Universal exponents and tail estimates in the enumeration of planar maps

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Abstract

It has been observed that for most classes of planar maps, the number of maps of size n grows asymptotically like $c \cdot n^{-5/2}\gamma^n$, for suitable positive constants c and γ. It has also been observed that, if d_k is the limit probability that the root vertex in a random map has degree k, then again for most classes of maps the tail of the distribution is asymptotically of the form $d_k \sim c \cdot k^{1/2} q^k$ as $k \to \infty$, for positive constants c, q with $q < 1$.

We provide a rationale for this universal behaviour in terms of analytic conditions on the associated generating functions. The fact that generating functions for maps satisfy as a rule a quadratic equation with one catalytic variable, allows us to identify a critical condition implying the shape of the above-mentioned asymptotic estimates. We verify this condition on several well-known families of planar maps.

1 Introduction

A planar map is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is rooted if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the outer face. All maps in this paper are rooted.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 1960's. He introduced the technique now called "the quadratic method" in order to compute the number M_n of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!}\, 3^n.$$

This was later extended by Tutte and his school to several classes of planar maps: 2-connected, 3-connected, bipartite, Eulerian, triangulations, quadrangulations, etc.

Using the previous formula, Stirling's estimate gives $M_n \sim c \cdot n^{-5/2} 12^n$, where $c > 0$ is a constant. In all cases where a "natural" condition is imposed on maps, the asymptotic estimates turn out to be of this kind:

$$c \cdot n^{-5/2}\gamma^n \tag{1.1}$$

The constants c and γ depend on the class under consideration, but one gets systematically an $n^{-5/2}$ term in the estimate. This phenomenon is discussed by Banderier

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et al. [1], where the generating function $M(z)$ of a given class of maps is parameterized in Lagrangean form as

$$M(z) = \psi(L(z)), \qquad \text{where} \;\; L(z) = z\phi(L(z)).$$

Usually this kind of parametrization gives rise to a square-root singularity and an asymptotic estimate with a subexponential term $n^{-3/2}$. However, as noted in [1], it appears that in all known map-related parameterizations of this form, the cancelation $\psi'(\tau) = 0$ holds, where τ is the positive solution of $\tau\phi'(\tau) - \phi(\tau) = 0$, so that the singular exponent is shifted to 3/2. The authors of [1] then observe: 'This generic asymptotic form is "universal" in so far as it is valid for all known "natural" families of maps'. If the singular exponent is 3/2 then, by singularity analysis, the subexponential term is $n^{-5/2}$ instead of the usual $n^{-3/2}$. The main goal of this paper is to provide an explanation for this universal phenomenon, based on the analysis of the associated counting generating functions.

We turn now to degree distributions. Given a class of planar maps, let $d_{n,k}$ be the probability that the root-vertex has degree k in a map with n edges, and assume that the following limit exists for all $k \geq 1$:

$$d_k = \lim_{n\to\infty} d_{n,k}.$$

Liskovets observed in [14] that for several natural classes of planar maps, the estimates of d_k for large k are of the form

$$d_k \sim c \cdot k^{1/2} q^k, \qquad k \to \infty, \tag{1.2}$$

where again the constants $c > 0$ and $0 < q < 1$ depend on the class, but the critical exponent 1/2 appears to be universal, in the sense that it is the same for all classes of planar maps. However we find no attempt to explain this universal phenomenon in [14]. Our main result provides again such an explanation.

It must be noted that there are exceptions to this behaviour. For instance, the class of rooted maps with a unique face: since they are in bijection with plane trees, the subexponential term is $n^{-3/2}$ in this case. Plane trees are also an exception for the degree of the root, since we have the well-known exact expression $d_k = k(1/2)^{k+1}$. Outerplanar maps, which can be encoded by trees, are another exception. Also, maps having some kind of symmetry usually do not follow the universal pattern.

In the following table we list several classes of maps that conform to the universal exponents $n^{-5/2}$ for the univariate enumeration, and to $k^{1/2}$ for the tail of the limit distribution of the root-vertex degree. We display in each case the constants γ and q that appear in (1.1) and (1.2).

Class of maps	γ	q
Arbitrary	12	5/6
Eulerian	8	$\sqrt{3}/2$
3-connected	4	1/2
Loopless	256/27	3/4
2-connected	27/4	2/3
Bipartite	8	3/4

The last two families have not been discussed explicitly with respect to the degree of the root-vertex, hence we treat them in some detail in Section 3 after proving our main result. To illustrate the applicability of the method, we also discuss the related problem of counting near-triangulations with respect to the degree of the root-face.

The goal of this paper is to provide an explanation for these universal phenomena, based on a detailed analysis of the quadratic method. In order to motivate the statements that follow, let us recall the basic technique for counting planar maps. Let $M_{n,k}$ be the number of maps with n edges and in which the degree of the root-face is equal k. Let $M(z,u) = \sum m_{n,k} u^k z^n$ be the associated generating function. As shown by Tutte [16], $M(z,u)$ satisfies the quadratic equation

$$M(z,u) = 1 + zu^2 M(z,u)^2 + uz \frac{uM(z,u) - M(z,1)}{u-1}. \tag{1.3}$$

Completing the square and setting $y(z) = M(z,1)$ for simplicity, the former equation can be rewritten as

$$[G_1(z,u,y(z))M(z,u) + G_2(z,u,y(z))]^2 = H(z,u,y(z)), \tag{1.4}$$

where the G_i and H depend on the variables indicated (in this particular case only H depends on y). The quadratic method consists on binding variables z and u, assuming that there exists a function $u(z)$ such that $H(z,u(z),y(z)) = 0$ identically. Because of the square in the left-hand side of (1.4), the derivative $H_u(z,u(z),y(z))$ with respect to u also vanishes. From the system of equations

$$H(z,u(z),y(z)) = 0, \qquad H_u(z,u(z),y(z)) = 0 \tag{1.5}$$

one eliminates $y(z)$ to find $u(z)$, and then find $y(z)$ from $H(z,u(z),y(z)) = 0$. Once we know $y(z) = M(z,1)$, from Equation (1.3) we obtain $M(z,u)$. If we carry out this program in this particular case, we find that

$$y(z) = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \cdots,$$

from which we can deduce the explicit form for the numbers M_n. An explicit expression is obtained also for $M(z,u)$, which encodes completely the distribution of the degree of the root-face. Since planar maps are closed under duality, this is the same distribution as the degree of the root-vertex.

In order to estimate M_n and $M_{n,k}$ we use singularity analysis. The singular expansion of $y(z)$ at its dominant singularity $z = 1/12$ is of the form

$$y(z) = y_0 + y_2(1-12z) + y_3(1-12z)^{3/2} + O((1-2z)^2).$$

By transfer theorems [10, 11], we obtain immediately that

$$M_n = [z^n]y(z) \sim c \cdot n^{-5/2} 12^n.$$

The key point is that there is no square-root term $(1-12z)^{1/2}$ in the singular expansion, hence the subexponential term is $n^{-5/2}$ instead of the classical $n^{-3/2}$ term that arises in the enumeration of trees.

It can be checked that, for u near 1, the singularity of $M(z,u)$ as a function of z does not change and we have a singular expansion

$$M(z,u) = y_0(u) + y_2(u)(1-12z) + y_3(u)(1-12z)^{3/2} + O((1-2z)^2),$$

where the $y_i(u)$ are analytic and can be computed explicitly. The probability that the root-face has degree k in a random map with n edges is equal to $M_{n,k}/M_n$. Hence the probability generating function (PGF) of the root-face degree is

$$p_n(u) = \frac{[z^n]M(z,u)}{[z^n]y(z)}.$$

It follows that the PGF of the limiting distribution is equal to

$$p(u) = \lim_{n\to\infty} p_n(u) = \frac{y_3(u)}{y_3} = \frac{u\sqrt{3}}{\sqrt{(2+u)\,(6-5\,u)^3}},$$

whose dominant singularity is at $u = 6/5$. Finally, again by singularity analysis, we obtain the tail estimate

$$d_k = [u^k]p(u) \sim c \cdot k^{1/2} \left(\frac{5}{6}\right)^k.$$

Our main result says (informally) that the above situation is typical for most families of planar maps: the univariate expansion of $y(z)$ has $(1 - z/z_0)^{3/2}$ as dominant term, where z_0 is the singularity of $y(z)$. Furthermore, the singularity in z of $M(z, u)$ does not change if u is small, and the singular expansion of the limit PFG $p(u)$ has $(1 - qu)^{-3/2}$ as the dominant term, where $1/q$ is the singularity of $p(u)$.

More precisely, we concentrate on situations where

$$F(z, u) = \sum_{n,k} f_{n,k} u^k z^n, \qquad y(z) = \sum_n y_n z^n$$

are the unique solutions of equation (1.4). Usually G_1, G_2, and H are polynomials, but the proof of our main result requires only that these functions are analytic in a proper range. In our applications we also have $y(z) = F(z, 1)$ or $y(z) = F(z, 0)$, but this is not be required either.

Nevertheless, we have to assume some a priori knowledge on the functions $y(z)$ and $F(z, u)$. In particular we have to check first that $y(z)$ has a finite radius of convergence z_0 such that $y_0 = y(z_0)$ is finite. Actually, this is relatively simple if H is a polynomial. We just have to eliminate u from the system

$$H(z, u, y) = 0, \qquad H_u(z, u, y) = 0 \tag{1.6}$$

and analyze a single equation $F(z, y) = 0$; see the comments in Section 2.2. It is also required that z_0 is the only singularity on the circle of convergence $|z| = z_0$, and that $y(z)$ can be analytically continued to a slit circle

$$C(z_0, \delta) = \{z : |z| < |z_0| + \delta\} \setminus [z_0, \infty)$$

for some $\delta > 0$. Again, since $y(z)$ is (usually) the solution of a single equation $F(z, y) = 0$, this is easy to establish.

Since

$$F(z, u) = \frac{-G_2(z, u, y(z)) + \sqrt{H(z, u, y(z))}}{G_1(z, u, y(z))},$$

the analytic behaviour of the mapping $u \mapsto F(y_0, u)$ is completely explicit and the radius of convergence u_0 can be determined. In our context it is natural to assume that u_0 satisfies $H(z_0, u_0, y_0) = 0$. Furthermore, since $y(z)$ is singular at $z = z_0$ the point (z_0, u_0, y_0) should be also a critical point of the system (1.6). This means that the solution $y(z)$ and $u(z)$ of (1.6) have a common singularity at $z = z_0$ with $u(z_0) = u_0$.

If (z_0, u_0, y_0) is a critical point of the system (1.6), then the Jacobian

$$\begin{vmatrix} H_y & H_u \\ H_{uy} & H_{uu} \end{vmatrix} = \begin{vmatrix} H_y & 0 \\ H_{uy} & H_{uu} \end{vmatrix} = H_y H_{uu}$$

must vanish, that is, $H_y H_{uu} = 0$ at (z_0, u_0, y_0). Interestingly enough, for all the natural classes of planar maps amenable to the quadratic method, it turns out that $H_{uu} = 0$ and $H_y \neq 0$.[1] The critical condition is then

$$H_{uu}(z_0, u_0, y_0) = 0.$$

This condition is easy to check, since we always work in the realm of algebraic functions and algebraic numbers. Actually the system $H = H_u = H_{uu} = 0$ has (usually) only finitely many solutions. For the running example we are using, we have $(z_0, u_0, y_0) = (1/12, 6/5, 4/3)$ and

$$H = 4(u-1)u^3z^2y + u^4z^2 - 4u^4z + 6u^3z - 2u^2z + u^2 - 2u + 1. \tag{1.7}$$

A simple check gives $H_{uu}(1/12, 6/5, 4/3) = 0$. It also gives $H_{uuu}(1/12, 6/5, 4/3) \neq 0$, which is necessary, together with other non-vanishing conditions, for the method to work properly.

We are now ready to state our main result.

Theorem 1. *Let $F(z,u) = \sum_{n,k} f_{n,k} u^k z^n$ and $y(z) = \sum_n y_n z^n$ with $f_{n,k} \geq 0$ and $y_n \geq 0$ be the unique solutions of the equation*

$$\left(G_1(z,u,y(z))F(z,u) + G_2(z,u,y(z))\right)^2 = H(z,u,y(z)), \tag{1.8}$$

with functions G_1, G_2 and H that are analytic for $|z| < z_0 + \eta$, $|u| < u_0 + \eta$, $|y| < y_0 + \eta$ for some $\eta > 0$, where $z_0 > 0$ denotes the radius of convergence of $y(z)$ that satisfies $0 < y_0 = y(z_0) < \infty$, and $u_0 > 0$ denotes the radius of convergence of the function $F(z_0, u)$. Assume also that $z = z_0$ and $u = u_0$ are the only singularities on the circles of convergence of $y(z)$ and $F(y_0, u)$ and that they can be continued analytically to slit circles $C(z_0, \delta')$ and $C(u_0, \delta'')$, respectively.

Furthermore assume that

$$H(z_0, u_0, y_0) = 0, \qquad H_u(z_0, u_0, y_0) = 0, \qquad H_{uu}(z_0, u_0, y_0) = 0,$$

together with

$$G_1 \neq 0, \quad H_y \neq 0, \quad H_{uy} \neq 0, \quad H_{uuu} \neq 0, \quad H_z H_{uy} \neq H_y H_{zu},$$
$$H_{uuu} H_{uy}^2 - H_y H_{uy} H_{uuuu} + 3 H_y H_{uuy} H_{uuu} \neq 0$$

evaluated at (z_0, u_0, y_0). Then

1. *The following asymptotic estimate holds for some constant $c > 0$:*

$$y_n \sim c \cdot n^{-5/2} z_0^{-n},$$

2. *For every integer $k \geq 0$ the limit*

$$d_k = \lim_{n\to\infty} \frac{f_{n,k}}{y_n}$$

   exists and we have, uniformly for $k \leq C \log n$,

$$\frac{f_{n,k}}{d_n} \sim \bar{c} \cdot k^{1/2} q^k$$

   for some $\bar{c} > 0$, $q = 1/u_0$ and any constant $C > 0$; in particular

$$d_k \sim \bar{c} \cdot q^k k^{1/2} \qquad (k \to \infty).$$

[1] Actually we have no combinatorial or analytic explanation for this phenomenon.

For completeness, we check all the condition in the statement for $H(z,u,y)$ as in (1.7), evaluated at the critical point $(z_0,u_0,y_0) = (1/12, 6/5, 4/3)$. In addition to H, we need $G_1 = 2(1-u)u^2 z$. Then

$$G_1 = \frac{-6}{125}, \quad H_y = \frac{6}{625}, \quad H_{uy} = \frac{9}{125}, \quad H_{uuu} = \frac{-50}{9}, \quad H_z H_{uy} - H_y H_{zu} = \frac{288}{15625},$$

$$H_{uuu} H_{uy}^2 - H_y H_{uy} H_{uuuu} + 3 H_y H_{uuy} H_{uuu} = \frac{-43}{625}.$$

The proof of the main result is presented in the next section. In Section 3 we present further examples of natural families of maps to which Theorem 1 applies. The paper concludes with some remarks on the degree distribution in maps and graphs.

2 Proof of the main result

We divide the proof of Theorem 1 into several parts. We start with a discussion of the (implicit) solution of a single equation $F(z,y) = 0$. The main parts are then the analysis of the system (1.6) (Section 2.2) and a bivariate asymptotic expansion for the coefficients $f_{n,k}$ using Cauchy's formula (Section 2.3).

2.1 A single equation

The first lemma on the singular structure of the solution of a single equation $F(z,y) = 0$ is classical (e.g., it can be found in [6] or [11]).

Lemma 1. *Suppose that y_0, z_0 are complex numbers and that $F(z,y)$ is a function analytic at (z_0,y_0) and satisfies the properties*

$$F(z_0,y_0) = 0, \quad F_y(z_0,y_0) = 0, \tag{2.1}$$

and

$$F_{yy}(z_0,y_0) \neq 0, \quad F_z(z_0,y_0) \neq 0.$$

Then the equation $F(z,y) = 0$ has precisely two solutions

$$y(z) = g(z) \pm h(z)\sqrt{1 - z/z_0} \tag{2.2}$$

in a neighbourhood of z_0 (except in the part where $1 - z/z_0 \in \mathbb{R}^-$), where the functions $g(z)$ and $h(z)$ are analytic at z_0 and satisfy $g(z_0) = y(z_0)$ and

$$h(z_0) = \sqrt{\frac{2z_0 F_z(z_0,y_0)}{F_{yy}(z_0,y_0)}} \neq 0.$$

Observe that, by the implicit function theorem, if z_0 is a singularity of a solution $y = y(z)$ with a finite value $y(z_0) = y_0$ of the equation $F(z,y) = 0$, then we have necessarily $F_y(z_0,y_0) = 0$. Thus, the above conditions are very natural.

Suppose now that the solution $y(z)$ of $F(z,y) = 0$ has a power series expansion $y(z) = \sum_{n\ge 0} y_n z^n$. It is not immediately clear how to compute the radius of convergence. For example, if $F(z,y)$ is a polynomial then $y(z)$ may have local singular expansions of the form $y(z) = y_0 + y_1(1 - z/z_0)^r + \cdots$, or of the form $y(z) \sim y_0(1 - z/z_0)^{-r}$ for some positive rational number r. Actually it is easy to exclude the second case by looking at the zeros (in z) of the leading coefficient of powers of y of F. In the remaining cases we have a finite value $y_0 = y(z_0)$ which can only appear at a singular point, that is, if $F_y(z_0,y_0) = 0$. Hence, we just have to consider the system (2.1) and to determine positive solutions (z_0, y_0). In particular, if there is just a single positive solution it follows that z_0 has to be the radius of convergence of $y(z)$ and $y(z_0) = y_0$. If there is more than one solution then one has to be more careful, but in any case it is possible (e.g., by using a priori bounds or just by using numerical analysis) to decide which solution is the proper one.

The above lemma assures that $y(z)$ has a square-root singularity if $F_{yy}F_z \neq 0$. If this condition is not satisfied there might be a different analytic behaviour. However, we will not encounter such situations.

Note also that any solution of the system (2.1) is a possible further singularity of $y(z)$ when it is extended analytically beyond the circle of convergence. In particular, if there is no other singularity on the circle of convergence $|z| = z_0$ then it follows that $y(z)$ is regular in a slit circle $C(z_0, \delta)$ for some $\delta > 0$. Together with the local representation (2.2) it thus follows by the transfer theorem (see [10]) that

$$y_n \sim c\, z_0^{-n} n^{-3/2} \qquad (n \to \infty),$$

where $c = \pm h(z_0)/(2\sqrt{\pi}) > 0$.

Since we use a similar technique to obtain a bivariate asymptotic expansion for $f_{n,k}$ (see Section 2.3), we sketch the main points of the proof method. It is based on Cauchy's formula

$$y_n = \frac{1}{2\pi i} \int_\gamma \frac{y(z)}{z^{n+1}}\, dz$$

with the following contours of integration: $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned}
\gamma_1 &= \left\{ z = z_0 \left(1 + \frac{-i + (\log n)^2 - t}{n} \right) : 0 \le t \le (\log n)^2 \right\}, \\
\gamma_2 &= \left\{ z = z_0 \left(1 - \frac{1}{n} e^{-i\phi} \right) : -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \right\}, \\
\gamma_3 &= \left\{ z = z_0 \left(1 + \frac{i + t}{n} \right) : 0 \le t \le (\log n)^2 \right\},
\end{aligned}$$

and γ_4 is a circular arc centered at the origin and making γ a closed curve. The integral over γ_4 is bounded by $O\left(z_0^{-n} e^{-(\log n)^2} \right)$ and is therefore negligible. Finally the integral over $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is approximated by the integral

$$\frac{\pm h(z_0) z_0^{-n}}{n^{3/2}} \frac{1}{2\pi i} \int_{\mathcal{H}} (-x)^{1/2} e^{-x}\, dx$$

where one uses the substitution $z = z_0(1 + x/n)$ and x varies on a so-called Hankel contour $\mathcal{H}$ (see Figure 1). By Hankel's integral representation of $1/\Gamma(s)$ we finally obtain

$$y_n \sim \frac{\pm h(z_0) z_0^{-n}}{\Gamma(-1/2)\, n^{3/2}} = \frac{\mp h(z_0) z_0^{-n}}{2\sqrt{\pi}\, n^{3/2}}.$$

It is clear that if $y(z)$ behaves like $(1 - z/z_0)^\alpha$ then the asymptotic leading term of the coefficient is $z_0^{-n} n^{-1-\alpha}/\Gamma(-\alpha)$, and it is also possible to transfer a $O\left((1 - z/z_0)^\alpha\right)$ term in a proper expansion of $y(z)$ into a $O\left(z_0^{-n} n^{-1-\alpha}\right)$ term of the Cauchy integral (see [10] or [11]).

2.2 Two equations

We recall that the quadratic methods requires to solve the system of equations (1.5). The next lemma provides a variant of Lemma 1 which will be applicable in this framework.

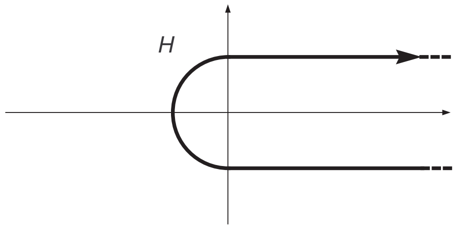

Figure 1: Hankel contour of integration

Lemma 2. *Suppose that z_0, u_0, y_0 are complex numbers and that $H(z,u,y)$ is a function analytic at (z_0,u_0,y_0) and that satisfies the properties*

$$H(z_0,u_0,y_0)=0, \quad H_u(z_0,u_0,y_0)=0, \quad H_{uu}(z_0,u_0,y_0)=0$$

and

$$H_y \neq 0, \quad H_{uy} \neq 0, \quad H_{uuu} \neq 0, \quad H_z H_{uy} \neq H_y H_{uz}$$

for $(z,u,y)=(z_0,u_0,y_0)$. Then the system of functional equations

$$H(z,u(z),y(z))=0, \tag{2.3}$$

$$H_u(z,u(z),y(z))=0, \tag{2.4}$$

has precisely two solutions $u(z)$ and $y(z)$ with $u(z_0)=u_0$ and $y(z_0)=y_0$, which are given by

$$u(z)=g_1(z)\pm g_2(z)\sqrt{1-z/z_0}, \tag{2.5}$$

$$y(z)=h_1(z)\pm h_2(z)\left(1-z/z_0\right)^{3/2} \tag{2.6}$$

in a neighbourhood of z_0 (except in the part, where $1-z/z_0\in\mathbb{R}^-$). The functions $g_1(z)$, $g_2(z)$, $h_1(z)$, and $h_2(z)$ are analytic at z_0 and satisfy

$$g_1(z_0)=u_0,$$

$$g_2(z_0)=\sqrt{\frac{2z_0(H_yH_{uz}-H_zH_{uy})}{H_yH_{uuu}}}\neq 0,$$

$$h_1(z_0)=y_0,$$

$$h_2(z_0)=g_2(z_0)\frac{2z_0(H_yH_{uz}-H_zH_{uy})(H_{uuu}H_{uy}^2-H_yH_{uy}H_{uuuu}+3H_yH_{uuy}H_{uuu})}{3H_{uuu}H_{uy}^2H_y^2},$$

where all derivatives of H have to be evaluated at $(z,u,y)=(z_0,u_0,y_0)$.

Proof. We solve the system (2.3)–(2.4) by first considering the equation $H_u(z,u,y)=0$, where z and u are considered as independent variables and $y=Y(z,u)$ is the unknown function. In a second step we solve the equation $H(z,u,Y(z,u))=0$, where z is the independent variable. Then the solution $u=u(z)$ is the function that we are looking for and $y(z)=Y(z,u(z))$.

Since we assume that $H_{uy}(z_0,u_0,y_0)\neq 0$, it follows from the implicit function theorem that there exists a function $Y(z,u)$ with $Y(z_0,u_0)=y_0$ that is analytic at

(z_0, u_0) and solves (locally) the equation $H_u(z, u, Y(z, u)) = 0$. Observe that

$$\begin{aligned}
Y_u(z_0, u_0) &= -\frac{H_{uu}}{H_{uy}} = 0, \\
Y_{uu}(z_0, u_0) &= -\frac{H_{uuu}}{H_{uy}} \neq 0, \\
Y_{uu}(z_0, u_0) &= \frac{3H_{uuy}H_{uuu} - H_{uy}H_{uuuu}}{H_{uy}^2}, \\
Y_z(z_0, u_0) &= -\frac{H_{uz}}{H_{uy}}, \\
Y_{uz}(z_0, u_0) &= \frac{H_{uuy}H_{uz} - H_{uuz}H_{uy}}{H_{uy}^2},
\end{aligned}$$

evaluated at $(z, u, y) = (z_0, u_0, y_0)$. Next we apply Lemma 1 with $F(z, u) = H(z, u, Y(z, u))$ (and z as the independent variable). By assumption we have

$$\begin{aligned}
F(z_0, u_0) &= H(z_0, u_0, y_0) = 0, \\
F_u(z_0, u_0) &= H_y(z_0, u_0, y_0)Y_u(z_0, u_0) + H_u(z_0, u_0, y_0) = 0, \\
F_{uu}(z_0, u_0) &= H_y(z_0, u_0, y_0)Y_{uu}(z_0, u_0) \neq 0, \\
F_z(z_0, u_0) &= H_y(z_0, u_0, y_0)Y_z(z_0, u_0) + H_z(z_0, u_0, y_0) \\
&= \frac{(H_zH_{uy} - H_yH_{uz})(z_0, u_0, y_0)}{H_{uy}(z_0, u_0, y_0)} \neq 0.
\end{aligned}$$

Hence, the only two solutions $u(z)$ have local expansions of the form

$$u(z) = g_1(z) \pm g_2(z)\sqrt{1 - z/z_0},$$

where $g_1(z)$ and $g_2(z)$ are analytic and satisfy $g_1(z_0) = u_0$ and

$$g_2(z_0) = \sqrt{\frac{2z_0F_z(z_0, u_0)}{F_{uu}(z_0, u_0)}} = \sqrt{\frac{2z_0(H_yH_{uz} - H_zH_{uy})}{H_yH_{uuu}}}.$$

A simple calculation (by using Taylor's theorem and comparing coefficients) we also obtain an expression for

$$\begin{aligned}
g_1'(z_0) &= \frac{F_zF_{uuu} - 3F_{uz}F_{uu}}{F_{uu}^2} \\
&= \frac{1}{3H_y^2H_{uuu}^2}\big(-3H_y^2H_{uuu}H_{uuz} + 2H_{uuu}H_{uy}H_yH_{uz} - 2H_{uuu}H_{uy}^2H_z \\
&\qquad + 3H_yH_{uuy}H_{uuu}H_z + H_y^2H_{uuuu}H_{uz} - H_yH_{uy}H_{uuuu}H_z\big).
\end{aligned}$$

Finally we use the expansion of $u(z)$ to derive the local behaviour of

$$\begin{aligned}
y(z) &= Y(z, u(z)) \\
&= y_0 + Y_z(z_0, u_0)(z - z_0) + \frac{1}{2}Y_{uu}(z_0, u_0)(u(z) - u_0)^2 \\
&\quad + Y_{uz}(z_0, u_0)(z - z_0)(u(z) - u_0) + \frac{1}{6}Y_{uuu}(z_0, u_0)(u(z) - u_0)^3 + O((z - z_0)^2).
\end{aligned}$$

Note that the property $Y_u(z_0, u_0) = 0$ implies that $y(z)$ has no $\sqrt{1 - z/z_0}$ term in its expansion. Precise expressions for the coefficients (like $h_2(z_0)$) can be determined easily. □

2.3 Bivariate asymptotics for the coefficients

Lemma 3. *Suppose that $G(z,u,y)$ is a function that is analytic for $|z| < z_0+\eta$, $|y| < y_0+\eta$, and $|u| < u_0+\eta$, where z_0, u_0, y_0 and η are positive real numbers. Furthermore suppose that $F(z,u) = \sum_{n,k} f_{n,k} u^k z^n$, with $f_{n,k} \ge 0$, is given by*

$$F(z,u) = \sqrt{G(z,u,y(z))},$$

where $y(z) = \sum_{n\ge 0} y_n z^n$, with $y_n \ge 0$, is analytic in a slit circle $C(z_0,\delta')$, with $\delta' > 0$, and can be represented locally by

$$y(z) = h_1(z) + h_2(z)\,(1-z/z_0)^{3/2}$$

with functions $h_1(z)$, $h_2(z)$, that are analytic at $z=z_0$ and satisfy $y_0 = h_1(z_0) \ne 0$ and $h_2(z_0) \ne 0$. Assume also that

$$G=0, \quad G_u = 0, \quad G_{uu} = 0$$

and

$$G_{uuu} \ne 0, \quad G_y \ne 0$$

evaluated at $(z,u,y) = (z_0,u_0,y_0)$, and $G(z_0,u,y_0) \ne 0$ for $u \in C(u_0,\delta'')$ for some $\delta'' > 0$.

Then for every integer $k \ge 0$ the limit

$$d_k = \lim_{n\to\infty} \frac{f_{n,k}}{y_n}$$

exists and we have uniformly for $k \le C \log n$

$$\frac{f_{n,k}}{y_n} \sim \overline{c}\, q^k k^{1/2}$$

for some $\overline{c} > 0$ and $q = 1/u_0$, for any constant $C > 0$. In particular we have

$$d_k \sim \overline{c}\, q^k k^{1/2}, \qquad k\to\infty.$$

Proof. First it follows from the methods described in Section 2.1 (or directly by the transfer theorem) that

$$y_n \sim c_1 z_0^{-n} n^{-5/2}$$

for some constant $c_1 > 0$.

Next we turn to the analysis of $F(z,u)$. By assumption it follows that $G(z,u,y(z))$ has a local expansion of the form

$$\begin{aligned} G(z,u,y(z)) &= G(z_0,u,y_0) + G_z(z_0,u,y_0)(z-z_0) \\ &\quad + G_y(z_0,u,y_0)(y(z)-y_0) + O\left(|1-z/z_0|^2\right) \\ &= \frac{1}{6} G_{uuu}(z_0,u_0,y_0)(u-u_0)^3 + O\left(|1-u/u_0|^4\right) \\ &\quad + \left(G_z(z_0,u,y_0) + G_y(z_0,u,y_0)h_1'(z_0)\right)(z-z_0) \\ &\quad + G_y(z_0,u_0,y_0)h_2(z_0)(1-z/z_0)^{3/2} \\ &\quad + O\left(|1-z/z_0|^{3/2}|1-u/u_0| + |1-z/z_0|^2\right) \\ &= A(u)(z-z_0) + B(1-z/z_0)^{3/2} + C(u)(1-u/u_0)^3 \\ &\quad + O\left(|1-z/z_0|^2 + |1-z/z_0|^{3/2}|1-u/u_0|\right) \end{aligned}$$

where $A(u)$ is a function that is analytic at $u = u_0$, B is a non-zero constant, and $C(u)$ is another function that is analytic at $u = u_0$ with $C(u_0) \neq 0$. In particular this shows that $u = u_0$ is a singularity of the mapping $u \mapsto F(z_0, u)$. By assumption it follows that $F(z_0, u)$ has no other singularities in a proper slit circle $C(u_0, \delta'')$ (with $\delta'' > 0$). Consequently, $u = u_0$ is the radius of convergence of $F(z_0, u)$. Since $f_{n,k} \geq 0$, it also follows that $F(z, u)$ is analytic for $|z| < z_0$ and $|u| < u_0$, there are no other singularities on the circles of convergence, and we have analytic continuation to proper slit circles $C(z_0, \delta')$ and $C(u_0, \delta'')$, respectively.

Next suppose that $|u| < u_0$ and consider the mapping $z \mapsto F(z, u)$. Due to the singularity of $y(z)$ and the non-vanishing of G it follows that the mapping $z \mapsto F(z, u)$ has also a singularity at $z = z_0$ and it can be analytically continued to the same slit circle as $y(z)$. Moreover we have a local expansion of the form

$$F(z, u) = F_0(u) + F_2(u)(1 - z/z_0) + F_3(u)(1 - z/z_0)^{3/2} + \cdots$$

which comes from the local behaviour of $y(z)$. By applying the transfer theorem it thus follows that the limit

$$\lim_{n\to\infty} \frac{[z^n]\, F(z, u)}{[z^n]\, y(z)} = c_3 F_3(u)$$

exists (with a proper constant $c_3 \neq 0$). By assumption this limit has to be a power series in u:

$$p(u) = c_3 F_3(u) = \sum_{k\geq 0} d_k u^k.$$

Furthermore we have for every $k \geq 0$

$$\lim_{n\to\infty} \frac{f_{n,k}}{y_n} = d_k.$$

It is easy to get an explicit form for $p(u)$ in terms of derivatives of G and to exploit this representation to observe that u_0 is the radius of convergence of this series with a local behaviour of the form $c'(1 - u/u_0)^{-3/2}$. By the transfer theorem this local behaviour implies an asymptotic equivalent for d_k of the form $d_k \sim \overline{c} u_0^{-k} k^{1/2}$.

However, we use a different approach to deduce this asymptotic relation by providing even a bivariate asymptotic relation for $f_{n,k}$. We use Cauchy's formula

$$f_{n,k} = \frac{1}{(2\pi i)^2} \int_\gamma \int_{\overline{\gamma}} \frac{F(z, u)}{u^{n+1} u^{k+1}}\, dz\, du$$

where γ is chosen as described in Section 2.1 and $\overline{\gamma}$ is a corresponding path for u. As in the univariate case, the dominant part of the integral comes from those parts that are close to z_0 and u_0. Therefore we neglect the other parts, since they do not contribute to the asymptotic leading term (a similar but more detailed analysis is worked out in [9]). An important observation is that we restrict ourselves to the case $k \leq C \log n$ (for an arbitrary constant $C > 0$). This implies that $|u - u_0|$ is larger than any power of $|z - z_0|$ in the range of interest; recall that $|1 - z/z_0| \approx 1/n$

and $|1-u/u_0| \approx 1/k$ in this range. Hence, in this range, $F(z,u)$ is given by

$$\begin{aligned} F(z,u) &= \sqrt{G(z,u,y(z))} \\ &= \sqrt{C(u)}\,(1-u/u_0)^{3/2}\left(1+\frac{A(u)(z-z_0)+B(1-z/z_0)^{3/2}}{C(u)(1-u/u_0)^3}\right. \\ &\qquad \left. +\,O\left(\frac{|1-z/z_0|^2}{|1-u/u_0|^3}+\frac{|1-z/z_0|^{3/2}}{|1-u/u_0|^2}\right)\right)^{1/2} \\ &= \sqrt{C(u)}\,(1-u/u_0)^{3/2}+\frac{A(u)(z-z_0)+B(1-z/z_0)^{3/2}}{2\sqrt{C(u)}\,(1-u/u_0)^{3/2}} \\ &\quad +O\left(\frac{|1-z/z_0|^2}{|1-u/u_0|^{3/2}}+\frac{|1-z/z_0|^{3/2}}{|1-u/u_0|^{1/2}}\right) \end{aligned}$$

By doing the double Cauchy integration the first two terms give no contribution, since they are analytic in z. However, the third term

$$\frac{B(1-z/z_0)^{3/2}}{2\sqrt{C(u)}\,(1-u/u_0)^{3/2}} = \frac{B(1-z/z_0)^{3/2}}{2\sqrt{C(u_0)}\,(1-u/u_0)^{3/2}} + O\left(\frac{|1-z/z_0|^{3/2}}{|1-u/u_0|^{1/2}}\right)$$

gives a non-trivial contribution. Since the dependence in z and u factors, the integration with respect to z and u can be done independently and we obtain the estimate (similarly to the procedure described in Section 2.1)

$$c_2\, z_0^{-n} u_0^{-k} n^{-5/2} k^{1/2} + O\left(z_0^{-n} u_0^{-k} n^{-5/2} k^{-1/2}\right)$$

for a certain constant $c_2 \neq 0$. Finally the contributions of the other error terms add up to

$$O\left(z_0^{-n} u_0^{-k}\left(n^{-3}k^{1/2}+n^{-5/2}k^{-1/2}\right)\right)$$

which is negligible for $k \le C\log n$.

This completes the proof of the lemma. □

2.4 Proof of Theorem 1

We are finally ready to prove Theorem 1. As already indicated we follow the principles of the quadratic method and in each step we also determine the kind of singularities of the corresponding generating functions.

First of all we apply Lemma 2 for H and obtain proper expansions for $u(z)$ and $y(z)$, which are given by (2.5) and (2.6). By assumption we have $g_2(z_0) \neq 0$ and $h_2(z_0) \neq 0$. Again by assumption $z = z_0$ is the only singularity of $y(z) = \sum_{n\ge 0} y_n z^n$ on the circle of convergence $|z| = z_0$ and there is an analytic continuation to a proper slit circle $C(z_0, \delta')$. By the transfer theorem, it follows that

$$y_n \sim \pm\frac{h_2(z_0)}{\sqrt{\pi}}\, z_0^{-n} n^{-\frac{5}{2}}.$$

Since G_1 and G_2 are analytic at (z_0, u_0, y_0) the functions $G_{1,2}(z,u,y(z))$ are analytic in u if (z,u) are in a neighbourhood of (z_0,u_0). Hence, the ratio $-G_2/G_1$ does not contribute to the asymptotic leading term of $f_{n,k}$. The dominant part comes from the function $\sqrt{H(z,u,y(z))}/G_1(z,u,y(z))$. Since $H(z_0,u_0,y_0) = 0$, this function gets singular. Actually we can apply Lemma 3 for $G = H/G_1^2$ and obtain the asymptotic expansion for $f_{n,k}$. It is easy to verify that all assumptions of Lemma 3 are satisfied.

This completes the proof of Theorem 1.

3 Examples

Our first example, arbitrary planar maps, has been discussed in the introduction. We continue with 2-connected maps (also called non-separable in the literature). In what follows, the size of a map is its number of edges.

2-connected maps. A maps is 2-connected if it has no cut vertex. The *core* of a map is the 2-connected component containing the root-edge. Given a map M of size n whose core C has size k, then M is obtained form C by attaching an arbitrary at each of the $2k$ corners of C. Let $B(z) = \sum B_n z^n$ be the counting generating function for 2-connected maps. The decomposition based on the core of a map gives, as shown by Tutte [16], the relation

$$M(z) = B(zM(z)^2),$$

where $M(z)$ is the generating function for arbitrary maps. If now we add variable u to take into account the degree of the root-face, then we have

$$M(z,u) = B\left(zM(z)^2, \frac{u\,M(z,u)}{M(z)}\right). \tag{3.1}$$

The proof uses again the decomposition into the core and the maps attached to the corners, refined to take into account the degree of the root-vertex (see [4]).

We change variables $x = zM(z)^2$ and $w = M(z,u)/M(z)$, and eliminate z, u and $M(z,u)$ from (3.1) and (1.3). If we notice that $M(z) = B(x)$, the resulting equation is for $B(x,w)$ is

$$B(x,w)^2 - (w^2x + wB(x) + 1)B(x,w) + (w^3 + w)B(x)x + w^2x - wx = 0.$$

Completing the square, the corresponding H function in (1.4) is equal to

$$H(x,w,y) = w^2y^2 - \left(2w^3x + 2w\right)y + w^4x^2 - 2w^2x + 4wx + 1.$$

The dominant singularity of $B(x)$ is at $x_0 = 4/27$. Then $y_0 = y(x_0) = 4/3$, and $w_0 = w(x_0) = 3/2$. Again we check that

$$H_{ww}(4/27, 3/2, 4/3) = 0,$$

together with the non-vanishing conditions. Thus we have proved the following.

Proposition 1. *Let d_k be the asymptotic probability that the root-vertex in 2-connected maps has degree k. Then, as $k \to \infty$,*

$$d_k \sim c \cdot k^{1/2} \left(\frac{2}{3}\right)^k.$$

As in the first section, one can compute the probability generating function and it turns out to be

$$p(w) = \frac{w}{2}\left(1 - \frac{27 - 36w + 4w^2}{\sqrt{(2w-27)(2w-3)^3}}\right).$$

The dominant term is $(1 - 2w/3)^{-3/2}$, which implies the previous estimate by singularity analysis. But the important point is that the critical condition $H_{ww} = 0$, together with the non-vanishing conditions, guarantees the result without knowledge of the probability generating function.

Bipartite and Eulerian maps. The enumeration of bipartite maps is classical and follows the same scheme used for arbitrary maps. If u marks the degree of the root-face, then the associated generating function $B(z,u)$ satisfies

$$B(z,u) = 1 + u^2 z B(z,u)^2 + u^2 z \frac{B(z,u) - B(z,1)}{u^2 - 1}. \tag{3.2}$$

The fact that the equation is in terms of u^2 is because in a bipartite map the root face has even degree. We then obtain

$$H(z,u,y) = (4u^6z^2 - 4u^4z^2)y + u^4 - 2u^2 + 6u^4z + 1 - 2u^2z + u^4z^2 - 4u^6z.$$

The usual computations give the critical values $z_0 = 1/8, u_0 = 2/\sqrt{3}, y_0 = 5/4$. Again we check the critical condition $H_{uu}(z_0,u_0,y_0) = 0$, together with the additional non-vanishing conditions. By duality, we obtain the distribution of the degree of the root-vertex degree in Eulerian maps.

In order to obtain the distribution of the degree of the root-vertex in bipartite maps, we can use a bijection between 2-colored maps and 3-colored triangulations [12]. Given a bipartite map M whose vertices are colored black and white, add a new vertex inside each face f of M and join it to all the vertices in the boundary of f. This produces a triangulation T that is 3-colored by given color red to the new vertices. The triangulation T has a unique 3-coloring (this is the same as saying that is Eulerian), so that the three color classes are indistinguishable. If v and f are, respectively, the root-vertex and the root-face of M, then f has the same degree in T as in M, but the degree of v in T is twice its degree in M. This implies that the number of bipartite maps of size n whose root-vertex has degree k is the same as the number of bipartite maps of size n whose root-face has degree $2k$. Hence both distributions are essentially the same, and the critical value u_0 for the degree of the root-vertex is the square of the corresponding value for the root-face, that is, $u_0 = (2/\sqrt{3})^2 = 4/3$.

Proposition 2. *Let d_k be the asymptotic probability that the root-vertex in bipartite maps has degree k. Then, as $k \to \infty$,*

$$d_k \sim c \cdot k^{1/2} \left(\frac{3}{4}\right)^k.$$

Triangulations. This example is a bit different and illustrates the flexibility of the method. In this section all maps are simple, that is, they have no multiple edges. A triangulation is a map in which every face is a triangle. A near-triangulation is a map in which all faces are triangles except possibly the outer face, and in addition there is no chord. Vertices in the outer face of a near-triangulation are called external, and internal otherwise. To enumerate triangulations, Tutte [15] proceeded as follows. Let $T_{n,k}$ be the number of near-triangulations with n internal vertices and $k+3$ external vertices, and let $T(z,u) = \sum T_{n,k} z^n u^k$. Notice that $T(z,0)$ enumerates triangulations according to the number of internal vertices.

By removing the triangle containing the root-edge in a near triangulation one obtains a sequence of near-triangulations. This decomposition gives rise to the equation

$$u^2\, T(z,u)^2 + (z + zuy(z) - u - u^2)T(z,u) + u - zy(z),$$

where $y(z) = T(z,0)$. The basic function to be analyzed is once more the discriminant of the quadratic equation, and is equal to

$$H(z,u,y) = z^2u^2y^2 + 2\left(z^2u + zu^2 - zu^3\right)y + \left(z - u - u^2\right)^2 - 4u^3.$$

The critical values are $z_0 = 27/256, u_0 = 3/16, y_0 = 32/27$. Notice that in this case $u_0 < 1$; this is because the univariate generating function is now obtained by setting $u = 0$. But this is only a detail and the analysis is the same. Indeed, we check the critical condition $H_{uu}(z_0, u_0, y_0) = 0$, together with the non-vanishing conditions.

If we denote by $S(z, u)$ the generating function of near-triangulations, where now z marks the total number of vertices minus 3, then $S(z, u) = T(z, zu)$. The critical values $(\widetilde{z}_0, \widetilde{u}_0)$ for $S(z, u)$ are determined by $\widetilde{z}_0 = z_0$ and $\widetilde{z}_0\widetilde{u}_0 = u_0$, so that $\widetilde{u}_0 = u_0/z_0 = 16/9$, and we obtain the following.

Proposition 3. *Let d_k be the asymptotic probability that the root-face in near-triangulations, counted according to the number of vertices, has degree k. Then, as $k \to \infty$,*

$$d_k \sim c \cdot k^{1/2} \left(\frac{9}{16}\right)^k.$$

4 Concluding remarks

For maps of genus g, the asymptotic estimates for natural families of maps are of the form

$$c \cdot n^{5/2(g-1)} \gamma^n.$$

Bender [2] posed as an open problem to find a combinatorial explanation for the presence of the exponent $5/2(g-1)$. This has been achieved recently by Chapuy [5], by constructing inductively maps of genus g from maps of genus $g - 1$. Our main result gives an analytic explanation for the base case $g = 0$.

Finally, one can also study these problems for *graphs* instead of maps, that is, graphs without an embedding. Building on results from [13] and [7], it has been shown [8] that for labelled planar graphs counted according to the number of vertices, there is a degree distribution d_k, and the estimate for the tail is of the from

$$c \cdot k^{-1/2} q^k,$$

for suitable constants c and $q < 1$. The reason for the exponent $-1/2$ instead of the exponent $1/2$ we have encountered for maps is that maps are rooted, hence there is an extra factor of k for the degree of the root vertex. The same kind of estimate holds for 2- and 3-connected planar graphs. However, for certain subclasses of planar graphs, such as series-parallel graphs, the estimates are of the form $c \cdot k^{-3/2} q^k$, due to a different critical behaviour of the corresponding generating functions at their singularities (see also [3] for a different approach).

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