BLOCK ADDITIVE FUNCTIONS ON GAUSSIAN INGEGERS

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Summary

- Block additive functions
- Asymptotics for generating functions
- Distributional results
- Mellin-Perron techniques

Block additive functions

$q = -a + i$ ... **basis** of digital expansion in $\mathbb{Z}[i]$ (for an integer $a > 0$)

$\mathcal{N} = \{0, 1, \ldots, a^2\}$ ... set of **digits**

$z \in \mathbb{Z}[i] \implies$

$$\boxed{z = \sum_{j \ge 0} \varepsilon_j(z)\, q^j} \qquad \text{with } \varepsilon_j(z) \in \mathcal{N}.$$

(Formally we set $\varepsilon_j(z) = 0$ for all negative integers $j < 0$.)

Block additive functions

$F : \mathcal{N}^{L+1} \to \mathbb{R}$ ... any given function (for some $L \geq 0$) with $F(\mathbf{0}) = 0$.

$$\boxed{s_F(z) = \sum_j F\left(\epsilon_j(n), \epsilon_{j+1}(z), \ldots, \epsilon_{j+L}(z)\right)}$$

$s_F(z)$ is a weighted sum over all subsequent digital patterns of length $L+1$ of the digital expansion of z.

• $L = 0$, $F(\epsilon) = \epsilon \Longrightarrow s_F(z) =$ sum-of-digits function.

• $L = 1$, $F(\epsilon, \eta) = 1 - \delta_{\epsilon,\eta} \Longrightarrow s_F(z) =$ is number of digit changes.

• $F(B) = 1$ for some specific block, $F(C) = 0$ for all blocks $C \neq B$ $\Longrightarrow s_F(n) =$ the number of occurances of B in the digital expansion of z.

Block additive functions

Recurrence

$z = \eta_0 + qv,\ (\epsilon_0(z), \ldots, \epsilon_L(z)) = B \Longrightarrow$

$$\boxed{s_F(z) = \epsilon(B) + s_F(q)}$$

with

$$\epsilon(B) = \sum_{i=0}^{L} \left(F(0,\ldots,0,\eta_0,\eta_1,\ldots,\eta_i) - F(0,\ldots,0,0,\eta_1,\ldots,\eta_i)\right).$$

Remark. $\epsilon_0(z) = 0 \Longrightarrow \epsilon(B) = 0$.

Asymptotics for generating functions

MAIN THEOREM

$$\boxed{\sum_{|z|^2<N} x^{s_F(z)} = \Psi(x, \log_{|q|^2} N) \cdot N^{\log_{|q|^2} \lambda(x)} \cdot \Big(1 + O(N^{-\kappa})\Big)}$$

uniformly in a complex neighborhood of $x = 1$. ($\kappa > 0$)

$\Psi(x,t)$ is analytic in x and periodic (with period 1) and Lipschitz continuous in t. ($\lambda(x)$ will be defined in a moment.)

$$\boxed{\sum_{|z|^2<N} x^{s_F(z)} \ll N^{\log_{|q|^2} \lambda(|x|)-\kappa}}$$

for x not close to the positive real line. ($\kappa > 0$)

Asymptotics for generating functions

Definition of $\lambda(x)$

$$A_{B,C}(x) = \begin{cases} x^{\epsilon(B)} & \text{if last } L \text{ digits of } B = \text{first } L \text{ digits of } C, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{A}(x) = (A_{B,C}(x))_{B,C \in \mathcal{N}^{L+1}}$$

$$\boxed{\lambda(x) = \text{largest eigenvalue of } \mathbf{A}(x).}$$

Distributional results

1. Asymptotics for moments:

$$\frac{1}{\pi N}\sum_{|z|^2<N} s_F(z)^k = \left(\frac{\lambda'(1)}{|q|^2}\right)^k \left(\log_{|q|^2} N\right)^k + \sum_{j=0}^{k-1}\left(\log_{|q|^2} N\right)^j \Psi_j(\log_{|q|^2} N) + O(N^{-\kappa}).$$

Take derivatives with respect to x and set $x = 1$.

Distributional results

2. Central limit theorem for $s_F(z)$.

$$\frac{1}{\pi N}\#\left\{|z|^2 < N : s_F(z) \le \frac{\lambda'(1)}{|q|^2}\log_{|q|^2} N + t\sqrt{\sigma^2 \log_{|q|^2} N}\right\} = \Phi(t) + o(1)$$

with

$$\sigma^2 = \lambda''(1)/|q|^2 - \lambda'(1)^2/|q|^4.$$

($\Phi(t)$ denotes the normal distribution function)

Setting $x = e^{iu}$ we get the characteristic function of the distribution of $s_F(z)$:

$$\frac{1}{\pi N}\sum_{|z|^2<N} e^{iu\, s_F(z)} = \mathbf{E}\, e^{iu\, S}$$

Distributional results

3. Local limit theorem:

Cauchy's formula (F is integer valued):

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\} = \frac{1}{2\pi i}\int_{|x|=x_0}\left(\sum_{|z|^2<N} x^{s_F(z)}\right) x^{-k-1}\,dx$$

$$\sim \frac{\Psi(x_0, \log_{|q|^2} N)}{\sqrt{2\pi \mathrm{const.}(x_0)\log_{|q|^2} N}}\lambda(x_0)^{\log_{|q|^2} N} x_0^{-k}$$

where x_0 is the *saddle point* defined by

$$\frac{x_0\lambda'(x_0)}{\lambda(x_0)} = \frac{k}{\log_{|q|^2} N}.$$

Distributional results

4. Uniform distribution of $s_F(z)$ in residue classes.

m ... positive integer with $(m, |q+1|^2) = 1$, F integer valued:

$$\frac{1}{\pi N} \#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) \equiv \ell \bmod m\} = \frac{1}{m} + O(N^{-\kappa}).$$

$x = e^{2\pi i j/m}$ ... m-th roots of unity + discrete Fourier analysis.

Distributional results

5. Uniform distribution of ($\alpha s_F(z)$ mod 1)

$$\alpha \text{ irrational} \implies \alpha s_F(z) \text{ is uniformly distributed modulo 1.}$$

$x = e^{2\pi i \alpha h}$:

$$\frac{1}{\pi N} \sum_{|z|^2 < N} e^{2\pi i h \alpha s_F(z)} = O(N^{-\kappa})$$

$+$ Weyl's criterion

Mellin-Perron techniques

$a(z)$ ... function on $\mathbb{Z}[i]$

$A(s) = \sum_{z\in\mathbb{Z}[i]\setminus\{0\}} \frac{a(z)}{|z|^{2s}}$ ... **Dirichlet series** of $a(z)$

Mellin-Perron $\Longrightarrow$

$$\boxed{\sum_{|z|<N} a(z) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} A(s) \frac{N^s}{s}\, ds}$$

for $c > \sigma_a$ (abscissa of absolute convergence of $A(s)$)

Mellin-Perron techniques

Dirichlet series

$$G_B(x,s) = \sum_{z\in\mathbb{Z}[i]\setminus\{0\},\,(\epsilon_0(z),\ldots,\epsilon_L(z))=B} \frac{x^{s_F(z)}}{|z|^{2s}}.$$

Substitution $\boxed{z = \eta_0 + qv}$.

Notation: $B = (\eta_0, \eta_1, \ldots, \eta_L) \longrightarrow B' = (\eta_1, \ldots, \eta_L)$

1st case: $\eta_0 = 0$ ($\Longrightarrow s_F(z) = s_F(q)$)

$$\begin{aligned} G_B(x,s) &= \frac{1}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\epsilon_0(v),\ldots,\epsilon_{L-1}(v))=B'} \frac{x^{s_F(v)}}{|v|^{2s}} \\ &= \frac{1}{|q|^{2s}} \sum_{\ell=0}^{a^2} G_{(B',\ell)}(x,s). \end{aligned}$$

Mellin-Perron techniques

2nd case: $\eta_0 > 0$ ($\Longrightarrow s_F(z) = \epsilon(B) + s_F(q)$)

$$
\begin{aligned}
G_B(x,s) &= x^{s_F(\eta_0)}|\eta_0|^{2s} + \frac{x^{\epsilon(B)}}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\epsilon_0(v),\ldots,\epsilon_{L-1}(v))=B'} \frac{x^{s_F(v)}}{|v+\eta_0/q|^{2s}} \\
&= \frac{x^{s_F(\eta_0)}}{|\eta_0|^{2s}} + \frac{x^{\epsilon(B)}}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\epsilon_0(v),\ldots,\epsilon_{L-1}(v))=B'} \frac{x^{s_F(v)}}{|v|^{2s}} + H_B(x,s) \\
&= \frac{x^{s_F(v)}}{|q|^{2s}} \sum_{\ell=0}^{a^2} G_{(B',\ell)}(x,s) + H_B(x,s),
\end{aligned}
$$

where

$$
H_B(x,s) = \frac{x^{s_F(\eta_0)}}{|\eta_0|^{2s}} + \frac{x^{\epsilon(B)}}{|q|^{2s}} \sum_{(\epsilon_0(v),\ldots,\epsilon_{L-1}(v))=B'} x^{s_F(v)} \left(\frac{1}{|v+\eta_0/q|^{2s}} - \frac{1}{|v|^{2s}} \right).
$$

Mellin-Perron techniques

$\mathbf{A}(x) = (A_{B,C}(x))_{B,C\in\mathcal{N}^{L+1}}$
$\mathbf{G}(x,s) = (G_B(x,s))_{B\in\mathcal{N}^{L+1}}$
$\mathbf{H}(x,s) = (H_B(x,s))_{B\in\mathcal{N}^{L+1}}$

$$\Longrightarrow \quad \mathbf{G}(x,s) = \frac{1}{|q|^{2s}}\mathbf{A}(x)\mathbf{G}(x,s) + \mathbf{H}(x,s)$$

or

$$\boxed{\mathbf{G}(x,s) = \left(\mathbf{I} - \frac{1}{|q|^{2s}}\mathbf{A}(x)\right)^{-1}\mathbf{H}(x,s)}$$

Mellin-Perron techniques

Dominant polar singularities of $G_B(x,s)$:

$$s_k = \log_{|q|^2} \lambda(x) + \frac{2\pi i k}{\log |q|^2} \qquad (k \in \mathbb{Z}).$$

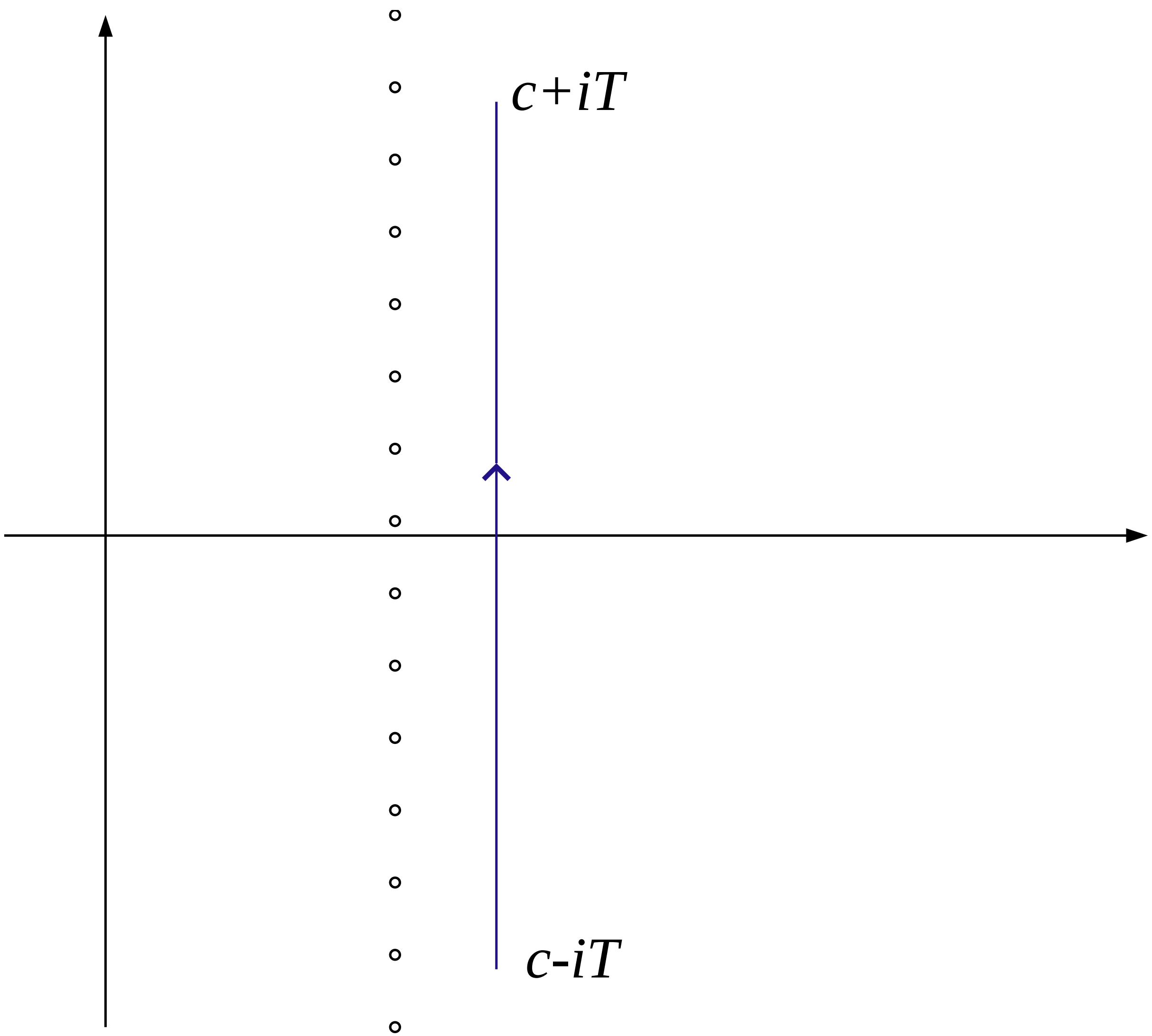
$(\det \left(\mathbf{I} - \frac{1}{|q|^{2s}}\mathbf{A}(x)\right) = 0)$

Perron-Frobenius: $G(x,s) = \sum_B G_B(x,s)$

$$\boxed{\sum_{|z|^2<N} x^{s_F(z)} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} G(x,s) \frac{N^s}{s}\, ds}$$

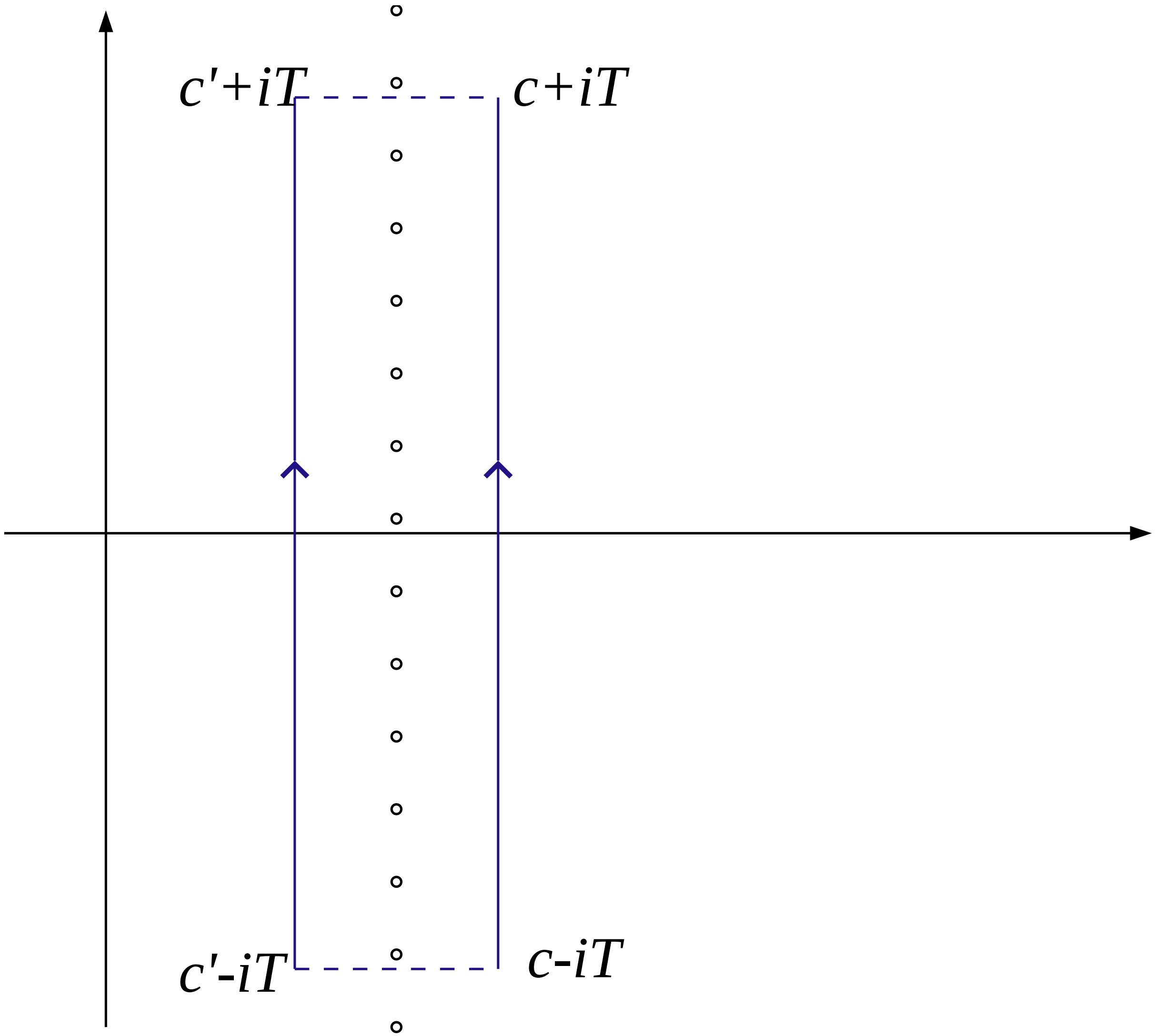
Mellin-Perron techniques

Shift of integration



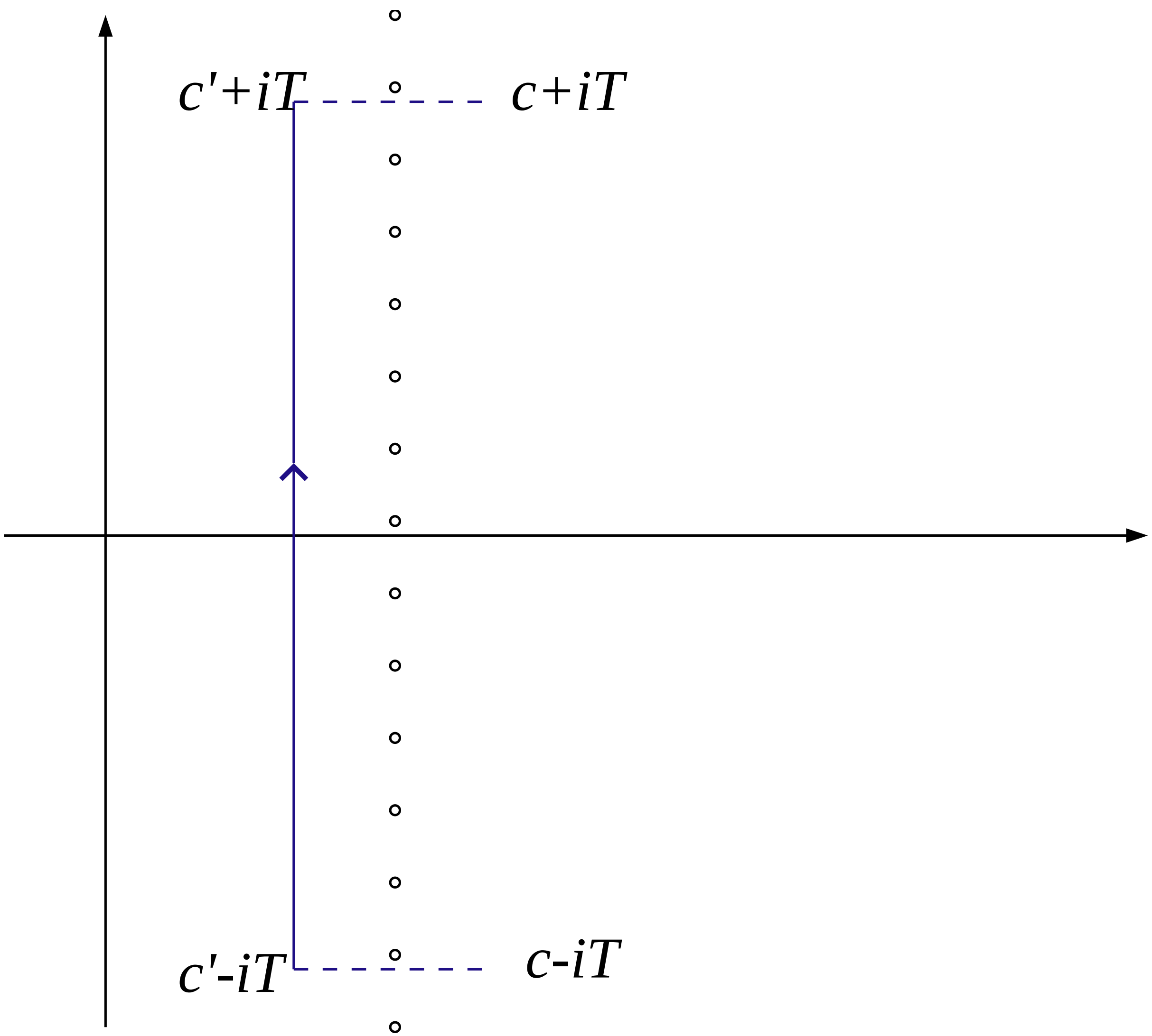
Mellin-Perron techniques

Shift of integration



Mellin-Perron techniques

Shift of integration



Mellin-Perron techniques

$$\sum_{|z|^2<N} x^{s_F(z)} = \lim_{T\to\infty} \boxed{\sum_{|k|\le T} D(s_k,x)\frac{N^{s_k}}{s_k}} + \frac{1}{2\pi i}\lim_{T\to\infty} \boxed{\int_{c'-iT}^{c'+iT} G(x,s)\frac{N^s}{s}\,ds}$$

with

$$D(s_k,x) = \mathrm{res}(G(x,s), s=s_k)$$

Problem: NO ABSOLUTE CONVERGENCE !!!

Mellin-Perron techniques

Notation: $e(x) = e^{2\pi i x}$, $\{x\} = x - \lfloor x \rfloor$, $\|t\| = \min\{\{x\}, \{-x\}\}$

Lemma 1

$$\boxed{\lim_{T\to\infty} \sum_{|k|\le T} \frac{e(kt)}{\alpha + 2\pi i k} = \frac{e^{-\alpha\{t\}}}{1 - e^{-\alpha}}}$$

More precisely, if $t \notin \mathbb{Z}$

$$\sum_{k\ge T} \frac{e(kt)}{\alpha + 2\pi i k} = \sum_{k>T} \frac{2\pi i}{(\alpha + 2\pi i k)(\alpha + 2\pi i (k-1))} \frac{e(Tt) - e(kt)}{1 - e(t)}$$
$$= O\left(\frac{1}{T\,\|t\|}\right),$$

and also

$$\sum_{|k|\ge T} \frac{1}{\alpha + 2\pi i k} = O\left(\frac{1}{T}\right)$$

Mellin-Perron techniques

Lemma 2

a and c are positive real numbers:

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} a^s \frac{ds}{s} - 1\right| \le \frac{a^c}{\pi T \log a} \qquad (a > 1),$$

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} a^s \frac{ds}{s}\right| \le \frac{a^c}{\pi T \log(1/a)} \qquad (0 < a < 1),$$

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} a^s \frac{ds}{s} - \frac{1}{2}\right| \le \frac{C}{T} \qquad (a = 1).$$

Further, if $0 < a, b < 1$ or if $a, b > 1$ then

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} (a^s - b^s)\frac{ds}{s} = \frac{1}{\pi i T}\left(a^c \frac{\sin(T \log a)}{\log a} - b^c \frac{\sin(T \log b)}{\log b}\right) + O\left(\frac{1}{T}\left(\frac{a^c}{\log a} - \frac{b^c}{\log b}\right)\right)$$

Thank You!