

UNIVERSAL SINGULAR EXPONENTS IN CATALYTIC VARIABLE EQUATIONS

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ABSTRACT. Catalytic equations appear in several combinatorial applications, most notably in the numeration of lattice path and in the enumeration of planar maps. The main purpose of this paper is to show that the asymptotic estimate for the coefficients of the solutions of (so-called) positive catalytic equations has a universal asymptotic behavior. In particular, this provides a rationale why the number of maps of size n in various planar map classes grows asymptotically like $c \cdot n^{-5/2} \gamma^n$, for suitable positive constants c and γ . Essentially we have to distinguish between linear catalytic equations (where the subexponential growth is $n^{-3/2}$) and non-linear catalytic equations (where we have $n^{-5/2}$ as in planar maps). Furthermore we provide a quite general central limit theorem for parameters that can be encoded by catalytic functional equations, even when they are not positive.

1. INTRODUCTION

A planar map is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is rooted if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the outer face. All maps in this paper are rooted.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 1960's. Tutte (and Brown) introduced the technique now called “the quadratic method” in order to compute the number M_n of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!} 3^n.$$

This was later extended by Tutte and his school to several classes of planar maps: 2-connected, 3-connected, bipartite, Eulerian, triangulations, quadrangulations, etc. Using the previous formula, Stirling's estimate gives $M_n \sim (2/\sqrt{\pi}) \cdot n^{-5/2} 12^n$. In all cases where a “natural” condition is imposed on maps, the asymptotic estimates turn out to be of this kind:

$$c \cdot n^{-5/2} \gamma^n.$$

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The constants c and γ depend on the class under consideration, but one gets systematically an $n^{-5/2}$ term in the estimate.

This phenomenon is discussed by Banderier et al. [2]: ‘This generic asymptotic form is “universal” in so far as it is valid for all known “natural families of maps”.’ The goal of this paper is to provide an explanation for this universal phenomenon, based on a detailed analysis of functional equations for generating functions with a catalytic variable.

In order to motivate the statements that follow, let us recall the basic technique for counting planar maps. Let $M_{n,k}$ be the number of maps with n edges and in which the degree of the root-face is equal k . Let $M(z, u) = \sum_{n,k} M_{n,k} u^k z^n$ be the associated generating function. As shown by Tutte [16], $M(z, u)$ satisfies the quadratic equation

$$M(z, u) = 1 + zu^2M(z, u)^2 + uz \frac{uM(z, u) - M(z, 1)}{u - 1}. \quad (1.1)$$

The variable u is called a “catalytic variable”.

It turns out that

$$M(z, 1) = \sum_{n \geq 0} M_n z^n = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \dots, \quad (1.2)$$

from which we can deduce the explicit form for the numbers M_n . The remarkable observation here is the singular part $(1 - 12z)^{3/2}$ that reflects the asymptotic behavior $c \cdot n^{-5/2} 12^n$ of M_n .

A general approach to equations of the form (1.1) was carried out by Bousquet-Mélou and Jehanne [4]. First one rewrites (1.1) into the form

$$P(M(z, u), M_1(z), z, u) = 0,$$

where $P(x_0, x_1, z, u)$ is a polynomial (or more generally an analytic function) and $M_1(z)$ abbreviates $M(z, 1)$. Next one searches for functions $f(z)$, $y(z)$ and $u(z)$ with

$$\begin{aligned} P(f(z), y(z), z, u(z)) &= 0, \\ P_{x_0}(f(z), y(z), z, u(z)) &= 0, \\ P_u(f(z), y(z), z, u(z)) &= 0. \end{aligned}$$

If $y(z)$ has a power series representation at $z = 0$, then one has $M_1(z) = y(z)$.

Bousquet-Mélou and Jehanne [4] considered in particular equations of the form¹

$$\boxed{M(z, u) = F_0(u) + zQ \left(M(z, u), \frac{M(z, u) - M(z, 0)}{u}, z, u \right)}, \quad (1.3)$$

where $F_0(u)$ and $Q(\alpha_0, \alpha_1, z, u)$ are polynomials, and showed that there is a unique power series solution $M(z, u)$ that is also an algebraic function. Actually all the examples that we will discuss can be rewritten into (almost) this form (possibly by replacing u by $u + 1$). For example, for the equation (1.1) we have

$$F_0(u) = 1 \quad \text{and} \quad Q(\alpha_0, \alpha_1, z, u) = (u + 1)^2 \alpha_0^2 + (u + 1) \alpha_0 + (u + 1) \alpha_1.$$

¹Actually Bousquet-Mélou and Jehanne [4] considered more general functional equations that contain also higher differences.

In the context of this paper we always assume that F_0 and Q have non-negative coefficients. This is natural since Equation (1.3) can be seen as a translation of a recursive combinatorial description of maps or other combinatorial objects. This also implies that $M(z, u)$ has non-negative coefficients, since the equation (1.3) can be written as an infinite system of equation for the functions $M_j(z) = [u^j] M(z, u)$ with non-negative coefficients on the right hand side.

Let us consider the first case, where Q is linear in α_0 and α_1 , that is, we can write (1.3) as

$$M(z, u) = Q_0(z, u) + zM(z, u)Q_1(z, u) + z \frac{M(z, u) - M(z, 0)}{u} Q_2(z, u). \quad (1.4)$$

Here we are in the framework of the so-called *kernel method*. We rewrite (1.4) as

$$M(z, u)(u - zuQ_1(z, u) - zQ_2(z, u)) = uQ_0(z, u) - zM(z, 0)Q_2(z, u), \quad (1.5)$$

where

$$K(z, u) = u - zuQ_1(z, u) - zQ_2(z, u)$$

is the *kernel*. The idea of the kernel method is to bind u and z so that $K(z, u) = 0$, that is, one considers a function $u = u(z)$ such that $K(z, u(z)) = 0$. Then the left hand side of (1.5) cancels and $M(z, 0)$ can be calculated from the right hand side by setting $u = u(z)$.

Proposition 1. *Suppose that Q_0 , Q_1 , and Q_2 are polynomials in z and u with non-negative coefficients and let $M(z, u)$ be the power series solution of (1.4). Furthermore let $u(z)$ be the power series solution of the equation*

$$u(z) = zQ_2(z, u(z)) + zu(z)Q_1(z, u(z)), \quad \text{with } u(0) = 0.$$

Then $M(z, 0)$ is given by

$$M(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))}.$$

There are three particular *degenerate* cases, where the solution function $M(z, 0)$ is a rational function (or even a polynomial). In these cases the asymptotic analysis of M_n is trivial:

- If $Q_0 = R_0(z)$ and $Q_1 = R_1(z)$ depend only on z , then

$$M(z, 0) = \frac{R_0(z)}{1 - zR_1(z)}.$$

- If $Q_1 = R_1(z)$ depends only on z and if $Q_2 = T_0(z) + T_1(z)u$ is at most linear in u then

$$u(z) = \frac{zT_0(z)}{1 - zR_1(z) - zT_1(z)} \quad \text{and} \quad M(z, 0) = \frac{Q_0(z, u(z))}{1 - zR_1(z)}$$

are rational functions.

- If Q_2 has u as a factor, then $u(z) = 0$ and we have

$$M(z, 0) = \frac{Q_0(z, 0)}{1 - zQ_1(z, 0)}$$

is a rational function.

In all other cases $M(z, 0)$ has universally a dominant square root singularity.

Theorem 1. *Suppose that Q_0 , Q_1 , and Q_2 are polynomials in z and u with non-negative coefficients such that none of the three above mentioned cases occurs.*

Let $M(z, u)$ be the power series solution of (1.4) and let $z_0 > 0$ denote the radius of convergence of $M(z, 0)$. Then the local Puiseux expansion of $M(z, 0)$ around z_0 is given by

$$M(z, 0) = a_0 + a_1(1 - z/z_0)^{1/2} + a_2(1 - z/z_0) + \dots, \quad (1.6)$$

where $a_0 > 0$ and $a_1 < 0$. Furthermore, there exist $b \geq 1$, a non-empty set $J \subseteq \{0, 1, \dots, b-1\}$ of residue classes modulo b and constants $c_j > 0$ such that for $j \in J$

$$M_n = [z^n]M(z, 0) = c_j n^{-3/2} z_0^{-n} \left(1 + O\left(\frac{1}{n}\right) \right), \quad (n \equiv j \pmod{b}, n \rightarrow \infty) \quad (1.7)$$

and $M_n = 0$ for $n \equiv j \pmod{b}$ with $j \notin J$.

This result is quite easy to prove (see Section 3). We just want to mention that there are variations of the above model, for example equations of the form

$$\begin{aligned} M(z, u) = & Q_0(z, u) + zM(z, u)Q_1(z, u) + z \frac{M(z, u) - M(z, 0)}{u} Q_2(z, u) \\ & + uM(z, 0)Q_3(z, u), \end{aligned}$$

that can be handled in the same way; see [14]. However, the asymptotics can be slightly different. For example one might have $n^{-1/2}$ instead of $n^{-3/2}$ in the subexponential growth of M_n (namely if $z_0 Q_1(z_0, u(z_0)) + u(z_0) Q_3(z_0, u(z_0)) = 1$; if $Q_3 = 0$ then we have $z_0 Q_1(z_0, u(z_0)) < 1$).

In the non-linear case the situation is more involved. Here we find the solution function $M(z, 0)$ in the following way.

Proposition 2. *Suppose that Q is a polynomial in α_0, α_1, z, u with non-negative coefficients that depends (at least) on α_1 , that is, $Q_{\alpha_1} \neq 0$, and let $M(z, u)$ be the power series solution of (1.3). Furthermore we assume that Q is not linear in α_0 and α_1 , that is, $Q_{\alpha_0 \alpha_0} \neq 0$, or $Q_{\alpha_0 \alpha_1} \neq 0$ or $Q_{\alpha_1 \alpha_1} \neq 0$.*

Let $f(z), u(z), w(z)$ be the power series solution of the system of equations

$$\begin{aligned} f(z) &= F_0(u(z)) + zQ(f(z), w(z), z, u(z)), \\ u(z) &= zu(z)Q_{\alpha_0}(f(z), w(z), z, u(z)) + zQ_{\alpha_1}(f(z), w(z), z, u(z)), \\ w(z) &= F'_0(u(z)) + zQ_u(f(z), w(z), z, u(z)) + zw(z)Q_{\alpha_0}(f(z), w(z), z, u(z)). \end{aligned} \quad (1.8)$$

with $f(0) = F_0(0)$, $u(0) = 0$, $w(0) = F'_0(0)$. Then

$$M(z, 0) = f(z) - w(z)u(z).$$

The meaning of $w(z)$ will become clear later in the proof of the proposition in Section 4. In Theorem 2 we assume that $Q_{\alpha_0 u} \neq 0$, which implies that the system (1.8) is strongly connected. This means that the dependency di-graph of the system is strongly connected as discussed in Section 4.

Again there are some *degenerate* cases. We do not provide a complete list and we just discuss some of them. We also comment on the case $Q_{\alpha_1} = 0$. Given a multivariate function f we replace one of its variables with a dot if f actually does not depend on this variable.

- Suppose that $Q_u = F'_0 = 0$, that is, F_0 is constant and Q does not depend on u . Here $w(z) = 0$ and consequently

$$M(z, 0) = f(z),$$

where $f(z)$ is the solution of the equation

$$f(z) = F_0 + zQ(f(z), 0, z, \cdot).$$

Thus, depending on the degree of α_0 in $Q(\alpha_0, 0, z, \cdot)$, the solution function $M(z, 0)$ is either a polynomial, a rational function, or it has a square-root singularity as in (1.6); see [1].

- Next suppose that $Q_u = Q_{\alpha_0} = 0$ but $F'_0 \neq 0$. Here we have we are left with the equations

$$f = F_0(u) + zQ(w, z), \quad u = zQ_{\alpha_1}(w, z), \quad w = F'_0(u).$$

Thus, we have to solve the equation $u = zQ_{\alpha_1}(F'_0(u), z)$ to obtain $u = u(z)$ and consequently $w(z) = F'_0(u(z))$ and $f(z) = F_0(u(z)) + zQ(w(z), z)$. Hence, depending on the structure of $zQ_{\alpha_1}(F'_0(u), z)$ we obtain a polynomial, a rational function, or a square-root singularity for

$$\begin{aligned} M(z, 0) &= f(z) - w(z)u(z) \\ &= zQ(w(z), z) + F_0(u(z)) - u(z)F'_0(u(z)). \end{aligned}$$

- Finally, if $Q_{\alpha_1} = 0$, then we have an equation of the form

$$M(z, u) = F_0(u) + zQ(M(z, u), z, u).$$

In this case the catalytic variable u is not necessary and we can set it to 0. Hence, depending on the structure of Q we just get a polynomial, a rational function, or a square-root singularity for $M(z, 0)$ (see [1]).

Theorem 2. *Suppose that Q is a polynomial in α_0, α_1, z, u with non-negative coefficients that depends (at least) on α_1 , that is, $Q_{\alpha_1} \neq 0$ and let $M(z, u)$ be the power series solution of (1.3). Furthermore, we assume that Q is not linear in α_0 and α_1 , that is, $Q_{\alpha_0\alpha_0} \neq 0$ or $Q_{\alpha_0\alpha_1} \neq 0$ or $Q_{\alpha_1\alpha_1} \neq 0$. We assume additionally that $Q_{\alpha_0u} \neq 0$.*

Let $z_0 > 0$ denote the radius of convergence of $M(z, 0)$. Then the local Puiseux expansion of $M(z, 0)$ around z_0 is given by

$$M(z, 0) = a_0 + a_2(1 - z/z_0) + a_3(1 - z/z_0)^{3/2} + O((1 - z/z_0)^2), \quad (1.9)$$

where $a_0 > 0$.

If we further assume that $F'_0(0) = 0$ and $Q_{\alpha_1}(F_0(0), 0, 0, 0) \neq 0$, then $a_3 > 0$. In this case there exists $b \geq 1$ and a residue class a modulo b such that

$$M_n = [z^n]M(z, 0)cn^{-5/2}z_0^{-n} \left(1 + O\left(\frac{1}{n}\right) \right), \quad (n \equiv a \pmod{b}, n \rightarrow \infty) \quad (1.10)$$

for some constant $c > 0$, and $M_n = 0$ for $n \not\equiv a \pmod{b}$.

The plan of the paper is as follows. In the next section we collect some examples of solving Theorems 1 and 2. We then prove Proposition 1 and Theorem 1 in Section 3, and Proposition 2 and Theorem 2 in Section 4. Finally we provide more information on the solution of catalytic equations. In particular, we formulate a quite general central limit theorem, involving an additional parameter, in Section 5.

2. EXAMPLES

2.1. The linear case. Natural examples for the linear case (Proposition 1 and Theorem 1) come from the enumeration of lattice path. We consider paths starting from the coordinate point $(0, 0)$ (or from $(0, t)$, $t \in \mathbb{N}$) and allowed to move only to the right (up, straight or down), but forbid going below the x -axis $y = 0$ at each step. Define a step set $\mathcal{S} = \{(a_1, b_1), (a_2, b_2), \dots, (a_s, b_s) \mid (a_j, b_j) \in \mathbb{N} \times \mathbb{Z}\}$, and let $f_{n,k}$ be the number of paths ending at point (n, k) , where each step is in \mathcal{S} . The associated generating function is then defined as

$$F(z, u) = \sum_{n, k \geq 0} f_{n,k} z^n u^k.$$

Example 1. (Motzkin Paths) We start from $(0, 0)$ with step set $\mathcal{S} = \{(1, 1), (1, 0), (1, -1)\}$. The functional equation of its associated generating function is as follows:

$$\begin{aligned} F(z, u) &= 1 + z \left(u + 1 + \frac{1}{u} \right) F(z, u) - \frac{z}{u} F(z, 0) \\ &= 1 + z(u + 1)F(z, u) + z \frac{F(z, u) - F(z, 0)}{u}, \end{aligned}$$

which in the notation of (1.4) corresponds to

$$Q_0(z, u) = 1, \quad Q_1(z, u) = u + 1, \quad \text{and} \quad Q_2(z, u) = 1.$$

We let $u(z)$ be the power series solution of the equation

$$u(z) = zQ_2(z, u(z)) + zu(z)Q_1(z, u(z)) = z + zu(z)(1 + u(z)),$$

that is,

$$u(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}.$$

Then $F(z, 0)$ is given by

$$F(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))} = \frac{1}{1 - z(1 + u(z))} = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2},$$

and

$$M_n^* = f_{n,0} = [z^n]F(z, 0) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!(k+1)!} \sim \frac{3\sqrt{3}}{2\sqrt{\pi}} n^{-3/2} 3^n.$$

These numbers are also called ‘‘Motzkin numbers’’.

Example 2. We start from $(0, k_0)$ with step set $\mathcal{S} = \{(2, 0), (1, -1)\}$. Here the functional equation is given by

$$\begin{aligned} F(z, u) &= u^{k_0} + (z^2 + \frac{z}{u})F(z, u) - \frac{z}{u}F(z, 0) \\ &= u^{k_0} + z^2F(z, u) + z \frac{F(z, u) - F(z, 0)}{u}, \end{aligned}$$

which corresponds to

$$Q_0(z, u) = u^{k_0}, \quad Q_1(z, u) = z \quad \text{and} \quad Q_2(z, u) = 1.$$

This is actually a *degenerate* case since Q_1 and Q_2 depend only on z . Here $u(z)$ is a rational function

$$u(z) = \frac{zQ_2(z, \cdot)}{1 - zQ_1(z, \cdot)} = \frac{z}{1 - z^2},$$

as well as

$$F(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))} = \frac{u(z)^{k_0}}{1 - z^2} = \frac{z^{k_0}}{(1 - z^2)^{k_0+1}}.$$

Example 3. We start again from $(0, 0)$ but now with step set $\mathcal{S} = \{(2, 0), (1, 1), (1, 0)\}$, and we also assume that the step $(1, 0)$ is forbidden on the x -axis $y = 0$. The functional equation in this case is

$$\begin{aligned} F(z, u) &= 1 + z(z + u + 1)F(z, u) - zF(z, 0) \\ &= 1 + z(z + u)F(z, u) + zu \frac{F(z, u) - F(z, 0)}{u}, \end{aligned}$$

that is, we have

$$Q_0(z, u) = 1, \quad Q_1(z, u) = z + u \quad \text{and} \quad Q_2(z, u) = u.$$

Here Q_2 has u as a factor so that we are again in a *degenerate* case. We have $u(z) = 0$ and consequently

$$F(z, 0) = \frac{1}{1 - z^2} = 1 + z^2 + z^4 + z^6 + \dots.$$

2.2. The non-linear case. We collect here some examples from the enumeration of planar maps. The starting point is the classical example of all planar maps.

Example 4. Let $M(z, u)$ be the generating function of planar maps with n edges and in which the degree of the root-face is equal k . We have already mentioned that $M(z, u)$ satisfies the non-linear catalytic equation (1.1). In order to apply Proposition 2 and Theorem 2 we use the substitution $u \rightarrow u + 1$ and obtain

$$\begin{aligned} M(z, u + 1) &= 1 + z(u + 1) \left((u + 1)M(z, u + 1)^2 + M(z, u + 1) \right. \\ &\quad \left. + \frac{M(z, u + 1) - M(z, 1 + 0)}{u} \right), \end{aligned}$$

that is, we have $F_0(u) = 1$, and $Q(\alpha_0, \alpha_1, z, u) = (u + 1)^2 \alpha_0^2 + (u + 1) \alpha_0 + (u + 1) \alpha_1$. Here $Q_{\alpha_1} = u + 1 \neq 0$, $Q_{\alpha_0, u} \neq 0$, and $Q_{\alpha_0, \alpha_0} \neq 0$, so that Theorem 2 applies. Of course this is in accordance with

$$M(z, 1) = \sum_{n \geq 0} M_n z^n = \frac{18z - 1 + (1 - 12z)^{3/2}}{54z^2},$$

and

$$M_n = [z^n]M(z, 1) \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n.$$

Example 5. Let $E(z, u)$ be the generating function of bipartite planar maps which satisfies the catalytic equation

$$E(z, u) = 1 + zu^2 E(z, u)^2 + u^2 z \frac{E(z, u) - E(z, 1)}{u^2 - 1}.$$

Here we use the substitution $u = \sqrt{1+v}$ and obtain

$$E(z, \sqrt{1+v}) = 1 + z(v+1)E(z, \sqrt{1+v})^2 + (v+1)z \frac{E(z, \sqrt{1+v}) - E(z, 1)}{v},$$

which is of a type where Theorem 2 applies:

$$F(v) = 1, \quad Q(\alpha_0, \alpha_1, z, v) = \alpha_0^2(v+1) + \alpha_1(v+1).$$

Example 6. Let $B(z, u)$ be the generating function of 2-connected planar maps. It satisfies

$$B(z, u) = z^2u + zuB(z, u) + u(z + B(z, u)) \frac{B(z, u) - B(z, 1)}{u-1}.$$

After substituting u by $u+1$ we obtain

$$\begin{aligned} B(z, u+1) &= z^2(u+1) + z(u+1)B(z, u+1) \\ &\quad + (u+1)(z + B(z, u+1)) \frac{B(z, u+1) - B(z, 1)}{u}, \end{aligned}$$

which is not exactly of the form (1.3). Nevertheless the same methods as in the proof of Theorem 2 apply – we just have to observe that the analogue of the system of equations (4.2) has proper positive power series solutions – and we obtain the same result.

Example 7. Let $T(z, u)$ be the generating function for planar triangulations, which satisfies

$$T(z, u) = (1 - uT(z, u)) + (z + u)T(z, u)^2 + z(1 - uT(z, u)) \frac{T(z, u) - T(z, 0)}{u}.$$

In order to get rid of the negative sign we use the substitution $\tilde{T}(z, u) = T(z, u)/(1 - uT(z, u))$ and we obtain

$$\tilde{T}(z, u) = 1 + u\tilde{T}(z, u) + z(1 + \tilde{T}(z, u)) \frac{\tilde{T}(z, u) - \tilde{T}(z, 0)}{u}.$$

Again this is not precisely of the form (1.3) but our methods apply once more. Note that $\tilde{T}(z, 0) = T(z, 0)$.

3. PROOFS OF PROPOSITION 1 AND THEOREM 1

3.1. Proof of Proposition 1. As already mentioned in the Introduction, we rewrite (1.4) as

$$M(z, u) (u - zuQ_1(z, u) - zQ_2(z, u)) = uQ_0(z, u) - zM(z, 0)Q_2(z, u).$$

It is clear that if $u = u(z)$ satisfies

$$u = zQ_2(z, u) + zuQ_1(z, u), \tag{3.1}$$

then the *kernel* $K(z, u) = u - zuQ_1(z, u) - zQ_2(z, u)$ is identically zero, which implies that $M(z, 0)$ is given by $M(z, 0) = u(z)Q_0(z, u(z))/(zQ_2(z, u(z)))$. Since $zQ_2(z, u(z)) = u(z)(1 - zQ_1(z, u(z)))$, we also have

$$M(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))},$$

as claimed.

Finally, we mention that Equation (3.1) has always a unique power series solution $u = u(z)$ with $u(0) = 0$. On a formal level this is immediately clear by comparing

coefficients and rewriting (3.1) as a recurrence for the coefficients of $u(z)$. However, (3.1) can be also seen as a fixed point equation, which is a contraction if z and u are sufficiently small. This means that the recurrence $u_0(z) = 0$, $u_{k+1}(z) = zQ_2(z, u_k(z)) + zu_k(z)Q_1(z, u_k(z))$, $k \geq 0$, has an analytic limit $u(z)$, provided that z is sufficiently small in modulus.

3.2. Proof of Theorem 1. Since $u(z)$ has non-negative coefficients the dominant singularity is positive and equals the radius of convergence of $u(z)$.

We do not comment on the *degenerate cases* that are discussed after Theorem 1, since there are only rational functions. In the non-degenerate case Equation (3.1) is a non-linear positive polynomial equation for $u(z)$. Here it follows by general considerations that $u(z)$ has a square-root singularity at the radius of convergence z_0 :

$$u(z) = u_0 + u_1(1 - z/z_0)^{1/2} + u_2(1 - z/z_0) + u_3(1 - z/z_0)^{3/2} + \dots,$$

where $z_0 > 0$ and $u_0 > 0$ are (uniquely) given by the system of equations

$$\begin{aligned} u_0 &= z_0 Q_2(z_0, u_0) + z_0 u_0 Q_1(z_0, u_0), \\ 1 &= z_0 Q_{2,u}(z_0, u_0) + z_0 Q_1(z_0, u_0) + z_0 u_0 Q_{1,u}(z_0, u_0). \end{aligned}$$

See [1] and [5] for details. In particular, we have $u_1 < 0$.

More precisely, $u(z)$ can be represented as $u(z) = z^a U(z^b)$, where $a \geq 0$, $b \geq 1$, and $U(z)$ has also a square-root singularity at $z = z_0^{1/b}$, that is the only singularity on the circle $|z| \leq z_0^{1/b}$. In particular it follows that the coefficients of $U_k = [z^k]U(z)$ are asymptotically given by $U_k \sim ck^{-3/2}z_0^{-k/b}$ for some $c > 0$. With the help of $U(z)$ we can now completely describe $M(z, 0)$ and directly deduce (1.7).

4. PROOF OF PROPOSITION 2 AND THEOREM 2

4.1. Proof of Proposition 2. As mentioned in the introduction, general catalytic equations can be solved with the help of the method of Bousquet-Mélou and Jehanne [4]. For this purpose we set

$$P(x_0, x_1, z, u) = F_0(u) + zQ(x_0, (x_0 - x_1)/u, z, u) - x_0. \quad (4.1)$$

The next step is to find functions $x_0 = f(z)$, $x_1 = y(z)$, and $u = u(z)$ such that $P = 0$, $P_{x_0} = 0$, and $P_u = 0$. In our situation this means that

$$\begin{aligned} f(z) &= F_0(u(z)) + zQ(f(z), (f(z) - y(z))/u(z), z, u(z)), \\ 1 &= zQ_{\alpha_0}(f(z), (f(z) - y(z))/u(z), z, u(z)) \\ &\quad + \frac{z}{u(z)}Q_{\alpha_1}(f(z), (f(z) - y(z))/u(z), z, u(z)), \\ 0 &= F'_0(u(z)) + zQ_u(f(z), (f(z) - y(z))/u(z), z, u(z)) \\ &\quad - z\frac{f(z) - y(z)}{u(z)^2}Q_{\alpha_1}(f(z), (f(z) - y(z))/u(z), z, u(z)). \end{aligned}$$

In order to simplify this system we set $w = w(z) = (f(z) - y(z))/u(z)$, multiply the second equation by $u(z)$ and replace $zQ_{\alpha_1}/u(z)$ by $1 - zQ_{\alpha_0}$ in the third equation. This leads to the system

$$\begin{aligned} f(z) &= F_0(u(z)) + zQ(f(z), w(z), z, u(z)), \\ u(z) &= zu(z)Q_{\alpha_0}(f(z), w(z), z, u(z)) + zQ_{\alpha_1}(f(z), w(z), z, u(z)), \\ w(z) &= F'_0(u(z)) + zQ_u(f(z), w(z), z, u(z)) + zw(z)Q_{\alpha_0}(f(z), w(z), z, u(z)), \end{aligned} \quad (4.2)$$

which is precisely (1.8). This is a (so-called) positive polynomial system of equations for the unknown functions $f(z)$, $w(z)$, and $u(z)$; recall that the coefficients of F_0 and Q are non-negative. It is easy to show that the system (4.2) has unique power series solutions with $f(0) = F_0(0)$, $w(0) = F_0'(0)$, $u(0) = 0$ and non-negative coefficients. Thus, $y(z) = f(z) - u(z)w(z)$ is a power series, too, and consequently $M(z, 0) = y(z) = f(z) - u(z)w(z)$.

4.2. Proof of Theorem 2. Positive polynomial systems of equations are discussed in detail in [1]. In particular if the system is strongly connected then we know that there is a common dominant singularity z_0 and $f(z)$, $w(z)$, and $u(z)$ have a square root singularity at z_0 of the form (1.6):

$$\begin{aligned} f(z) &= f_0 + f_1Z + f_2Z^2 + f_3Z^3 + \dots, \\ u(z) &= u_0 + u_1Z + u_2Z^2 + u_3Z^3 + \dots, \\ w(z) &= w_0 + w_1Z + w_2Z^2 + w_3Z^3 + \dots, \end{aligned} \quad (4.3)$$

with $Z = \sqrt{1 - z/z_0}$ and where $f_1 < 0$, $u_1 < 0$ and $w_1 < 0$. Thus, it follows that $M(z, 0) = y(z) = f(z) - w(z)u(z)$ has also the same kind of singularity:

$$y(z) = y_0 + y_1Z + y_2Z^2 + y_3Z^3 + \dots. \quad (4.4)$$

Hence, in order to complete the proof of Theorem 2 we have to show the following properties:

- (1) If $Q_{\alpha_0 u} \neq 0$ then the system (4.2) is strongly connected.
- (2) We have $y_1 = 0$ in the expansion (4.4).
- (3) If $F_0'(0) = 0$ and $Q_{\alpha_1}(F_0(0), 0, 0, 0) \neq 0$ then $y_3 > 0$ in the expansion (4.4).

With these properties the singular structure of $M(z, 0)$ at z_0 is precisely of the form (1.9). Furthermore, the asymptotics of M_n follows in the following way. Since the system (4.2) is non-linear and strongly connected we know from [1] that there exists $a_1, a_2, a_3 \geq 0$ and $b \geq 1$ such that $f(z) = z^{a_1}F(z^b)$, $w(z) = z^{a_2}W(z^b)$, $u(z) = z^{a_3}U(z^b)$, where F , W , and U have square-root singularities at $z = z_0^{1/b}$ but no other singularities on the circle $|z| \leq z_0^{1/b}$. It also follows that $M(z, 0) = z^{a_1}F(z^b) - z^{a_2+a_3}W(z^b)U(z^b)$. If $a_1 \not\equiv a_2 + a_3 \pmod{b}$ then $M(z, 0)$ would have negative coefficients M_n for $n \equiv a_2 + a_3 \pmod{b}$ which is impossible. Thus, $a_1 \equiv a_2 + a_3 \pmod{b}$ and we have positive coefficients for $n \equiv a_1 \pmod{b}$ (if n is sufficiently large) and zero coefficients else. From $M(z, 0) = z^{a_1}\widetilde{M}(z^b)$, where $\widetilde{M}(z)$ has $z = z_0^{1/b}$ as a singularity of type (1.9) and no other singularities on the circle $|z| \leq z_0^{1/b}$, we obtain the asymptotics (1.10).

Finally we comment on the computation of z_0 . Let $\mathbf{J} = \mathbf{J}(f, w, u, z)$ denote the Jacobian matrix (with derivatives with respect to f, w, u) of the right hand side of (4.2). Then we consider the extended system of equations

$$\begin{aligned} f_0 &= F_0(u_0) + z_0Q(f_0, w_0, z_0, u_0), \\ u_0 &= z_0u_0Q_{\alpha_0}(f_0, w_0, z_0, u_0) + z_0Q_{\alpha_1}(f_0, w_0, z_0, u_0), \\ w_0 &= F_0'(u_0) + z_0Q_u(f_0, w_0, z_0, u_0) + z_0w_0Q_{\alpha_0}(f_0, w_0, z_0, u_0), \\ 0 &= \det(\mathbf{I} - \mathbf{J}(f_0, w_0, u_0, z_0)), \end{aligned} \quad (4.5)$$

and search for the unique positive solution (f_0, w_0, u_0, z_0) such that the spectral radius of $\mathbf{J}(f_0, w_0, u_0, z_0)$ equals 1. This gives the correct value z_0 .

4.2.1. **Strong connectedness.** Let $y_j = F_j(z, y_1, \dots, y_d)$ a d -dimensional system of equations. The dependency di-graph of such a system consists of vertices $\{y_1, \dots, y_d\}$ and there is an oriented edge from y_i to y_j if F_j depends on y_i , that is, $F_{j,y_i} \neq 0$. We say that the system is strongly connected if the dependency di-graph is strongly connected (see [1, 5]). In our present situation our vertex set is $\{f, u, w\}$. By assumption we have $Q_{\alpha_1} \neq 0$. Thus, there is always an edge from w to f .

Suppose first the $Q_{\alpha_0\alpha_0} \neq 0$. Then by the second equation there is an edge from f to u . By assumption we always have $Q_{\alpha_0u} \neq 0$ which implies that there is an edge from u to w . This implies a circle $w \rightarrow f \rightarrow u \rightarrow w$ and consequently strongly connectedness.

Second suppose that $Q_{\alpha_0\alpha_1} \neq 0$ or $Q_{\alpha_1\alpha_1} \neq 0$. In this case there is certainly an edge from w to u . Furthermore, since $Q_{\alpha_0u} \neq 0$ there is an edge from u to w and another one from f to w . This again leads to a strongly connected di-graph and completes the proof of the first assertion.

4.2.2. **The condition $y_1 = 0$.** In order to prove that y_1 vanishes we recall first the approach by Bousquet-Mélou and Jehanne [4]. Starting with the function $P(x_0, x_1, z, u)$ that is given by (4.1) we have to solve the system

$$\begin{aligned} P(f(z), y(z), z, u(z)) &= 0, \\ P_{x_0}(f(z), y(z), z, u(z)) &= 0, \\ P_u(f(z), y(z), z, u(z)) &= 0. \end{aligned} \tag{4.6}$$

Instead of searching for the functions $f(z)$, $u(z)$ and $w(z)$ we equivalently search for $f(z)$, $y(z)$ and $u(z)$. It is also immediately clear that the singular condition for the system (1.8) implies that the system (4.6) gets singular too. Consequently the functional determinant has to be zero, evaluated at $(f(z_0), y(z_0), z_0, u(z_0))$. Since $P_{x_0} = P_u = 0$, we get

$$\det \begin{pmatrix} P_{x_0} & P_{x_1} & P_u \\ P_{x_0x_0} & P_{x_0x_1} & P_{x_0u} \\ P_{x_0u} & P_{x_1u} & P_{uu} \end{pmatrix} = -P_{x_1} (P_{x_0x_0}P_{uu} - P_{x_0u}^2) = 0.$$

Otherwise the implicit function theorem would imply that there is an analytic continuation. Since $P_{x_1} = -zQ_{\alpha_1}/u \neq 0$ (by assumption $Q_{\alpha_1} \neq 0$) we obtain the relation $P_{x_0x_0}P_{uu} = P_{x_0u}^2$.

We now discuss the analytic function P at the point $(f_0, y_0, z_0, u_0) = (f(z_0), y(z_0), z_0, u(z_0))$ in more detail. We already know that $P_{x_0} = 0$. However, by differentiating (4.1) and using $F'_0(0) = 0$ it follows that

$$P_{x_0x_0} = z_0Q_{\alpha_0\alpha_0} + 2\frac{z_0}{u_0}Q_{\alpha_0\alpha_1} + \frac{z_0}{u_0^2}Q_{\alpha_1\alpha_1} > 0.$$

Hence by the Weierstrass preparation theorem² [12] it follows that P can be locally written as

$$P(x_0, x_1, z, u) = K(x_0, x_1, z, u) \left((x_0 - G(x_1, z, u))^2 - H(x_1, z, u) \right),$$

²The Weierstrass preparation theorem says that every non-zero function $F(z_1, \dots, z_d)$ with $F(0, \dots, 0) = 0$ that is analytic at $(0, \dots, 0)$ has a unique factorisation $F(z_1, \dots, z_d) = K(z_1, \dots, z_d)W(z_1; z_2, \dots, z_d)$ into analytic factors, where $K(0, \dots, 0) \neq 0$ and $W(z_1; z_2, \dots, z_d) = z_1^d + z_1^{d-1}g_1(z_2, \dots, z_d) + \dots + g_d(z_2, \dots, z_d)$ is a so-called Weierstrass polynomial, that is, all g_j are analytic and satisfy $g_j(0, \dots, 0) = 0$.

where K , G and H are analytic function with the properties that $K(f_0, y_0, z_0, u_0) \neq 0$, $G(y_0, z_0, u_0) = f_0$, and $H(y_0, z_0, u_0) = 0$.

Since $P = 0$ if and only if $(f - G)^2 = H$ and

$$\begin{aligned} P_{x_0} &= K_{x_0} ((f - G)^2 - H) + 2K(f - G), \\ P_u &= K_u ((f - G)^2 - H) + K(-2(f - G)G_u - H_u) \end{aligned}$$

it follows from (4.6) that

$$H(y(z), z, u(z)) = 0 \quad \text{and} \quad H_u(y(z), z, u(z)) = 0$$

for z close to z_0 . We note that this is precisely a system of equations that appears in the context of the quadratic method (see [4, 8]).

Next we will show how the singular condition $P_{x_0x_0}P_{uu} = P_{x_0u}^2$ translates into $H_{uu}(y_0, z_0, u_0) = 0$. Since

$$\begin{aligned} P_{x_0x_0} &= K_{x_0x_0} ((f - G)^2 - H) + 4K_{x_0}(f - G) + 2K, \\ P_{uu} &= K_{uu} ((f - G)^2 - H) + 2K_u(-2(f - G)G_u - H_u) \\ &\quad + K(2G_u^2 - 2(f - G)G_{uu} - H_{uu}), \\ P_{x_0u} &= K_{x_0u} ((f - G)^2 - H) + 2K_u(f - G) + K_{x_0}(-2(f - G)G_u - H_u) \\ &\quad + K(-2G_u - 2(f - G)G_{x_0u}) \end{aligned}$$

it follows that we have

$$\begin{aligned} P_{x_0x_0} &= 2K, \\ P_{uu} &= (2G_u^2 - H_{uu})K, \\ P_{x_0u} &= -2G_uK \end{aligned}$$

for $(y, z, u) = (y_0, z_0, u_0)$. Consequently the condition $P_{x_0x_0}P_{uu} = P_{x_0u}^2$ implies $H_{uu}(y_0, z_0, u_0) = 0$.

In a similar (but much easier way) it also follows that that $P_{x_1} = -KH_{x_1}$. This also implies that $H_{x_1} \neq 0$ since $P_{x_1} \neq 0$ (by assumption $Q_{\alpha_1} \neq 0$).

Nest we recall that $u(z)$ and $y(z) = f(z) - u(z)w(z)$ have singular (and convergent) expansions of the form

$$\begin{aligned} u(z) &= u_0 + u_1Z + u_2Z^2 + u_3Z^3 + \dots, \\ y(z) &= y_0 + y_1Z + y_2Z^2 + y_3Z^3 + \dots, \end{aligned}$$

where $Z = \sqrt{1 - z/z_0}$ and $u_1 < 0$. By using the Taylor expansion of H at (y_0, z_0, u_0) and the property $H(y(z), z, u(z)) = 0$ it follows that

$$\begin{aligned} 0 &= H_{x_1}(y_1Z + y_2Z^2 + y_3Z^3 + \dots) - z_0H_zZ^2 + \frac{1}{2}H_{x_1x_1}(y_1^2Z^2 + 2y_1y_2Z^3 + \dots) \\ &\quad + H_{x_1u}(y_1u_1Z^2 + (y_1u_2 + y_2u_1)Z^3 + \dots) - z_0H_{zu}(u_1Z^3 + \dots) \\ &\quad - z_0H_{x_1z}(y_1Z^3 + \dots) + \frac{1}{6}H_{uuu}(u_1^3Z^3 + \dots) + \frac{1}{2}H_{x_1uu}(u_1^2y_1Z^3 + \dots) \\ &\quad + \frac{1}{2}H_{x_1x_1u}(u_1y_1^2Z^3 + \dots) + \frac{1}{6}H_{x_1x_1x_1}(y_1^3Z^3 + \dots) + O(Z^4). \end{aligned}$$

By comparing coefficients of Z this implies

$$\begin{aligned} 0 &= H_{x_1} y_1, \\ 0 &= H_{x_1} y_2 - z_0 H_z + \frac{1}{2} H_{x_1 x_1} y_1^2 + H_{x_1 u} y_1 u_1, \\ 0 &= H_{x_1} y_3 + H_{x_1 x_1} y_1 y_2 + H_{x_1 u} (y_1 u_2 + y_2 u_1) - z_0 H_{z u} u_1 - z_0 H_{y z} y_1 + \frac{1}{6} H_{u u u} u_1^3 \\ &\quad + \frac{1}{2} H_{x_1 u u} u_1^2 y_1 + \frac{1}{2} H_{x_1 x_1 u} u_1 y_1^2 + \frac{1}{6} H_{x_1 x_1 x_1} y_1^3. \end{aligned}$$

In particular, since $H_{x_1} \neq 0$ it follows that $y_1 = 0$, which completes the proof of the second property.

4.2.3. **The condition** $y_3 > 0$. By taking also into account the second and third relations above and using $y_1 = 0$ we get

$$\begin{aligned} y_2 &= \frac{z_0 H_z}{H_{x_1}}, \\ y_3 &= \frac{u_1}{H_{x_1}} \left(z_0 H_{z u} - H_{x_1 u} y_2 - \frac{1}{6} H_{u u u} u_1^2 \right). \end{aligned}$$

By doing the same procedure as above for H_u (where we can only use that $H_{u u} = 0$) we also get the relation

$$H_{x_1 u} y_2 - z_0 H_{z u} + \frac{1}{2} H_{u u u} u_1^2 = 0. \quad (4.7)$$

Hence y_3 can be also represented as

$$y_3 = \frac{2u_1 z_0}{3H_{x_1}^2} (H_{x_1} H_{z u} - H_z H_{x_1 u}).$$

We already know that $u_1 < 0$ and $H_{x_1} \neq 0$. Thus it remains to show that

$$H_{x_1} H_{z u} - H_z H_{x_1 u} \neq 0, \quad (4.8)$$

where we evaluate at (y_0, z_0, u_0) . This will show that $y_3 \neq 0$.

By slightly more involved computations as above it follows that (4.8) holds if and only if the functional determinant

$$\begin{aligned} \Delta = \begin{vmatrix} P_{x_0} & P_{x_1} & P_z \\ P_{x_0 x_0} & P_{x_0 x_1} & P_{x_0 z} \\ P_{x_0 u} & P_{x_1 u} & P_{u z} \end{vmatrix} &= -P_{x_1} (P_{x_0 x_0} P_{u z} - P_{x_0 z} P_{x_0 u}) \\ &\quad + P_z (P_{x_0 x_0} P_{u x_1} - P_{x_0 x_1} P_{u x_0}) \neq 0, \end{aligned} \quad (4.9)$$

where we evaluate at (f_0, y_0, z_0, u_0) , so that $P_{x_0} = 0$.

Let us assume for a moment that (4.9) is satisfied. Then by the implicit function theorem it follows that the system

$$P = 0, \quad P_{x_0} = 0, \quad P_u = 0 \quad (4.10)$$

has a unique solution $x_0 = \tilde{f}(u)$, $x_1 = \tilde{y}(u)$, $z = \tilde{z}(u)$ with $\tilde{f}(u_0) = y_0$, $\tilde{y}(u_0) = y_0$, $\tilde{z}(u_0) = z_0$. Actually the converse is *almost true* and this is the strategy of our proof.

First we observe that assuming (4.9) the system (4.10) has a unique solution $x_0 = \tilde{f}(u)$, $x_1 = \tilde{y}(u)$, $z = \tilde{z}(u)$ with $\tilde{f}(u_0) = y_0$, $\tilde{y}(u_0) = y_0$, $\tilde{z}(u_0) = z_0$. For this purpose we go back to the system (4.2), which is equivalent (after the substitution $w = (f - y)/u$). The essential difference is that the *free variable* of (4.2) is z and in

(4.10) it is u . From the theory of positive strongly connected polynomial systems it follows that at the critical point $z = z_0$ there are precisely two local solutions, namely

$$\begin{aligned} u_+(z) &= u_0 + u_1 Z + u_2 Z^2 + u_3 Z^3 + \dots \quad \text{and} \\ u_-(z) &= u_0 - u_1 Z + u_2 Z^2 - u_3 Z^3 + \dots \end{aligned} \quad (4.11)$$

with $Z = \sqrt{1 - z/z_0}$, and similarly for $f(z)$ and $w(z)$. (The method reduces the system first to a single non-linear positive equation for which this can be easily observed; see [1, 5].) It is immediate that the two solutions of (4.11) correspond to a single function $z = \tilde{z}(u)$ with $\tilde{z}'(u_0) = 0$ with $z = \tilde{z}(u_+(z)) = \tilde{z}(u_-(z))$, and it is clear that $\tilde{f}(u)$ and $\tilde{y}(u)$ are unique. This is a strong indication that (4.9) should be satisfied. However, there might be exceptional situations as we argue next.

We start by solving the simpler system

$$P = 0, \quad P_{x_0} = 0$$

with solutions $x_0 = \bar{f}(z, u)$, $x_1 = \bar{y}(z, u)$ with $\bar{f}(z_0, u_0) = f_0$, $\bar{y}(z_0, u_0) = y_0$. This is certainly possible since the functional determinant $P_{x_0} P_{x_0 x_1} - P_{x_1} P_{x_0 x_0} = -P_{x_1} P_{x_0 x_0} \neq 0$ as long as x_0, x_1, z, u are positive.

Now we substitute these solution functions into the equation $P_u = 0$ and obtain

$$P_u(\bar{f}(z, u), \bar{y}(z, u), z, u) = \bar{P}(z, u) = 0,$$

that has the unique solution $z = \tilde{z}(u)$ with $\tilde{z}(u_0) = z_0$. Note that

$$\begin{aligned} \bar{P}_z &= P_{ux_0} \bar{f}_z + P_{ux_1} \bar{y}_z + P_{uz} \\ &= P_{ux_0} \frac{P_{x_0 x_1} P_z - P_{x_0 z} P_{x_1}}{P_{x_1} P_{x_0 x_0}} - P_{ux_1} \frac{P_z}{P_{x_1}} + P_{uz} \\ &= -\frac{\Delta}{P_{x_1} P_{x_0 x_0}} \end{aligned}$$

so that $\bar{P}_z \neq 0$ if and only if (4.9) holds.

Thus, it suffices to consider an equation of the form $\bar{P}(z, u) = 0$ for which we know that there is a unique solution $z = \tilde{z}(u)$ with $\tilde{z}(u_0) = z_0$, for which we now assume that $\bar{P}_z(z_0, u_0) = 0$. Since \bar{P} is non-zero there exists $r \geq 1$ such that $\bar{P}_z(z_0, u_0) = \bar{P}_{z^2}(z_0, u_0) = \dots = \bar{P}_{z^r}(z_0, u_0) = 0$, but $\bar{P}_{z^{r+1}}(z_0, u_0) \neq 0$. By the Weierstrass preparation theorem there exists an analytic function $\bar{K}(z, u)$ with $\bar{K}(z_0, u_0) \neq 0$ and analytic functions $c_0(u), \dots, c_r(u)$ with $c_j(u_0) = 0$, $0 \leq j \leq r$, such that

$$\bar{P}(z, u) = \bar{K}(z, u) ((z - z_0)^{r+1} + c_r(u)(z - z_0)^r + \dots + c_0(u)). \quad (4.12)$$

Since we know that there exists a unique solution $z = \tilde{z}(u)$ with $\tilde{z}(u_0) = z_0$ of the equation $\bar{P}(z, u) = 0$, it follows that the polynomial is an $(r + 1)$ -st power:

$$(z - z_0)^{r+1} + c_r(u)(z - z_0)^r + \dots + c_0(u) = (z - \tilde{z}(u))^{r+1}.$$

Next we note that the function $u(z)$ is strictly increasing for $0 \leq z \leq z_0$. Thus the inverse function $\tilde{z}(u)$ exists for $0 \leq u \leq u_0$ and can be analytically continued to a region G that covers the real interval $[0, u_0]$. (Note that by assumption $u'(0) = Q_{\alpha_1}(F_0(0), 0, 0, 0) \neq 0$ so that the inverse function $\tilde{z}(u)$ is analytic at $u = 0$, too.) Since

$$\bar{P}(z, u) = \bar{K}(z, u)(z - \tilde{z}(u))^{r+1}.$$

holds in a neighborhood of (z_0, u_0) it follows that $\bar{P}_z(\tilde{z}(u), u) = 0$ holds in a neighborhood of u_0 and, thus, for all $u \in G$.

Summing up, if (4.9) does not hold at (y_0, z_0, u_0) , that is, $\Delta = 0$ at (f_0, y_0, z_0, u_0) then $\Delta = 0$ evaluated at $(\tilde{f}(u), \tilde{y}(u), \tilde{z}(u), u)$ for all $u \in [0, u_0]$.

We can now finalize the proof by showing that $\Delta \neq 0$ for u sufficiently close to 0 provided that $F'_0(0) = 0$. Note first that the condition $F'_0(0) = 0$ implies $w(0) = 0$, where $w = (\tilde{f}(u) - \tilde{y}(u))/u$. By using (4.1) it follows that

$$\begin{aligned} u^4 \Delta &= \tilde{z}(u) Q_{\alpha_1} (Q_{\alpha_1} + w Q_{\alpha_1 \alpha_1} - 2w Q_{\alpha_0 \alpha_1}) + O(u^2) \\ &= \tilde{z}'(0) u Q_{\alpha_1} (F_0(0), 0, 0, 0)^2 + O(u^2) \\ &= Q_{\alpha_1} (F_0(0), 0, 0, 0) u + O(u^2). \end{aligned}$$

By assumption $Q_{\alpha_1} (F_0(0), 0, 0, 0) \neq 0$. Hence Δ is not identically 0, so that $\Delta \neq 0$ for u sufficiently close to 0. As argued above this also implies that (4.9) holds, and consequently $y_3 \neq 0$. Since $y(z) = M(z, 0)$ has non-negative coefficients this implies $y_3 > 0$.

5. CENTRAL LIMIT THEOREMS FOR ADDITIONAL PARAMETERS

Let $M(z, x, u)$ denote the generating function of rooted planar maps, where the variable z corresponds to the number of edges, x to the number of vertices and u to the root face valency. Then by the usual combinatorial decomposition of maps we have

$$M(z, x, u) = x + zu^2 M(z, x, u)^2 + zu \frac{M(z, x, 1) - uM(z, x, u)}{1 - u}.$$

Thus, for every positive x this is a catalytic equation of the form (1.3) so that Proposition 2 and Theorem 2 apply. In particular we obtain an expansion and asymptotics of the form

$$M(z, x, 1) = a_0(x) + a_2(x) \left(1 - \frac{z}{\rho(x)}\right) + a_3(x) \left(1 - \frac{z}{\rho(x)}\right)^{3/2} + \dots, \quad (5.1)$$

where $z = \rho(x)$ satisfies the equation

$$\begin{aligned} 768 x^4 z^4 - 1536 x^3 z^4 - 512 x^3 z^3 + 2304 x^2 z^4 + 768 x^2 z^3 - 1536 x z^4 \\ + 96 x^2 z^2 + 768 x z^3 + 768 z^4 - 96 x z^2 - 512 z^3 + 96 z^2 - 1 = 0 \end{aligned}$$

with $\rho(1) = \frac{1}{12}$ and where $a_0(1) = \frac{4}{3}$, $a_2(1) = -\frac{4}{3}$, $a_3(1) = \frac{8}{3}$, and consequently

$$[z^n] M(z, x, 1) = c(x) n^{-5/2} \rho(x)^{-n} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Actually all the functions $\rho(x)$, $c(x)$, and $a_j(x)$ are not only defined for positive x but extend to analytic functions around the positive real axis, and by inspection of the proof even the asymptotics can be extended to non-real x that are close to the positive real axis.

Let X_n denote the random variable equal to the number of vertices in a random planar rooted map with n edges, where each map of size n is considered to be equally likely. Then the probability generating function $\mathbb{E}[x^{X_n}]$ can be written as

$$\mathbb{E}[x^{X_n}] = \frac{[z^n] M(z, x, 1)}{[z^n] M(z, 1, 1)} = \frac{c(x)}{c} \left(\frac{\rho(1)}{\rho(x)}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

At this stage we can apply standard tools (see [5, Chapter 2]) to obtain a central limit theorem for X_n of the form $(X_n - \mu n)/\sqrt{\sigma^2 n} \rightarrow \mathcal{N}(0, 1)$, where $\mu = -\rho'(1)/\rho(1)$ and $\sigma^2 = \mu + \mu^2 - \rho''(1)/\rho(1)$. Since $\rho'(1) = -\frac{1}{24}$ and $\rho''(1) = \frac{19}{384}$ we immediately obtain $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{5}{32}$. We also have $\mathbb{E}[X_n] = \mu n + O(1)$ and $\text{Var}[X_n] = \sigma^2 n + O(1)$. In this special case Euler's relation and duality can be used to obtain (the even more precise representation) $\mathbb{E}[X_n] = n/2 + 1$.³

Actually we can easily generalize Proposition 2 and Theorem 2 in order to obtain the following central limit theorem.

Theorem 3. *Suppose that Q is a polynomial in $\alpha_0, \alpha_1, z, x, u$ with non-negative coefficients that depends (at least) on α_1 , that is, $Q_{\alpha_1} \neq 0$, and $F_0(x, u)$ is another polynomial with non-negative coefficients. Let $M(z, x, u)$ be the power series solution of the equation*

$$M(z, x, u) = F_0(x, u) + zQ \left(M(z, x, u), \frac{M(z, x, u) - M(z, x, 0)}{u}, z, x, u \right).$$

Furthermore assume that Q is not linear in α_0 and α_1 , that is, $Q_{\alpha_0 \alpha_0} \neq 0$, or $Q_{\alpha_0 \alpha_1} \neq 0$ or $Q_{\alpha_1 \alpha_1} \neq 0$. Additionally assume that $Q_{\alpha_0 u} \neq 0$, $\frac{\partial F_0}{\partial u}(x, 0) = 0$, $Q_{\alpha_1}(F_0(1, 0), 0, 0, 1, 0) \neq 0$ and that (according to Theorem 2) $[z^n]M(z, 1, 0) > 0$ for $n \equiv a \pmod{b}$, whereas $[z^n]M(z, 1, 0) = 0$ for $n \not\equiv a \pmod{b}$.

Let X_n with $n \equiv a \pmod{b}$ be a sequence of random variables defined by

$$\mathbb{E}[x^{X_n}] = \frac{[z^n]M(z, x, 0)}{[z^n]M(z, 1, 0)}.$$

For positive x , let $\rho(x) > 0$ denote the radius of convergence of $z \mapsto M(z, x, 0)$. Then $\rho(x)$ can be extended to an analytic function around the positive real axis and we have with

$$\mu = -\frac{\rho'(1)}{\rho(1)}, \quad \sigma^2 = \mu + \mu^2 - \frac{\rho''(1)}{\rho(1)}$$

the following asymptotic moment properties:

$$\mathbb{E}[X_n] = \mu n + O(1) \quad \text{and} \quad \text{Var}[X_n] = \sigma^2 n + O(1),$$

for $n \equiv a \pmod{b}$. Furthermore, if $\sigma^2 \neq 0$ then we also have a central limit theorem of the form

$$\frac{X_n - \mathbb{E}[X_n]}{\sqrt{\text{Var}[X_n]}} \rightarrow \mathcal{N}(0, 1) \quad (n \equiv a \pmod{b}).$$

Proof. It is easy to show that, for every positive x , we can apply Proposition 2 and Theorem 2 and obtain (for $n \equiv a \pmod{b}$)

$$[z^n]M(z, x, 1) = c(x) n^{-5/2} \rho(x)^{-n} \left(1 + O\left(\frac{1}{n}\right) \right) \quad (5.2)$$

for some positive valued function $c(x)$. Note that the error term comes from the remainder terms

$$a_4(x) \left(1 - \frac{z}{\rho(x)} \right)^2 + O\left(\left(1 - \frac{z}{\rho(x)} \right)^{5/2} \right)$$

³This central limit theorem seems to be a folklore result. However, to the best of our knowledge it was first explicitly mentioned by the second author at the Alea-meeting 2010 in Luminy: <https://www-apr.lip6.fr/alea2010/>.

in the singular expansion (5.1) of $M(z, x, 1)$. The term $a_4(x)(1 - z/\rho(x))^2$ has no asymptotic contribution, whereas the other term gives rise to the error term $O(n^{-7/2}\rho(x)^{-n})$; see also [5]. This proves (5.2). Furthermore, since F_0 and Q are polynomials it follows that $\rho(x)$ is an algebraic function since it is determined by the algebraic system of equations (4.5), where we just have to add the algebraic dependence on x .

Actually it can be shown that $\rho(x)$ has no singular point for $x > 0$. As explained in the proof of Proposition 2, we can reduce the solution of the catalytic equation to a system of three positive polynomial equations. Such a system can be reduced to a single equation $u(x, z) = F(x, z, u(x, z))$ in one unknown function $u = u(x, z)$, where $u = F(x, z, u)$ has a power series expansion with non-negative coefficients (see [5]). Note that we certainly have $F_z \neq 0$ and $F_{uu} \neq 0$. The system of equations that determines the values $z = \rho(x)$ and $u = u(x, \rho(x))$, where the solution function $z \mapsto u(x, z)$ gets singular, is given by

$$u = F(x, z, u), \quad 1 = F_u(x, z, u).$$

The functional determinant of this system, when we solve it for $z = \rho(x)$ and $u = u(x, \rho(x))$, is given by

$$F_z F_{uu} - (F_u - 1)F_{uz} = F_z F_{uu} \neq 0.$$

By the implicit function theorem $z = \rho(x)$, as well as $u = u(x, \rho(x))$ are analytic. Moreover $\rho'(x) = -F_x/F_z < 0$, since $F_x > 0$ and $F_z > 0$.

By the methods of [5] it also follows that the singular expansion (4.3), where Z has to be replaced by $\sqrt{1 - z/\rho(x)}$ and all coefficient functions f_j, u_j, w_j depend on x , can be extended to complex x that are sufficiently close to the positive real axis. Accordingly the asymptotic expansion (5.2) holds uniformly if x varies in a compact subset of the complex plane, where $\rho(x)$ is well defined.

As mentioned above this is sufficient to prove the asymptotic expansion for $\mathbb{E}[X_n], \text{Var}[X_n]$, as well the central limit theorem. \square

We note the crucial point in the proof of Theorem 3 was to prove a singular expansion of the form (5.1) that holds in a complex neighborhood of $x = 1$. We finally add a theorem for catalytic equations, where we do not necessarily have a polynomial equation with non-negative coefficients. Again, we assume that there is an additional variable x , where we non necessarily assume that the defining catalytic equations contains only non-negative coefficients. This kind of approach was first applied in [7], where the number of faces of given valency in random planar maps was discussed; see below. (It was first stated without a proof in [9]).

Theorem 4. *Suppose that $M(z, x, u)$ and $M_1(z, x)$ are the solutions of the catalytic equation $P(M(z, x, u), M_1(z, x), z, x, u) = 0$, where the function $P(x_0, x_1, z, x, u)$ is analytic and $M_1(z, 1)$ has a singularity at $z = z_0$ of form*

$$M_1(z, 1) = y_0 + y_2 \left(1 - \frac{z}{z_0}\right) + y_3 \left(1 - \frac{z}{z_0}\right)^{3/2} + \dots, \quad (5.3)$$

with $y_3 \neq 0$ such that for $x_0 = M(z_0, 1, u_0)$, $x_1 = M_1(z_0, 1)$, $z = z_0$, $x = 1$, and $u = u_0$ we have

$$P = 0, \quad P_u = 0, \quad P_{x_0} = 0, \quad P_{x_1} \neq 0, \quad P_{x_0 x_0} P_{uu} = P_{x_0 u}^2. \quad (5.4)$$

Furthermore, let $z = \rho(x)$, $u = u_0(x)$, $x_0 = x_0(x)$, $x_1 = x_1(x)$ for x close to 1 be defined by $\rho(1) = z_0$, $u_0(1) = u_0$, $x_0(1) = M(z_0, 1, u_0)$, $x_1(1) = M_1(z_0, 1)$ and by the system

$$P = 0, \quad P_u = 0, \quad P_{x_0} = 0, \quad P_{x_0 x_0} P_{uu} = P_{x_0 u}^2. \quad (5.5)$$

Then for x close to 1 the function $M_1(z, x)$ has a local singular representation of the form

$$M_1(z, x) = a_0(x) + a_2(x) \left(1 - \frac{z}{\rho(x)}\right) + a_3(x) \left(1 - \frac{z}{\rho(x)}\right)^{3/2} + \dots \quad (5.6)$$

where the functions $a_j(x)$ are analytic at $x = 1$ and satisfy $a_j(1) = y_j$.

Proof. As in the proof of Theorem 2 we can replace the (catalytic) equation $P(M(z, x, u), M_1(z, x), z, x, u) = 0$ by

$$(M(z, x, u) - G(M_1(z, x), z, x, u))^2 = H(M_1(z, x), z, x, u)$$

around $z = z_0$, $x = 1$, $u = u_0$. In particular we have

$$H = 0, \quad H_u = 0, \quad H_{uu} = 0, \quad H_{x_1} \neq 0$$

for $x_0 = M(z_0, 1, u_0)$, $x_1 = M_1(z_0, 1)$, $z = z_0$, $x = 1$, and $u = u_0$.

In the next step we set $x = 1$ and apply the methods from [8, Lemma 2] that ensure that there exist precisely two (local) solutions $u(z)$ and $y(z)$ of the system of equations

$$H(y(z), z, 1, u(z)) = 0, \quad H_u(y(z), z, 1, u(z)) = 0$$

with $y(z_0) = M_1(z_0, 1)$ and $u(z_0) = u_0$ and with local expansions

$$u(z) = u_0 \pm u_1 Z + u_2 Z^2 \pm u_3 Z^3 + \dots, \quad y(z) = y_0 + y_2 Z^2 \pm y_3 Z^3 + \dots,$$

where $Z = \sqrt{1 - z/z_0}$ (and the signs are either all positive or all negative). By assumption, one of these two solutions has to be equal to $M_1(z, 1)$ which implies that $y_3 \neq 0$.

By the methods of the proof of Theorem 2 we also have

$$y_3 = \frac{2u_1 z_0}{3H_{x_1}^2} (H_{x_1} H_{zu} - H_z H_{x_1 u}).$$

Recall that $H_{x_1} \neq 0$. Thus it follows that $z_0 \neq 0$, $u_1 \neq 0$, and $H_{x_1} H_{zu} - H_z H_{x_1 u} \neq 0$. Furthermore, since $y_2 = z_0 H_z / H_{x_1}$ it follows from the relation (4.7) that

$$H_{uuu} = \frac{2z_0}{u_1^2} (H_{x_1} H_{zu} - H_z H_{x_1 u}) \neq 0.$$

We want to do a similar analysis for x close to 1. For this purpose we have to check whether the conditions (5.4) can be extended to x different from 1, that is, whether it is possible to solve the system (5.5) for $z = \rho(x)$, $u = u_0(x)$, $x_0 = x_0(x)$, $x_1 = x_1(x)$ (if x is close to 1). Note that the condition $P_{x_1} \neq 0$ certainly extends to a neighborhood. By the same procedure as in the proof of Theorem 2 it follows that the system (5.5) is equivalent to the system

$$H = 0, \quad H_u = 0, \quad H_{uu} = 0 \quad (5.7)$$

for $x_1 = x_1(x)$, $z = \rho(x)$, $u = u_0(x)$. Note that $x_1(x)$, $\rho(x)$, $u_0(x)$ are the same functions as above; and the function $x_0(x)$ can be recovered by $x_0(x) =$

$M(\rho(x), x, u_0(x))$. Now the functional determinant of the system (5.7) is given by

$$\begin{vmatrix} H_{x_1} & H_{ux_1} & H_{uux_1} \\ H_z & H_{uz} & H_{uuz} \\ H_u & H_{uu} & H_{uuu} \end{vmatrix} = H_{uuu} (H_{x_1} H_{uz} - H_z H_{ux_1})$$

which is non-zero at $x_1 = M_1(z_0, 1)$, $z = z_0$, $x = 1$, and $u = u_0$. Hence by the implicit function theorem the system (5.7) has an analytic (and unique) local solution $x_1 = x_1(x)$, $z = \rho(x)$, $u = u_0(x)$ with $x_1(1) = M_1(z_0, 1)$, $\rho(1) = z_0$, $u_0(1) = u_0$.

Summing up, we can apply the same techniques as in [8, Lemma 2] that are now valid uniformly in a small (complex) neighborhood of $x = 1$ and leads to an expansion of the form (5.6). \square

Expansions of the form (5.3) or (5.6), respectively, are in particular useful if $z = z_0$ or $z = \rho(x)$ is the only singularity on the slit disc

$$\{z \in \mathbb{C} : |z| < z_0 + \varepsilon\} \setminus [z_0, \infty) \quad \text{or} \quad \{z \in \mathbb{C} : |z| < |\rho(x)| + \varepsilon\} \setminus [z_0, \infty)$$

for some $\varepsilon > 0$. In this case it follows directly that, as $n \rightarrow \infty$,

$$[z^n] M_1(z, 1) = \frac{3y_3}{4\sqrt{\pi}} z_0^{-n} n^{-5/2} \left(1 + O\left(\frac{1}{n}\right) \right)$$

or

$$[z^n] M_1(z, x) = \frac{3a_3(x)}{4\sqrt{\pi}} \rho(x)^{-n} n^{-5/2} \left(1 + O\left(\frac{1}{n}\right) \right).$$

Thus, if

$$M_1(z, x) = \sum_{n \geq 0} \mathbb{E}[x^{X_n}] \cdot [z^n] M_1(z, 1) \cdot z^n$$

encodes the distribution of a sequence of random variables X_n it follows that

$$\mathbb{E}[x^{X_n}] = \frac{[z^n] M_1(z, x)}{[z^n] M_1(z, 1)} = \frac{a_3(x)}{y_3} \left(\frac{z_0}{\rho(x)} \right)^n \left(1 + O\left(\frac{1}{n}\right) \right).$$

and we obtain a central limit theorem by standard tools (see [5] and the above discussion).

Example 8. Let $k \geq 2$ be a fixed integer and let $M(z, x, u)$ be the ordinary generating function enumerating rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces of degree k , and u to the root-face degree. In [7, Lemma 2] it was shown that $M(z, x, u)$ satisfies the equation

$$\begin{aligned} M(z, x, u) (1 - z(x-1)u^{-k+2}) = & 1 + zu^2 M(z, x, u) + zu \frac{uM(z, x, u) - M(z, x, 1)}{u-1} \\ & - z(x-1)u^{-k+2} G(z, x, M(z, x, 1), u), \end{aligned}$$

where $G(z, x, y, u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z, x, y) for $|z| \leq 1/10$, $|x-1| \leq 2^{1-k}$, and $|y| \leq 2$. It should be noted that the function G is not explicitly given but is (one of) the solution(s) of an infinite system of equations that can be solved with the help of Banach's fixed point theorem.

Clearly, $M(z, 1, u)$ is just the usual planar map counting generating function for which we know that $M(z, 1, 1)$ is explicitly given by (1.2) so that all assumptions of Theorem 4 are satisfied. Alternatively we could have used Theorem 2 to obtain the local expansion of $M(z, 1, 1)$. Furthermore a central limit theorem follows, where

X_n is just the number of non-root faces of valency k in a random planar map with n edges. (This is also one of the main results of [7].)

Example 9. We say that a face is a pure k -gon ($k \geq 2$) if it is incident exactly to k different edges and k different vertices. Let $P(z, x, u)$ be the ordinary generating function enumerating rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces that are pure k -gons, and u to the root-face degree. Similarly to the previous case it can be shown (see [17]) that $P(z, x, u)$ satisfies an equation of the form

$$P(z, x, u) = 1 + zu^2P(z, x, u) + zu \frac{uP(z, x, u) - P(z, x, 1)}{u - 1} - z(x - 1)u^{-k+2}\tilde{G}(z, x, P(z, x, 1), u),$$

where $\tilde{G}(z, x, y, u)$ is a polynomial of degree $k - 2$ in u with coefficients that are analytic functions in (z, x, y) for $|z| \leq 1/10$, $|x - 1| \leq 2^{1-k}$, and $|y| \leq 2$.

Again, if we set $x = 1$ we recover $M(z, u) = P(z, 1, u)$ so that all assumptions of Theorem 4 are satisfied. Hence, for fixed $k \geq 2$, the number of pure k -gons in a random planar map satisfies a central limit theorem.

Example 10. A planar map is simple if it has no loops and no multiple edges. The corresponding generating function $S(z, u)$ (where z corresponds to the number of edges and u to the root face valency) satisfies the catalytic equation (see [17])

$$S(z, u) = 1 + zu^2S(z, u)^2 + zu \frac{uS(z, u) - S(z, 1)}{u - 1} - zuS(z, u)S(z, 1) - (S(z, u) - 1)(S(z, 1) - 1)$$

and the solution $S(z, 1)$ is explicitly given by

$$S(z, 1) = \frac{1 + 20z - 8z^2 + (1 - 8z)^{3/2}}{2(z + 1)^3}.$$

Let $k \geq 2$ be a fixed integer and let $S(z, x, u)$ be the ordinary generating function enumerating simple rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces of degree k , and u to the root-face degree. In [17] it was shown that $S(z, x, u)$ satisfies the equation

$$S(z, x, u) = 1 + zu^2S(z, x, u) + zu \frac{uS(z, x, u) - S(z, x, 1)}{u - 1} - zuS(z, x, u)S(z, x, 1) - (S(z, x, u) - 1)(S(z, x, 1) - 1) + (x - 1) \left(zu^{-k+2}S(z, x, u)G_1(z, x, S(z, x, 1), u) - zuS(z, x, u)G_2(z, x, S(z, x, 1)) - (S(z, x, u) - 1)G_3(z, x, S(z, x, 1)) \right),$$

where $G_1(z, x, y, u)$ is a polynomial of degree $k - 2$ in u with coefficients that are analytic functions in (z, x, y) for $|z| \leq 2/25$, $|x - 1| \leq 2^{-k-5}$, and $|y - 1| \leq 2/5$. Similarly the functions $G_2(z, x, y)$ and $G_3(z, x, y)$ are analytic functions in (z, x, y) for $|z| \leq 2/25$, $|x - 1| \leq 2^{-k-5}$, and $|y - 1| \leq 2/5$.

Again all assumptions of Theorem 4 are satisfied. Hence, for fixed $k \geq 2$, the number of faces of valency k in a random simple planar map satisfies a central limit theorem.

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