INCREASING TREE FAMILIES

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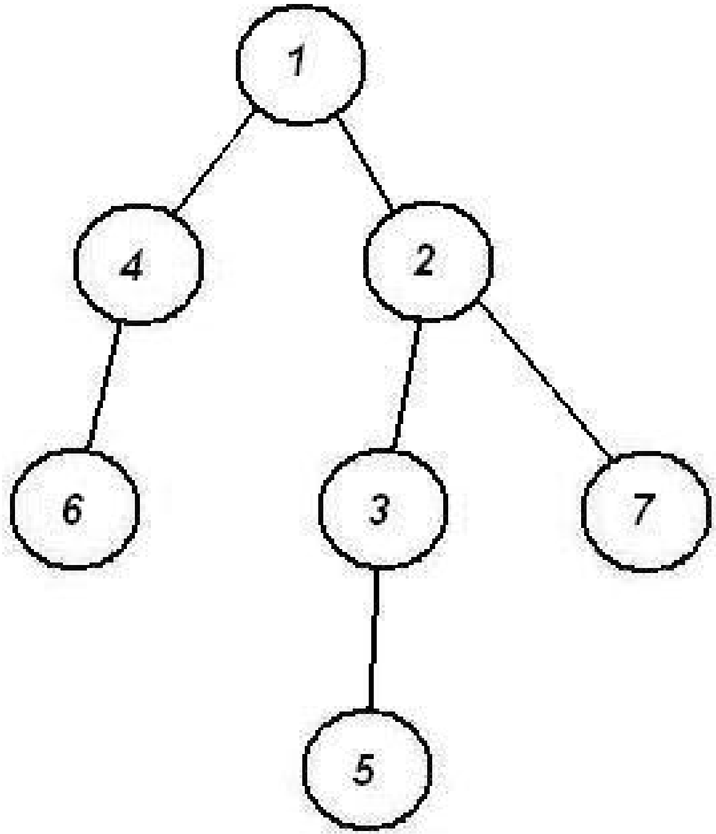
Complex Networks and Random Graphs

Physikzentrum Bad Honnef, July 11–13, 2005

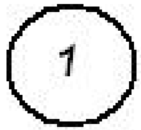
Outline of the Talk

- Recursive Trees
- Plane Oriented Trees
- General Increasing Trees
- Degree Distribution
- Conclusion

Recursive Trees



Recursive Trees



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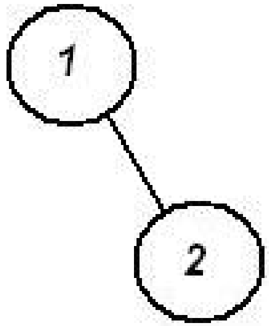
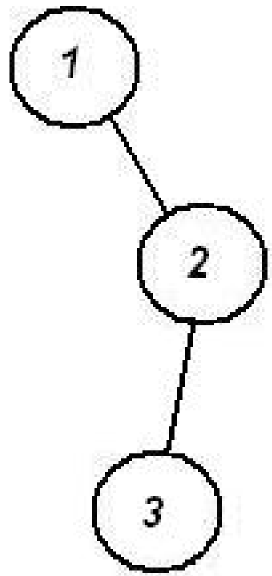
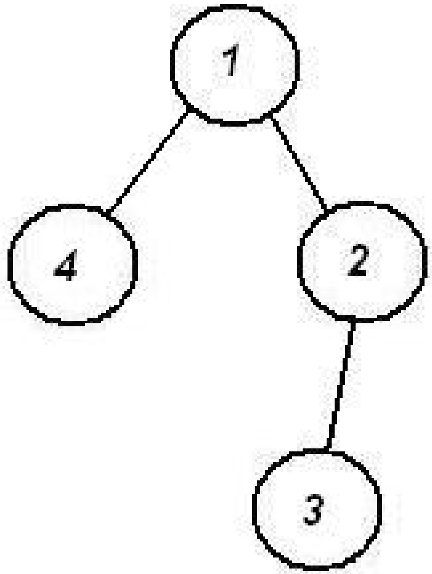


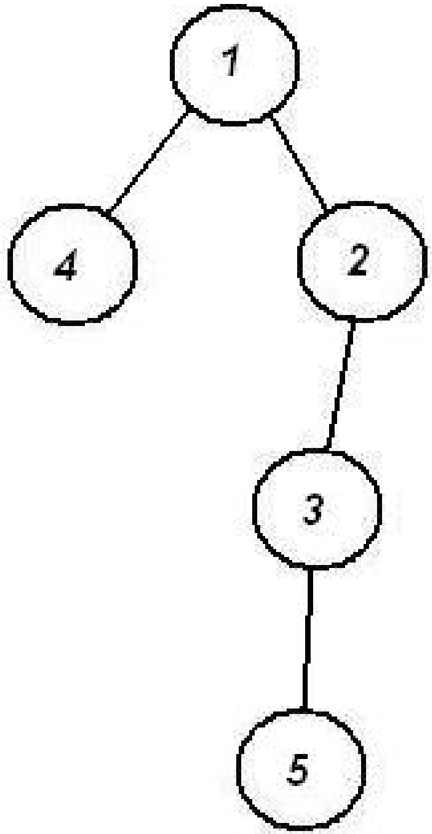
Recursive Trees



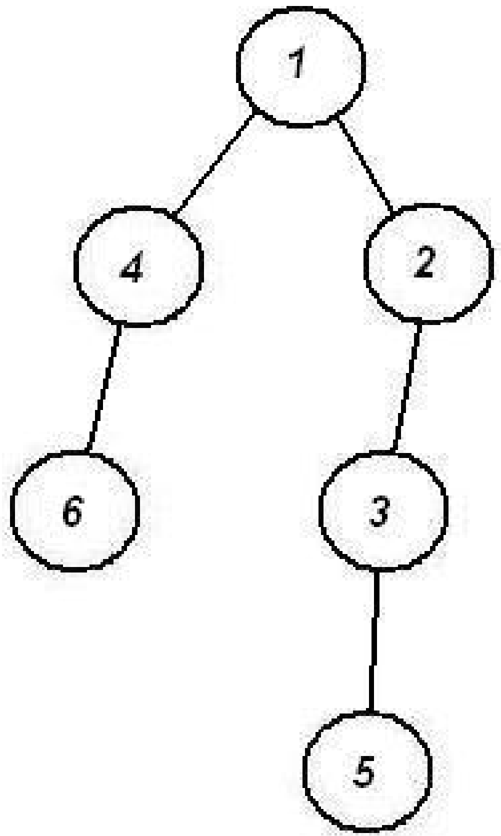
Recursive Trees



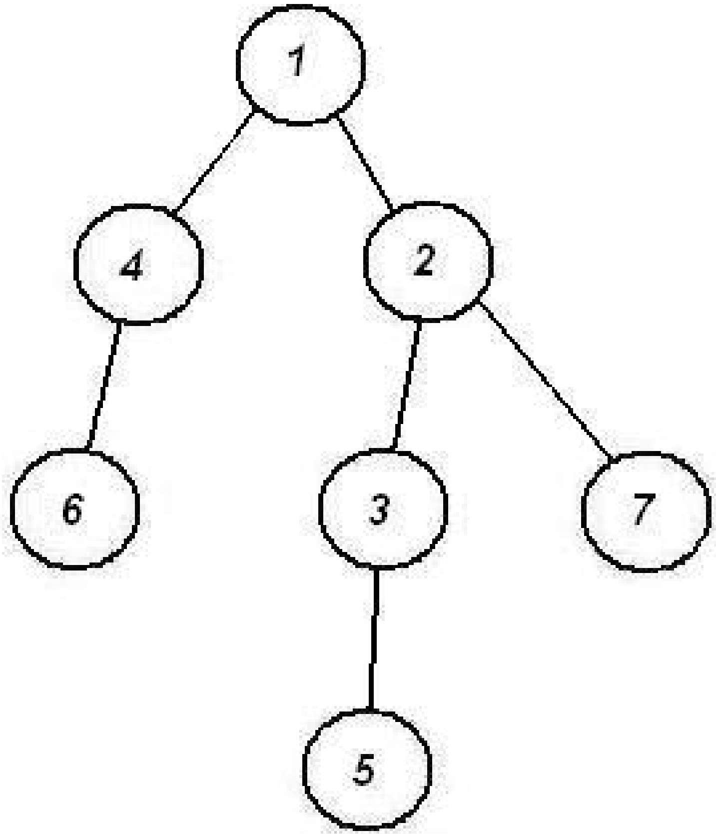
Recursive Trees



Recursive Trees



Recursive Trees



Recursive Trees

Combinatorial Description:

- labeled rooted tree
- labels are strictly increasing (starting at the root)
- no left-to-right order (non-planar)

Recursive Trees

Number of Recursive Trees:

$$\begin{aligned} y_n &= \text{number of recusive trees of size } n \\ &= (n-1)! \end{aligned}$$

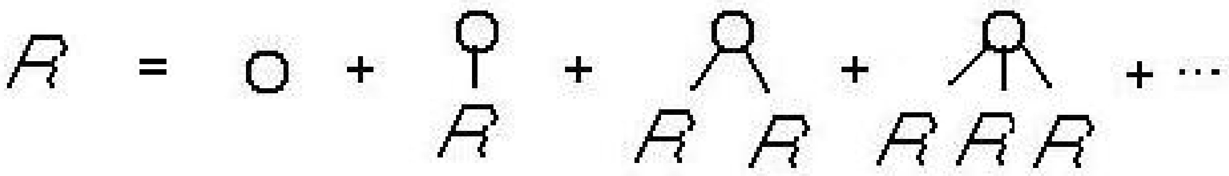
The node with label j has exactly $j-1$ possibilities to be inserted $\Longrightarrow y_n = 1 \cdot 2 \cdots (n-1)$.

Recursive Trees

Generating Functions:

$$y(x) = \sum_{n\geq 1} y_n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n} = \log \frac{1}{1-x}$$

$$\boxed{y'(x) = 1 + y(x) + \frac{y(x)^2}{2!} + \frac{y(x)^3}{3!} + \cdots = e^{y(x)}}$$



A recursive tree can be interpreted as a root followed by an **unordered** sequence of recursive trees. ($y'(x) = \sum_{n\geq 0} y_{n+1} x^n / n!$)

Recursive Trees

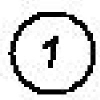
Probability Model:

Process of growing trees

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node with **equal** probability $1/(j-1)$.

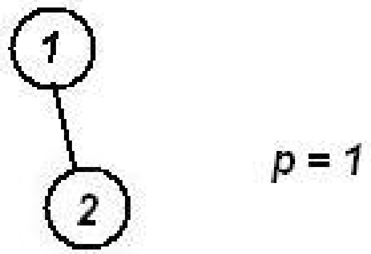
After n steps every tree (of size n) has equal probability $\frac{1}{(n-1)!}$.

Recursive Trees

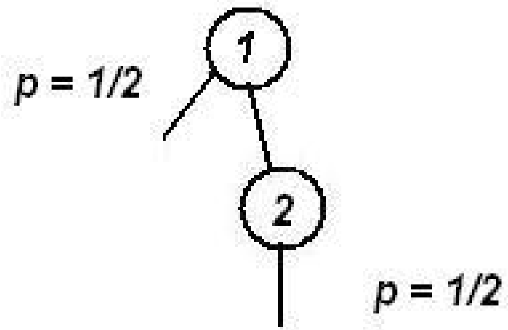


Recursive Trees

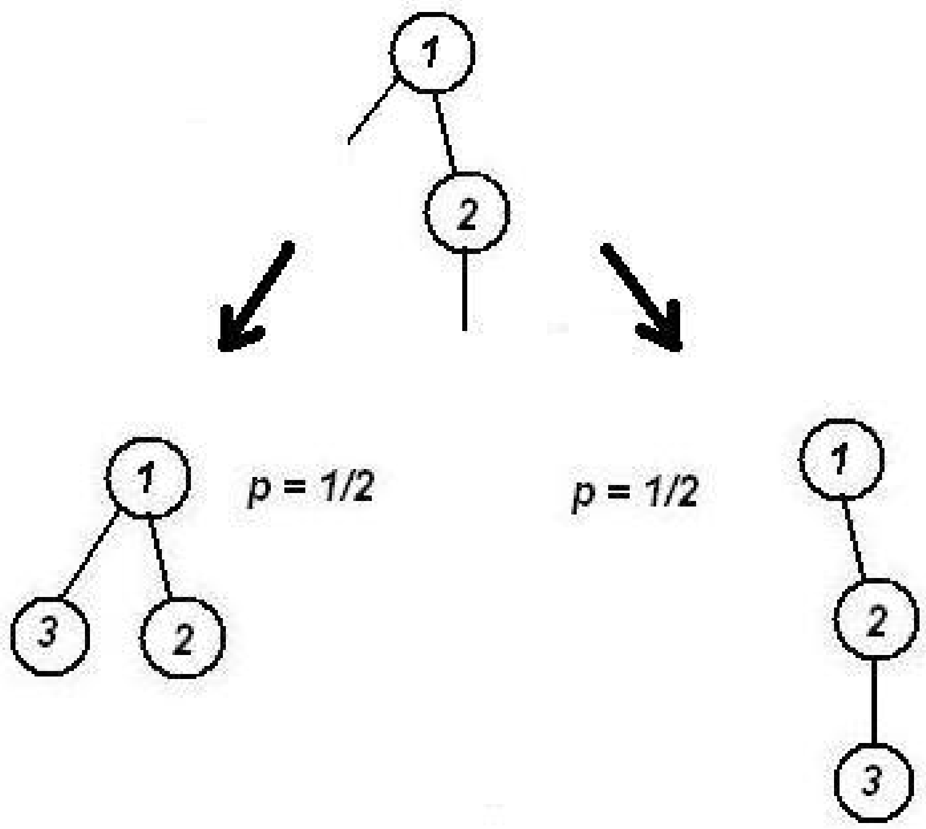
Recursive Trees



Recursive Trees

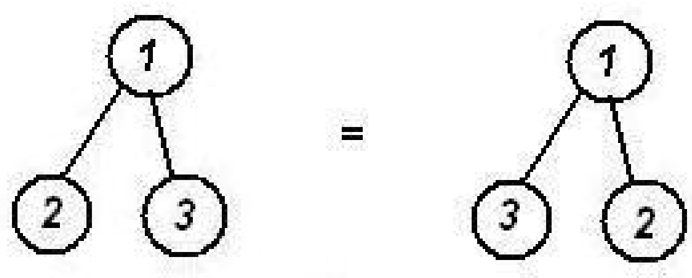


Recursive Trees



Recursive Trees

Remark



Results for Recursive Trees

Depth D_n of the n-th node [Devroye, 1988; Mahmoud, 1991]

$$\begin{aligned} \mathbf{E}\, D_n &= H_{n-1} = \log n + O(1) \\ \mathbf{Var}\, D_n &= H_{n-1} - H_{n-1}^{(2)} = \log n + O(1) \end{aligned}$$

Central limit theorem:

$$\boxed{\frac{D_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)}$$

Harmonic numbers: $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, $H_n^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$

Results for Recursive Trees

Number L_n **of leaves** [Najock & Heyde, 1982]

$$\mathbf{E}\, L_n = \frac{n}{2}$$

$$\mathbf{Var}\, L_n = \frac{7}{12} n + \frac{1}{3}$$

Central limit theorem:

$$\boxed{\frac{L_n - \frac{n}{2}}{\sqrt{\frac{7}{12} n}} \to \mathcal{N}(0,1)}$$

Results for Recursive Trees

Distribution of out-degrees [Gastwirth, 1977]

$$\begin{aligned}\lambda_d &= \lim_{n\to\infty} \text{probability that a random node of a trees of size } n \\ &\qquad \text{has out-degree } d \\ &= \lim_{n\to\infty} \frac{\text{expected number of nodes with out-degree } d}{n} \\ &= \frac{1}{2^{d+1}}\end{aligned}$$

E.g.: Number of leaves = number of nodes with out-degree 0 $= \frac{n}{2}$.

Results for Recursive Trees

Root degree $d_{0,n}$

$$\begin{aligned} \mathbf{E}\, d_{0,n} &= H_{n-1} = \log n + O(1) \\ \mathbf{Var}\, d_{0,n} &= \log n + O(1) \end{aligned}$$

Central limit theorem:

$$\boxed{\frac{d_{0,n} - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)}$$

This result follows from the correspondance to (random) permutations.

Results for Recursive Trees

Height H_n

[Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \to e \qquad (a.s.)}$$

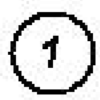
$$\mathbf{E}\, H_n \sim e \cdot \log n$$

Exponential tails [Drmota 200?]

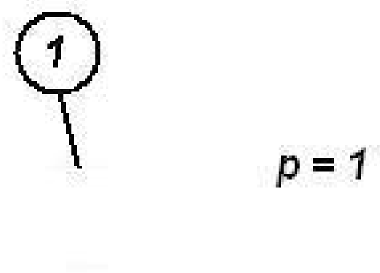
$$\boxed{\mathbf{Pr}\,\{|H_n - \mathbf{E}\, H_n| \ge z\} = O(e^{-\eta z})}$$

for some $\eta > 0$. ($\Longrightarrow \quad \mathbf{Var}\, H_n = O(1)$ etc.)

Plane Oriented Trees



Plane Oriented Trees



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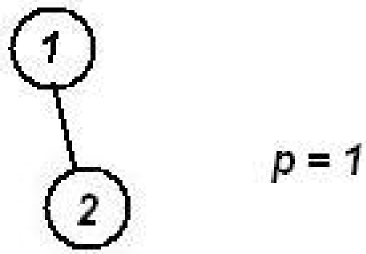

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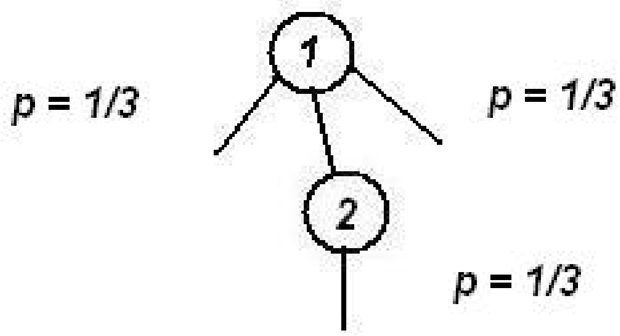
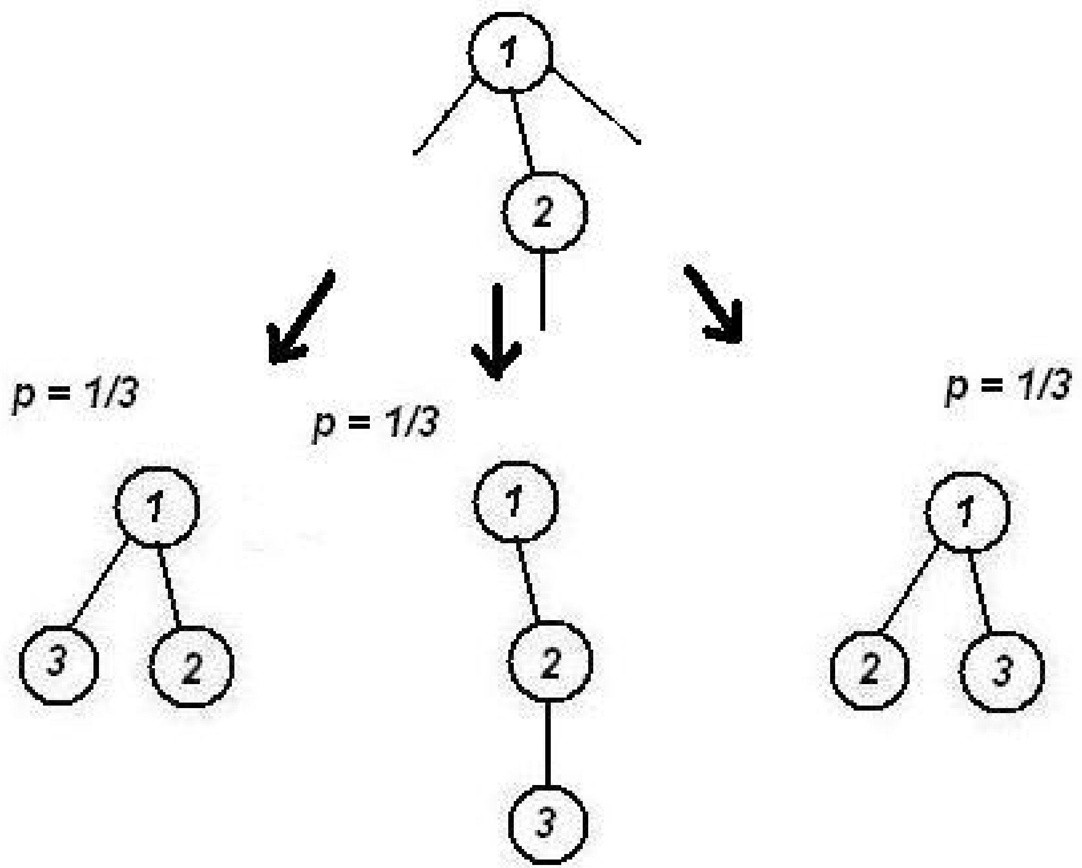
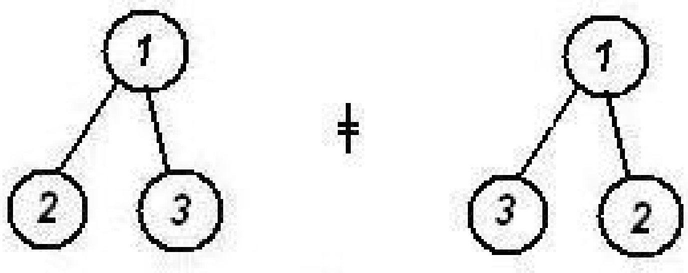


Plane Oriented Trees



Plane Oriented Trees

Remark



Plane Oriented Trees

Number of Plane Oriented Trees:

$$\begin{aligned} y_n &= \text{number of plane oriented trees of size } n \\ &= 1 \cdot 3 \cdot 5 \cdots (2n-3) = (2n-3)!! \\ &= \frac{(2n-2)!}{2^{n-1}(n-1)!} \end{aligned}$$

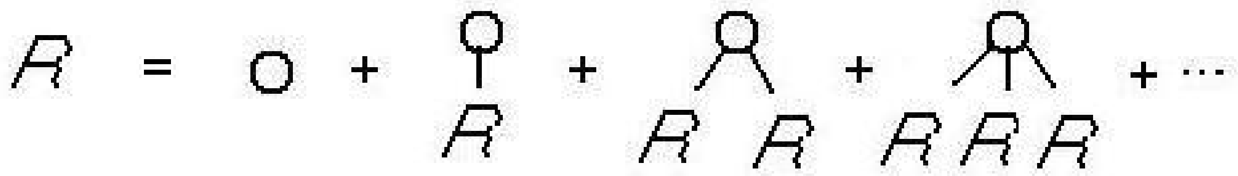
The node with label j has exactly $2j-3$ possibilities to be inserted $\Longrightarrow y_n = 1 \cdot 3 \cdots (2n-3)$.

Plane Oriented Trees

Generating Functions:

$$y(x) = \sum_{n\geq 1} y_n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{1}{2^{n-1}} \binom{2(n-1)}{n-1} \frac{x^n}{n} = 1 - \sqrt{1-2x}$$

$$\boxed{y'(x) = 1 + y(x) + y(x)^2 + y(x)^3 + \cdots = \frac{1}{1-y(x)}}$$



A plane oriented tree can be interpreted as a root followed by an **ordered** sequence of plane oriented trees. ($y'(x) = \sum_{n\geq 0} y_{n+1} x^n / n!$)

Plane Oriented Trees

Probability Model:

Process of growing trees

- The process starts with the root that is labeled with 1.

- At step j a new node (with label j) is attached to any previous node of outdegree d with probability $(d+1)/(2j-3)$.
  (**Barabási-Albert model**)

After n steps every tree (of size n) has equal probability $\frac{1}{(2n-3)!!}$.

Results for Plane Oriented Trees

Depth D_n of the n-th node [Mahmoud, 1992]

$$\begin{aligned}\mathbf{E}\, D_n &= H_{2n-1} - \frac{1}{2} H_{n-1} = \frac{1}{2}\log n + O(1) \\ \mathbf{Var}\, D_n &= H_{2n-1} - \frac{1}{2} H_{n-1} - H^{(2)}_{2n-1} + \frac{1}{4} H^{(2)}_{n-1} \\ &= \frac{1}{2}\log n + O(1)\end{aligned}$$

Central limit theorem:

$$\boxed{\frac{D_n - \frac{1}{2}\log n}{\sqrt{\frac{1}{2}\log n}} \to \mathcal{N}(0,1)}$$

Results for Plane Oriented Trees

Number L_n **of leaves** [Mahmoud, Smythe & Szymanski, 1993]

$$\mathbf{E}\, L_n = \frac{2n-1}{3}$$

$$\mathbf{Var}\, L_n = \frac{n}{9} - \frac{1}{18} - \frac{1}{6(2n-1)}$$

Central limit theorem:

$$\boxed{\frac{L_n - \frac{2}{3}n}{\sqrt{\frac{n}{9}}} \to \mathcal{N}(0,1)}$$

Results for Plane Oriented Trees

Distribution of out-degrees [Bergeron, Flajolet & Salvy, 1992]

$$\begin{aligned}
\lambda_d &= \lim_{n\to\infty} \text{probability that a random node in } \mathcal{P}_n \text{ has out-degree } d \\
&= \lim_{n\to\infty} \frac{\text{expected number of nodes with out-degree } d}{n} \\
&= \frac{4}{(d+1)(d+2)(s+3)}
\end{aligned}$$

Remark. $\lambda_d \sim 4\,d^{-3}$ as $d \to \infty$.

Results for Plane Oriented Trees

Root degree $d_{0,n}$ [Bergeron, Flajolet & Salvy, 1992]

$$\mathbf{Pr}\{d_{0,n} = k\} = \frac{(2n-3-k)!}{2^{n-1-k}(n-1-k)!} \sim \sqrt{\frac{2}{\pi n}} e^{-k^2/(4n)}$$

$$\mathbf{E}\, d_{0,n} = \sqrt{\pi n} + O(1)$$

Results for Plane Oriented Trees

Height H_n

[Pittel 1994]

$$\boxed{\frac{H_n}{\log n} \to \frac{1}{2s} = 1.79556\ldots \qquad (a.s.)},$$

where $s = 0.27846\ldots$ is the positive solution of $se^{s+1} = 1$.

$$\mathbf{E}\, H_n \sim \frac{1}{2s} \log n$$

Exponential tails [Drmota 200?]

$$\boxed{\mathbf{Pr}\,\{|H_n - \mathbf{E}\, H_n| \ge z\} = O(e^{-\eta z})}$$

for some $\eta > 0$. ($\Longrightarrow$ $\mathbf{Var}\, H_n = O(1)$ etc.)

Results for Plane Oriented Trees

Distance E_n between 2 random points

[Bollobas & Riordan, 200?; Morris, Panholzer & Prodinger, 2004]

$$\mathbf{E}\, E_n = \log n + O(1)$$

$$\mathbf{Var}\, E_n = \log n + O(1)$$

Central limit theorem:

$$\boxed{\frac{E_n - \log n}{\sqrt{\log n}} \to \mathcal{N}(0,1)}$$

General Increasing Trees

[Bergeron, Flajolet & Salvy, 1992]

$\mathcal{P}_n$: set of all *plane oriented trees* of size n

$\phi_0, \phi_1, \ldots$: weight sequence ($\phi_0 > 0$, $\phi_j > 0$ for some $j \geq 2$)

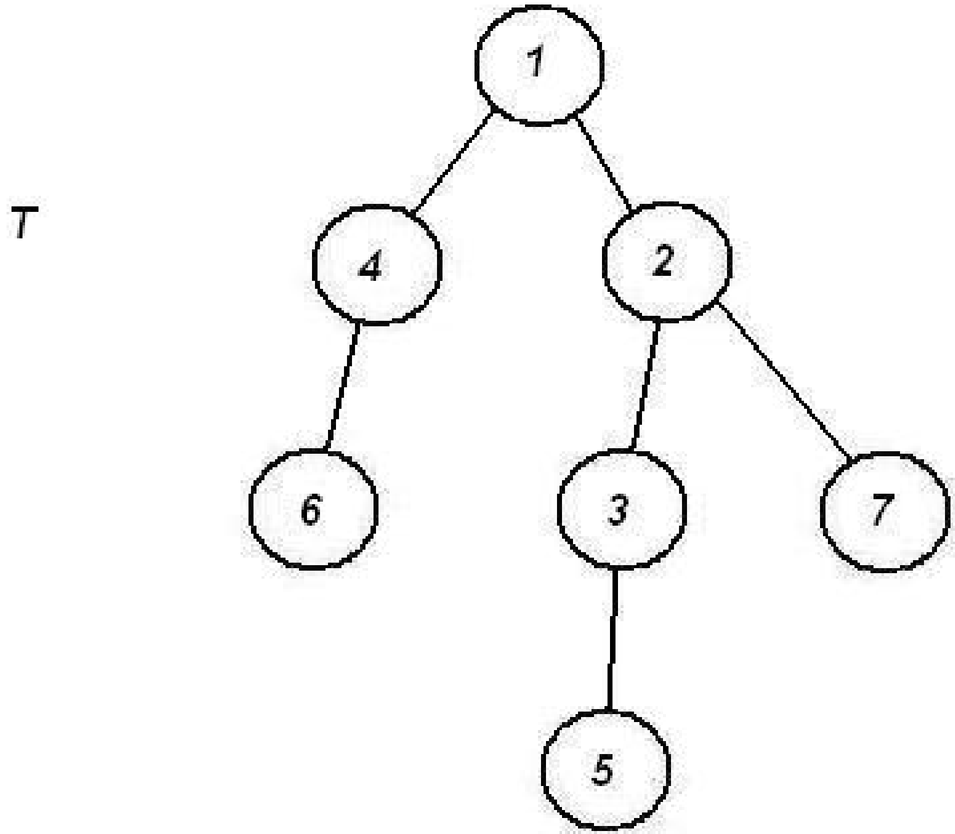
$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$$

Weight of a tree $T \in \mathcal{P}_n$:

$$\boxed{\omega(T) = \prod_{j \geq 0} \phi_j^{N_j(T)}},$$

where $N_j(T)$ = the number of nodes in T with **outdegree** j.

General Increasing Trees



$$\boxed{\omega(T) = \phi_0^3 \phi_1^2 \phi_2^2}$$

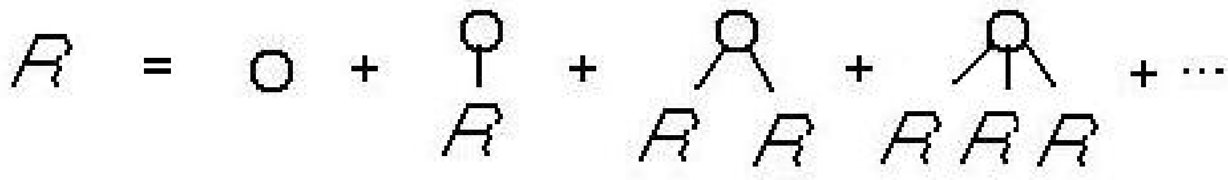
General Increasing Trees

Generating Functions:

$$y_n = \sum_{T \in \mathcal{P}_n} \omega(T)$$

$$y(x) = \sum_{n \geq 1} y_n \frac{x^n}{n!}$$

$$\boxed{y'(x) = \phi_0 + \phi_1\, y(x) + \phi_2\, y(x)^2 + \cdots = \phi(y(x))}$$



General Increasing Trees

Probability distribution on $\mathcal{P}_n$

For $T \in \mathcal{P}_n$ set:

$$\boxed{\pi_n(T) := \frac{\omega(T)}{y_n}}$$

Remark. In general it is **not** clear whether π_n is induced by **a tree evolution process**. It is just a sequence of probability measures.

General Increasing Trees

Examples

- **Recursive Trees:** $\boxed{\phi(t) = \sum_{j\geq 0} \frac{t^j}{j!} = e^t}$, $\phi_j = \frac{1}{j!}$

  The factor $1/j!$ "reduces" planar trees to non-planar ones.

- **Plane Oriented Trees:** $\boxed{\phi(t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}}$, $\phi_j = 1$

- **Binary Search Trees:** $\boxed{\phi(t) = (1+t)^2}$, $\phi_0 = 1$, $\phi_1 = 2$, $\phi_2 = 1$.

For all these three examples, π_n is induced by a tree evolution process.

General Increasing Trees

Theorem [Panholzer & Prodinger, 200?]

The sequence π_n of probability measures on $\mathcal{P}_n$ is induced by a tree evolution process if and only if $\phi(t)$ has one of the three forms:

- $\phi(t) = \phi_0 \left(1 + \frac{\phi_1}{D\phi_0} t\right)^D$ for some $D \in \{2, 3, \ldots\}$ and $\phi_0 > 0$, $\phi_1 > 0$.

- $\phi(t) = \phi_0 e^{\frac{\phi_1}{\phi_0} t}$ with $\phi_0 > 0$, $\phi_1 > 0$.

- $\phi(t) = \frac{\phi_0}{\left(1 - \frac{\phi_1}{r\phi_0} t\right)^r}$ for some $r > 0$ and $\phi_0 > 0$, $\phi_1 > 0$.

General Increasing Trees

Probabilistic tree evolution model

- The process starts with the root that is labeled with 1.
- At step j a new node (with label j) is attached to any previous node (with out-degree d) with probability **proportional** to

$$\frac{(d+1)\phi_{d+1}\phi_0}{\phi_d}$$

In order to obtain all possible π_n it is sufficient to work with $\phi_0 = \phi_1 = 1$:

$$\phi(t) = (1+t)^D, \quad \phi(t) = e^t, \quad \phi(t) = 1/(1-t)^r$$

General Increasing Trees

Recursive Trees: $\phi(t) = e^t$

$$\phi_d = \frac{1}{d!} \quad \Longrightarrow \quad \frac{(d+1)\phi_{d+1}\phi_0}{\phi_d} = 1$$

A new node is attached to previous nodes with **equal** probability.

General Increasing Trees

Generalized Plane Oriented Trees: $\phi(t) = 1/(1-t)^r$ for some $r > 0$

$$\phi_d = \binom{r+d-1}{d} \quad \Longrightarrow \quad \frac{(d+1)\phi_{d+1}\phi_0}{\phi_d} = d + r$$

A new node is attached to a previous nodes with probability **proportional** to $\boxed{d+r}$, where d is the **out-degree**.

For $r = 1$ this these are (usual) plane oriented trees.

The Degree Distribution

Theorem

Let $\phi(t) = 1/(1-t)^r$ for some $r > 0$ and set

$$\begin{aligned}\lambda_d &= \lim_{n\to\infty} \text{probability that a random node in } \mathcal{P}_n \text{ has out-degree } d \\ &= \lim_{n\to\infty} \frac{\text{expected number of nodes with out-degree } d}{n}\end{aligned}$$

Then

$$\boxed{\boxed{\lambda_d = \frac{(r+1)\Gamma(2r+1)\Gamma(r+d)}{\Gamma(r)\Gamma(2r+d+2)}}}.$$

In particular

$$\boxed{\lambda_d \sim \frac{(r+1)\Gamma(2r+1)}{\Gamma(r)} \cdot d^{-2-r}}.$$

The Degree Distribution

Remark 1

There is also a **central limit theorem** for each d.

Remark 2

This result is in accordance to [Dorogovtsev, Mendes & Samukhin, 2000] and [Buckley & Osthus, 2004].
$(r = A,\ m = 1)$

The Degree Distribution

Generating Functions

$N_d(T)$ number of nodes of T with out-degree d
$N_{d,n}$ (random) number of nodes in $\mathcal{P}_n$ with out-degree d.

$$y_{n,k} = \sum_{T \in \mathcal{P}_n, N_d(T)=k} \omega(T) = y_n \cdot \mathbf{Pr}\,(N_{d,n} = k)$$

$$\begin{aligned} y(x,u) &= \sum_{n,k} y_{n,k}\, u^k \frac{x^n}{n!} = \sum_T \omega(T) \frac{x^{|T|}}{|T|!} u^{N_d(T)} \\ &= \sum_{n \ge 1} y_n \cdot \mathbf{E}\, u^{N_{d,n}} \cdot \frac{x^n}{n!} \end{aligned}$$

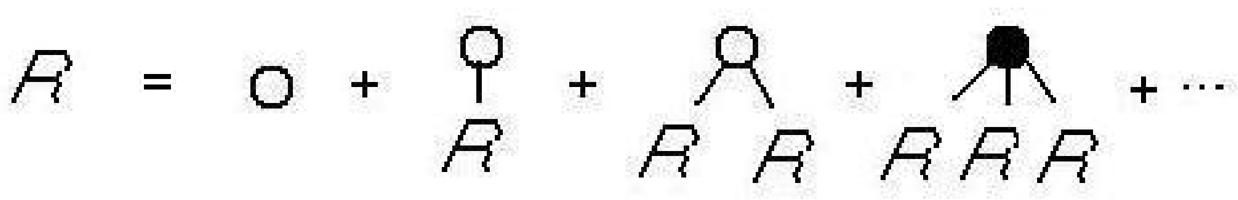
The Degree Distribution

Generating Functions

$$\frac{\partial y(x,u)}{\partial x} = \phi(y(x,u)) + \phi_d \cdot (u-1) \cdot y(x,u)^d$$

$d = 3$:

$$\phi(y) + \phi_d \cdot (u-1) \cdot y^d = \phi_0 + \phi_1\, y + \phi_2\, y^2 + u\, \phi_3\, y^3 + \phi_4\, y^4 + \cdots$$



The Degree Distribution

Expected value of nodes of degree d

$N_{d,n}$ denotes the random variable that counts the number of nodes in $\mathcal{P}_n$ with out-degree d.

$$\Longrightarrow \quad y(x,u) = \sum_{n,k} y_{n,k} \frac{x^n}{n!} u^k = \sum_{n \geq 1} y_n \cdot \mathbf{E}\, u^{N_{d,n}} \cdot \frac{x^n}{n!}$$

$$\Longrightarrow \quad \left. \frac{\partial y(x,u)}{\partial u} \right|_{u=1} = \sum_{n \geq 1} y_n \cdot \mathbf{E}\, N_{d,n} \cdot \frac{x^n}{n!}$$

The Degree Distribution

Expected value of nodes of degree d

Set $S(x) = \left.\frac{\partial y(x,u)}{\partial u}\right|_{u=1} = \sum_{n\ge 1} y_n \cdot \mathbf{E}\, N_{d,n} \cdot \frac{x^n}{n!}$.

Recall that $\frac{\partial y(x,u)}{\partial x} = \phi(y(x,u)) + \phi_d(u-1)y(x,u)^d$

$$\Longrightarrow \quad S'(x) = \phi'(y(x))S(x) + \phi_d y(x)^d$$

$$\Longrightarrow \quad \boxed{S(x) = \phi_d\, y'(x) \int_0^x \frac{y(t)^d}{y'(t)}\, dt}$$

The Degree Distribution

Singular behaviour of $y(x)$

$\phi(t) = 1/(1-t)^r$, $y'(x) = \phi(y(x))$

$$\Longrightarrow \quad \boxed{y(x) = 1 - (1-(r+1)x)^{\frac{1}{r+1}}}$$

$x_0 = \frac{1}{r+1}$ is the (only) singularity of $y(x)$.

$$\Longrightarrow \quad \boxed{\frac{y_n}{n!} = (-1)^{n-1}(r+1)^n \binom{\frac{1}{r+1}}{n} \sim \frac{-1}{\Gamma(-\frac{1}{r+1})}(r+1)^n n^{-\frac{r+2}{r+1}}}$$

Algebraic Singularities

Lemma

Suppose that

$$\boxed{y(x) = (1-x)^{-\alpha}}.$$

Then

$$\boxed{y_n = (-1)^n \binom{-\alpha}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} + \mathcal{O}\left(n^{\alpha-2}\right).}$$

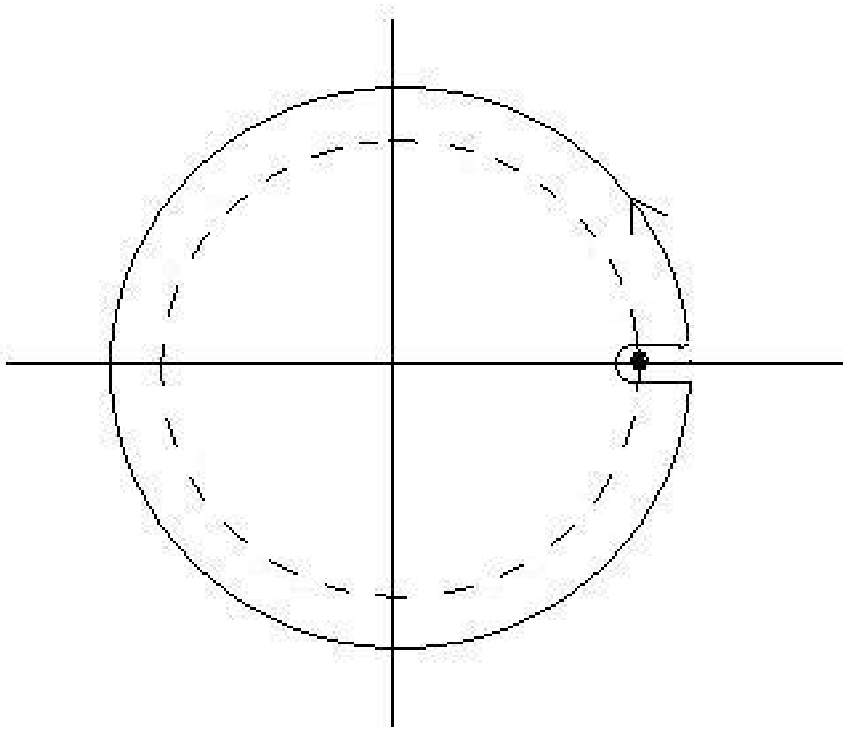
Remark. This lemma applies in the previous situation for $\alpha = -\frac{1}{r+1}$.

Algebraic Singularities

Proof.

Cauchy's formula:

$$(-1)^n \binom{-\alpha}{n} = \frac{1}{2\pi i} \int_\gamma (1-x)^{-\alpha} x^{-n-1}\, dx.$$



Algebraic Singularities

More precisely ...

$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

$$\gamma_1 = \left\{ x = 1 + \frac{t}{n} \,\middle|\, |t| = 1, \Re t \le 0 \right\}$$

$$\gamma_2 = \left\{ x = 1 + \frac{t}{n} \,\middle|\, 0 < \Re t \le \log^2 n, \Im t = 1 \right\}$$

$$\gamma_3 = \overline{\gamma_2}$$

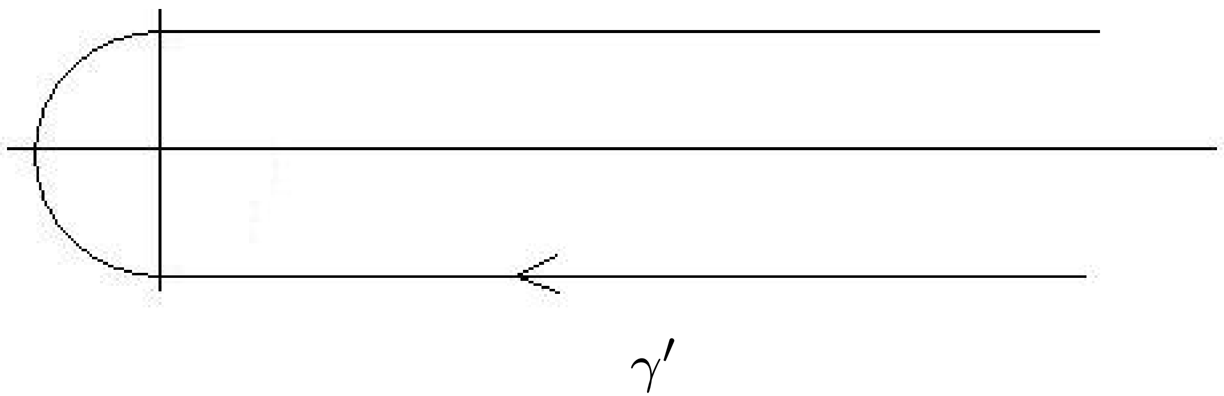
$$\gamma_4 = \left\{ x \,\middle|\, |x| = \left| 1 + \frac{\log^2 n + i}{n} \right|, \arg(1 + \frac{\log^2 n + i}{n}) \le |\arg(x)| \le \pi \right\}.$$

Algebraic Singularities

Substitution for $x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$:

$$x = 1 + \frac{t}{n} \Longrightarrow x^{-n-1} = e^{-t}\left(1 + \mathcal{O}\left(\frac{t^2}{n}\right)\right)$$

$$\Longrightarrow \quad t \in \gamma' = \{t \,|\, |t| = 1, \Re t \le 0\} \cup \{t \,|\, 0 < \Re t \le \log^2 n, \Im t = \pm 1\}:$$



Algebraic Singularities

With Hankel's integral representation for $1/\Gamma(\alpha)$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3}(1-x)^{-\alpha}x^{-n-1}\,dx &= \frac{n^{\alpha-1}}{2\pi i}\int_{\gamma'}(-t)^{-\alpha}e^{-t}\,dt \\
&+ \frac{n^{\alpha-2}}{2\pi i}\int_{\gamma'}(-t)^{-\alpha}e^{-t}\cdot\mathcal{O}\left(t^2\right)\,dt \\
&= n^{\alpha-1}\frac{1}{\Gamma(\alpha)}+\mathcal{O}\left(n^{\alpha-2}\right).
\end{aligned}$$

Algebraic Singularities

Lemma [Flajolet and Odlyzko, 1992]

Let

$$y(x) = \sum_{n \geq 0} y_n \, x^n$$

be analytic in a region

$$\Delta = \{x : |x| < x_0 + \eta, \; |\mathrm{arg}(x - x_0)| > \delta\},$$

$x_0 > 0$, $\eta > 0$, $0 < \delta < \pi/2$.

Suppose that for some real α

$$\boxed{y(x) = \mathcal{O}\left((1 - x/x_0)^{-\alpha}\right) \qquad (x \in \Delta).}$$

Then

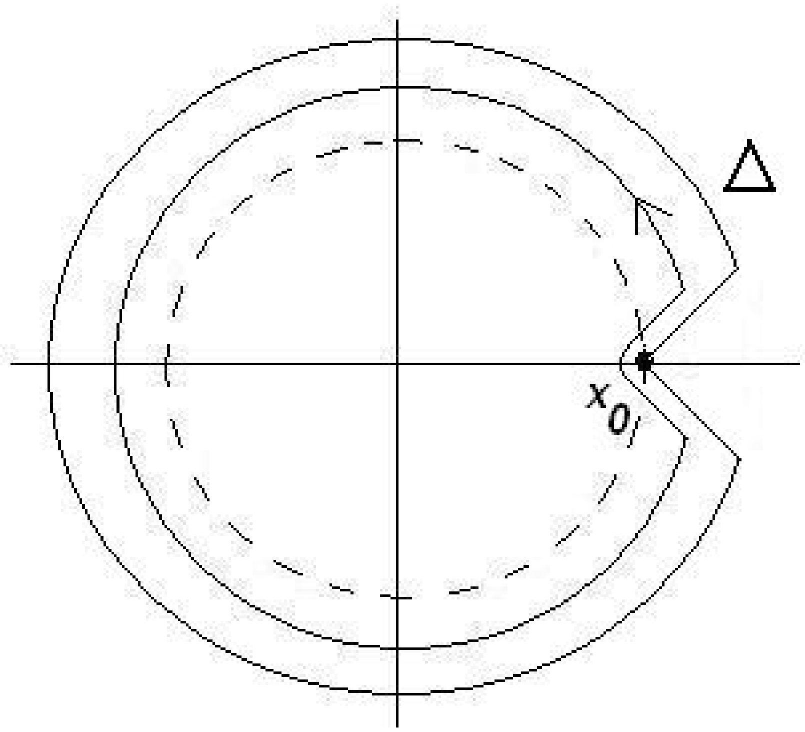
$$\boxed{y_n = \mathcal{O}\left(x_0^{-n} n^{\alpha - 1}\right).}$$

Algebraic Singularities

Proof

Cauchy's formula:

$$y_n = \frac{1}{2\pi i} \int_\gamma y(x)\, x^{-n-1}\, dx,$$



Algebraic Singularities

Asymptotic Transfer

Suppose that a function $y(x)$ is analytic in a region of the form Δ and that it has an expansion of the form

$$y(x) = C\left(1 - \frac{x}{x_0}\right)^{-\alpha} + \mathcal{O}\left(\left(1 - \frac{x}{x_0}\right)^{-\beta}\right) \qquad (x \in \Delta),$$

where $\beta < \alpha$. Then we have (as $n \to \infty$)

$$y_n = [x^n]y(x) = C\frac{n^{\alpha-1}}{\Gamma(\alpha)}x_0^{-n} + \mathcal{O}\left(x_0^{-n} n^{\max\{\alpha-2,\beta-1\}}\right).$$

The Degree Distribution (cont.)

Singular behaviour of $S(x) = \sum_{n\ge 1} y_n \, \mathbf{E}\, N_{d,n} \frac{x^n}{n!} = \phi_d \, y'(x) \int_0^x \frac{y(t)^d}{y'(t)} \, dt$

- $y'(x) = \dfrac{1}{(1-(r+1)x)^{1-\frac{1}{r+1}}}$

- $\displaystyle\int_0^x \frac{y(t)^d}{y'(t)} \, dt = \int_0^{1/(r+1)} \frac{y(t)^d}{y'(t)} \, dt + O\left((1-(r+1)x)^{\frac{1}{r+1}}\right)$

$$\Longrightarrow \quad S(x) \; = \; \phi_d \int_0^{1/(r+1)} \frac{y(t)^d}{y'(t)} \, dt \cdot \boxed{\frac{1}{(1-(r+1)x)^{1-\frac{1}{r+1}}}} + O\left((1-(r+1)x)^{\frac{2}{r+1}-1}\right)$$

The Degree Distribution (cont.)

$$\Longrightarrow \quad \frac{y_n}{n!} \cdot \mathbf{E}\, N_{d,n} \;=\; \phi_d \int_0^{1/(r+1)} \frac{y(t)^d}{y'(t)}\, dt \cdot (-1)^n (r+1)^n \binom{\frac{1}{r+1} - 1}{n} + O\left((r+1)^n n^{-\frac{2}{r+1}}\right)$$

Recall: $\frac{y_n}{n!} \sim \frac{-1}{\Gamma(-\frac{1}{r+1})} (r+1)^n n^{-\frac{r+2}{r+1}}$

$$\Longrightarrow \quad \mathbf{E}\, N_{d,n} \;=\; \phi_d \cdot (r+1) \int_0^{1/(r+1)} \frac{y(t)^d}{y'(t)}\, dt \cdot n + O\left(n^{1-\frac{1}{r+1}}\right)$$

The Degree Distribution (cont.)

$$\begin{aligned}
\Longrightarrow \quad \lambda_d &= \lim_{n\to\infty} \frac{\mathbf{E}\, N_{d,n}}{n} \\
&= \phi_d \cdot (r+1) \int_0^{1/(r+1)} \frac{y(t)^d}{y'(t)}\, dt \\
&= \frac{(r+1)\Gamma(2r+1)\Gamma(r+d)}{\Gamma(r)\Gamma(2r+d+2)} \\
&\sim \frac{(r+1)\Gamma(2r+1)}{\Gamma(r)} \cdot d^{-2-r}
\end{aligned}$$

Conclusion

General Plane Oriented Trees (defined by $\phi(t) = \frac{1}{(1-t)^r}$) have the following properties:

- Average distance between 2 nodes is of order $\log n$ (+ **central limit theorem**)
- Height is order $\log n$ (+ **exponential tails**)
- Degree distribution is of the form $\lambda_d \sim cd^{-2-r}$ (**scale free**)

There is **no clustering** (it is a tree !!!) but these kind of trees can be used as a **prototype for scale free random graphs** (trees) where several asymptotic properties can be proved rigorously.

Thank You!