}{n-1}\end{aligned}$$

Cutting down Recursive Trees

Theorem [Drmota+Iksander+Möhle+Rösler]

$$\boxed{\frac{X_n - \frac{n}{\log n} - \frac{n \log\log n}{(\log n)^2}}{\frac{n}{(\log n)^2}} \to Y},$$

where Y is a stable random variable with characteristic function

$$\mathbb{E}\, e^{i\lambda Y} = e^{i\lambda \log|\lambda| - \frac{\pi}{2}|\lambda|}.$$

$$\mathbb{E}\, X_n = \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right), \quad \mathbb{V}\, X_n = \frac{n^2}{2\log^3 n} + O\left(\frac{n^2}{\log^4 n}\right)$$

Cutting down Recursive Trees

Stable distributions

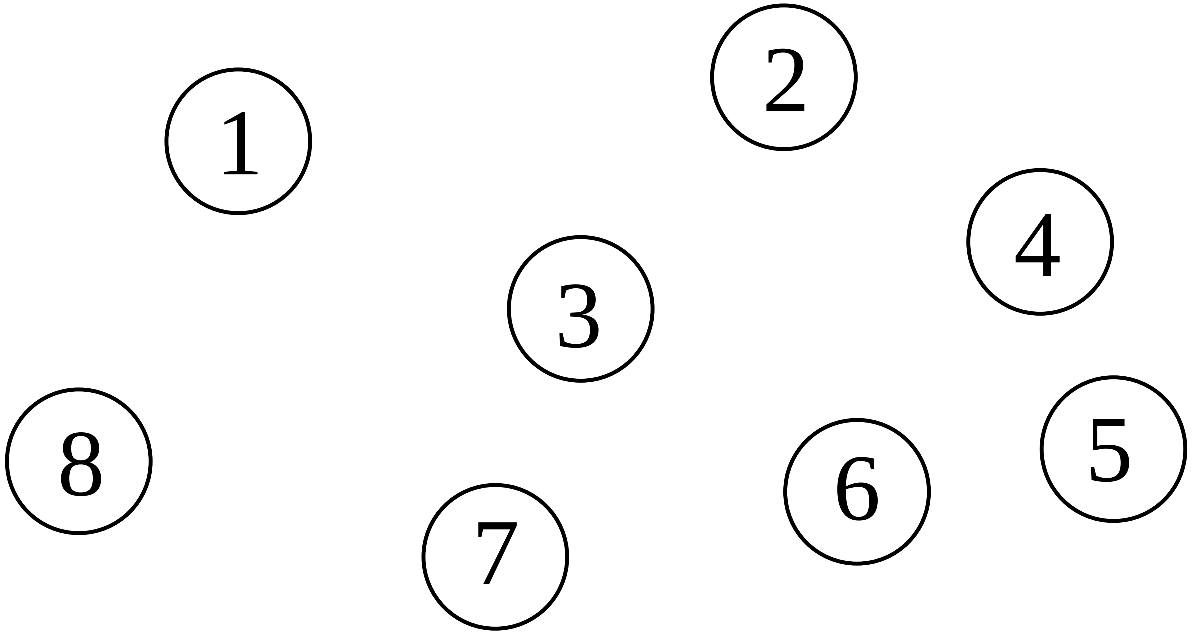
The distribution of random variable X is **stable**, if for all real a, b and independent copies X_1, X_2 of X there exists c, d with

$$aX_1 + bX_2 \equiv cX + d$$

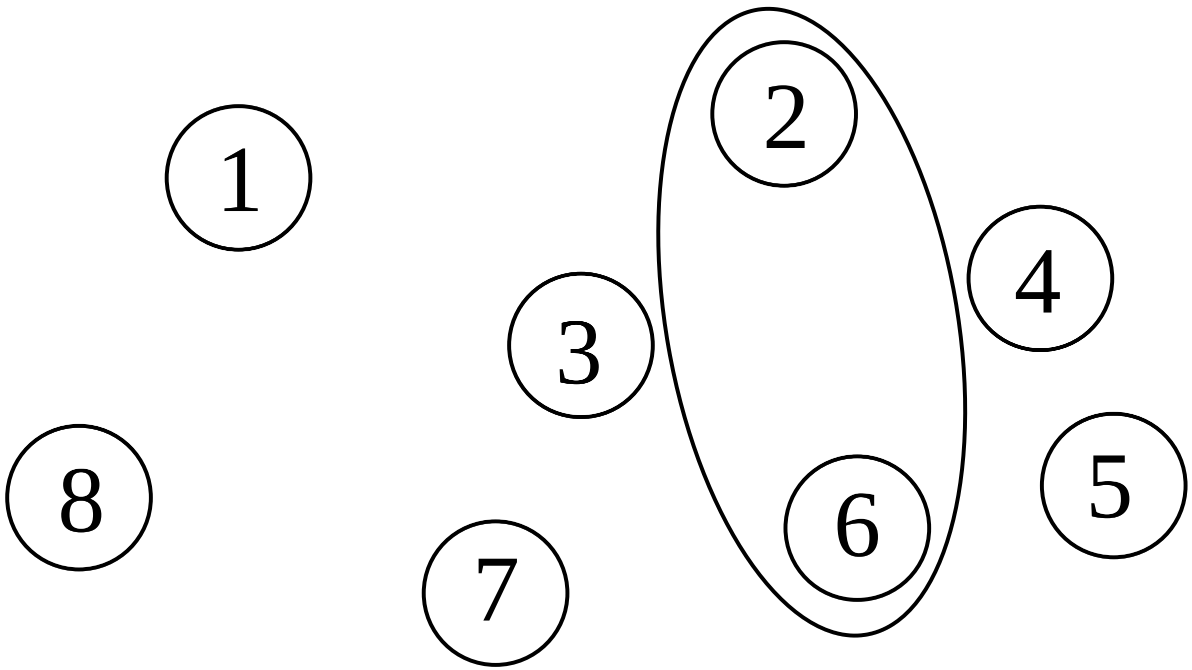
Examples: normal distribution, Cauchy distribution, Levy distribution

All stable distributions can be characterized in term of the characteristic function $\mathbb{E}\, e^{i\lambda X}$.

Coalescent Process



Coalescent Process



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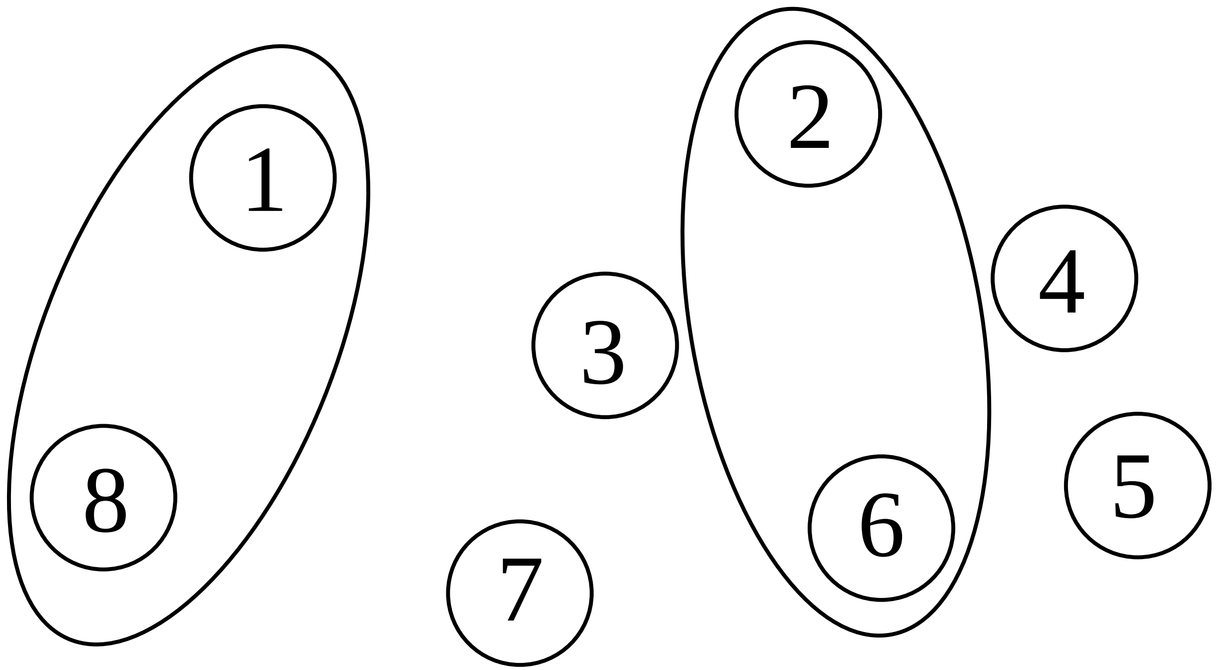

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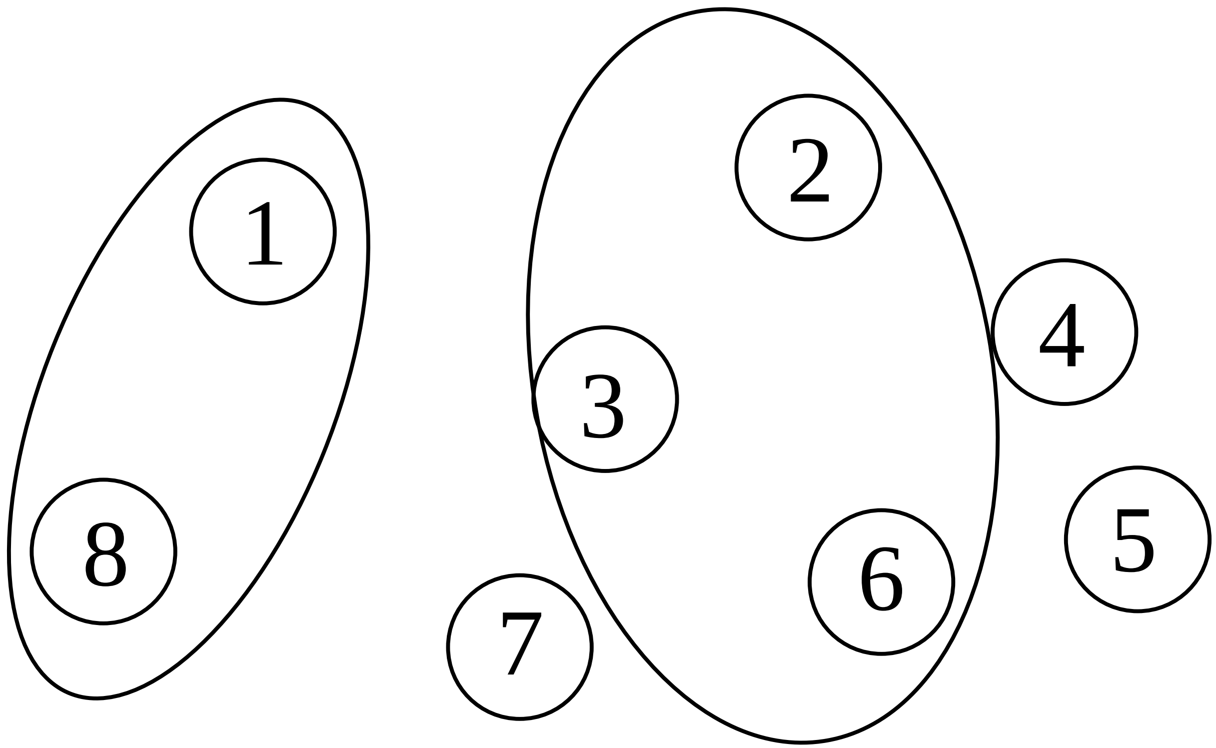

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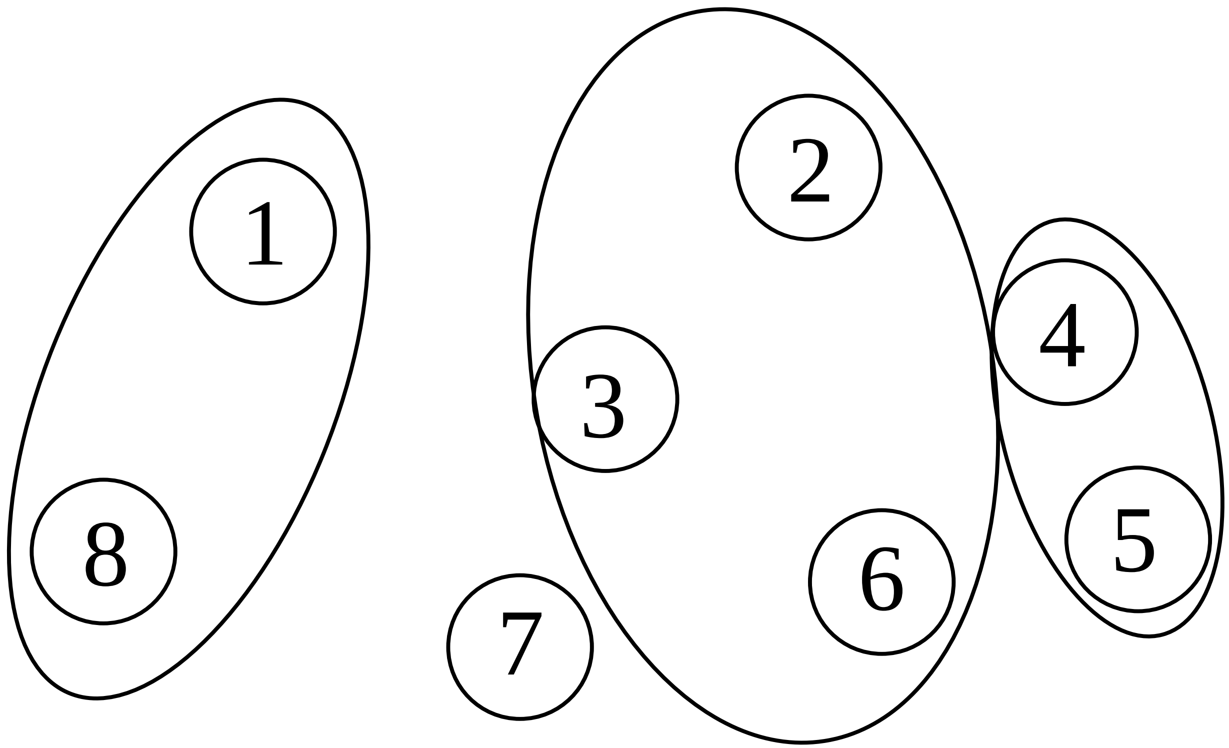
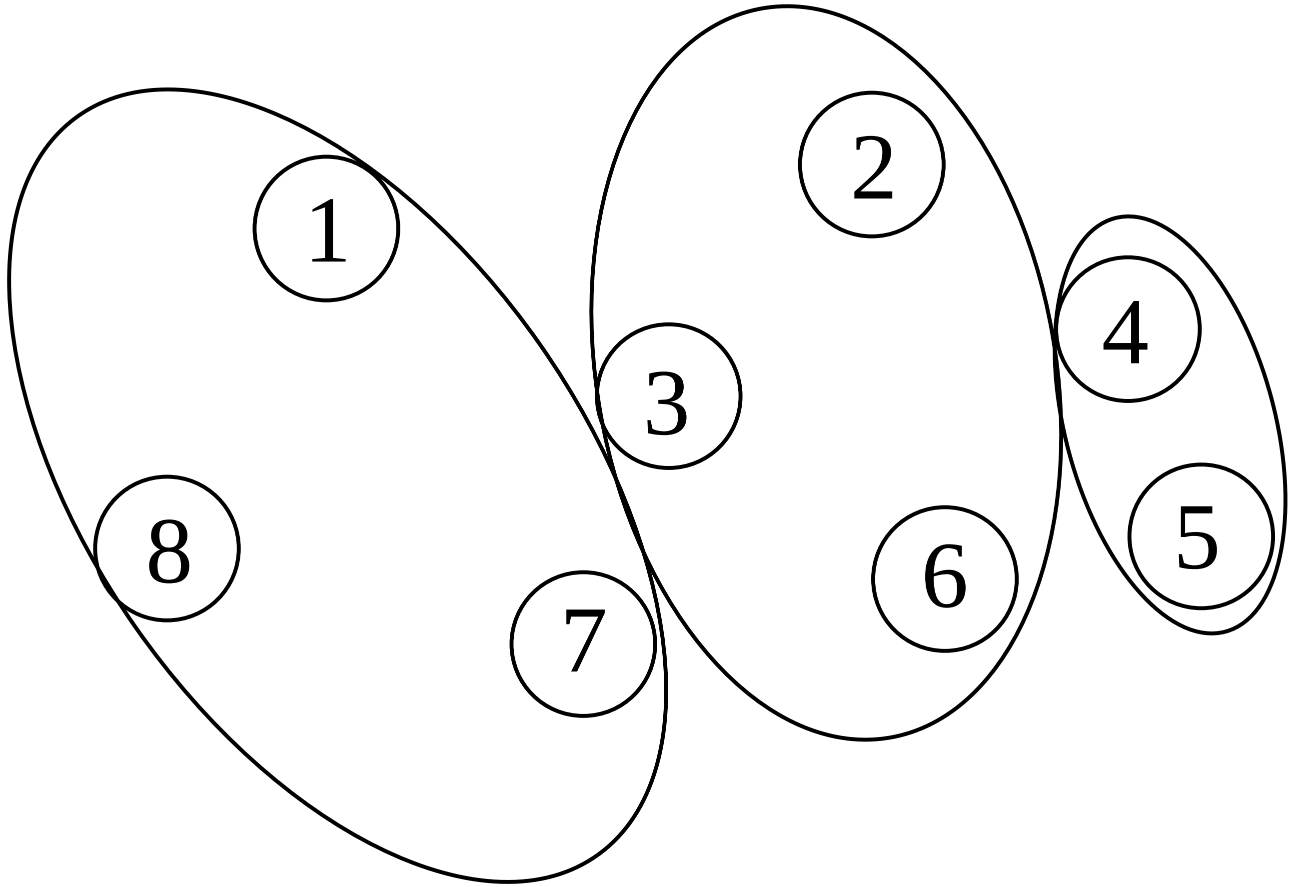


Coalescent Process



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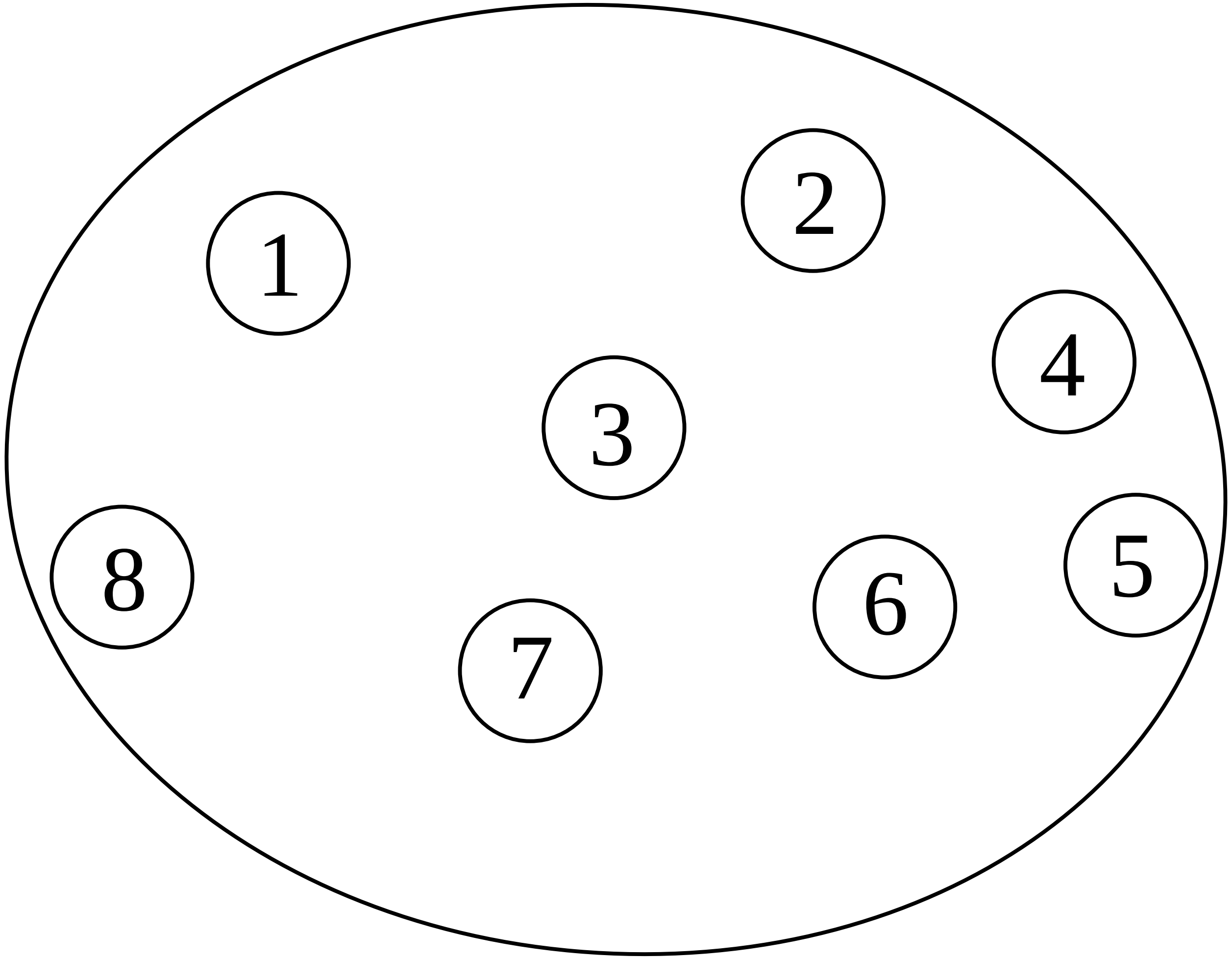

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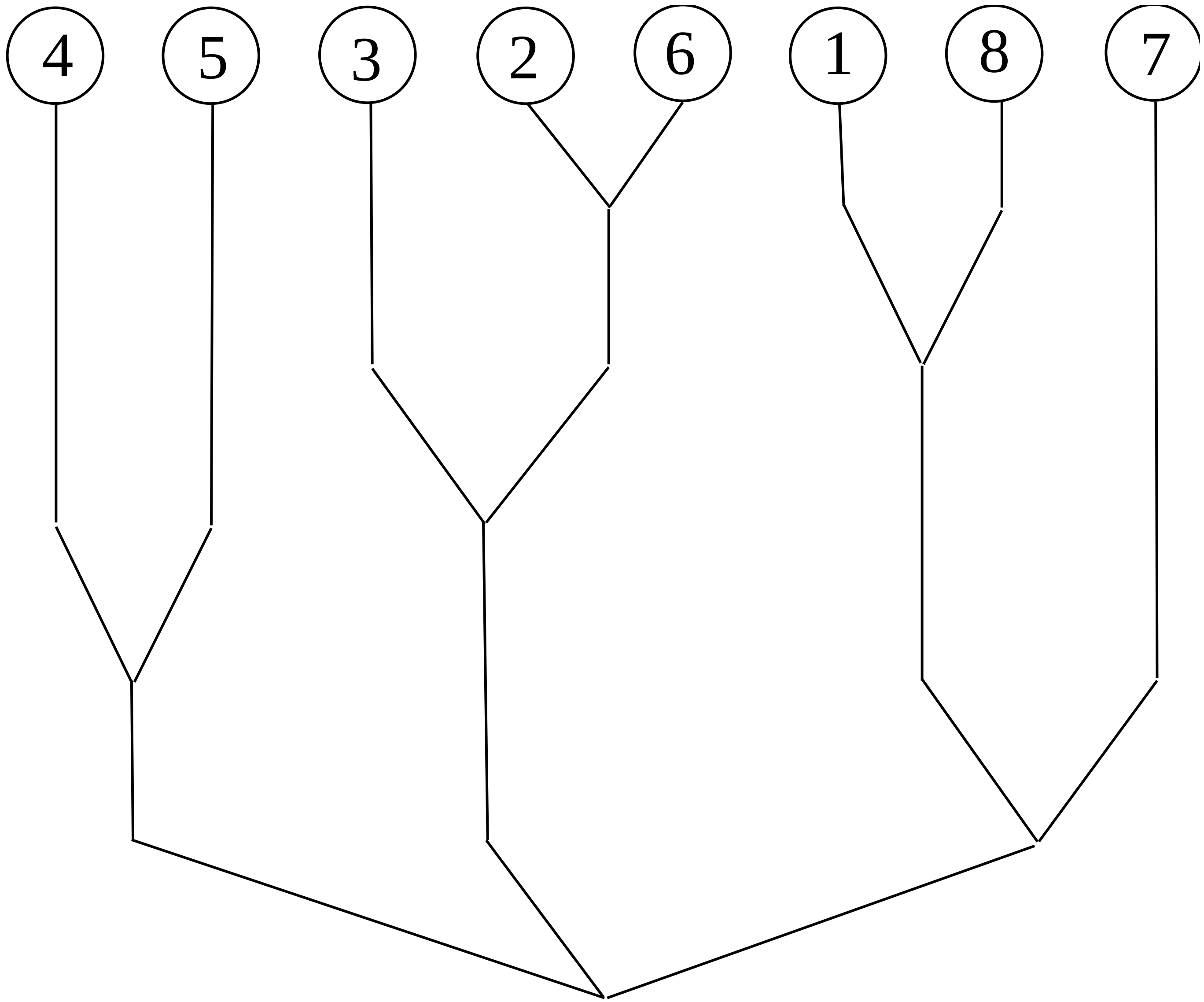

Coalescent Process

Stochastic model

Let Λ be a measure on $[0,1]$.

- Continuous Markov process of partitions of $\{1,2\ldots,n\}$,
  Initial partition: $\{\{1\},\{2\},\ldots,\{n\}\}$.

- If ξ and η are two partitions with a resp. b equivalence classes, where $b-a+1$ classes of ξ are merged to obtain η.
  Then the rate $q_{\xi,\eta}$ that ξ merges to η is

$$q_{\xi,\eta} = \begin{cases} \int_{[0,1]}(1-(1-x)^b - bx(1-x)^{b-1})x^{-2}d\Lambda(x) & \text{if } \xi = \eta, \\ \int_{[0,1]} x^{b-a-1}(1-x)^{a-1}d\Lambda(x) & \text{if } \xi \neq \eta. \end{cases}$$

Coalescent Process

Kingman-coalescent

$$\Lambda = \delta_0$$

Bolthausen-Sznitman-coalescent

$$\Lambda = \mathrm{univ}[0,1]$$

Coalescent Process

Remark

The process of number of classes is also a Markov process with rates

$$g_{ba} = \binom{b}{a-1} \int_{[0,1]} x^{b-a-1}(1-x)^{a-1} d\Lambda(x)$$

$(1 \leq a < b \leq n)$

Coalescent Process

Bolthausen-Sznitman-coalescent

X_n ... **number of collisions** until there is a single block:

$X_0 = X_1 = 0$,

$$\boxed{X_n \equiv X_{I_n} + 1} \qquad (n \ge 2),$$

where I_n is a discrete random variable with

$$\mathbb{P}\{I_n = k\} = \frac{1}{(n-k)(n-k+1)} \frac{n}{n-1} \qquad (0 \le k < n)$$

that is independent of $(X_0, X_1, \ldots, X_{n-1})$.

Cutting down Recursive Trees

Lemma

$$f(s,t) = \sum_{n\ge 1} \mathbb{E}\, s^{X_n}\, t^{n-1}$$

satisfies the partial differential equation

$$\boxed{\frac{\partial f(s,t)}{\partial t}\left(1 - t + \frac{t}{\log(1-t)}\left(1-\frac{1}{s}\right)\right) = f(s,t)}$$

with initial condition $f(s,0) = 1$.

Cutting down Recursive Trees

Expected Value

$$g(t) := \frac{\partial f(s,t)}{\partial s}\bigg|_{s=1} = \sum \mathbb{E}\, X_n\, t^{n-1}$$

$$\Longrightarrow \quad \boxed{g'(t) - \frac{g(t)}{1-t} = \frac{t}{(1-t)^3 \log \frac{1}{1-t}}}$$

$$\Longrightarrow \quad g(t) = \frac{1}{(1-t)^2 \log \frac{1}{1-t}} - \frac{\log\log \frac{1}{1-t}}{1-t} + O\left(\frac{1}{(1-t)^2 \log^2 \frac{1}{1-t}}\right)$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, X_n = \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right)}.$$

Thank You!