Distribution Properties of Generalized van der Corput Sequences

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September 10, 2009

Contents

Zusammenfassung

Folgen mit kleiner Diskrepanz erhalten ihre Bedeutung durch ihre Anwendbarkeit in den verschiedensten Gebieten, wie zum Beispiel im Bereich der nummerischen Integration oder der Optimierung. Das Ziel der nummerischen Integration ist es, ein Integral durch eine Summe von gewichteten Funktionswerten zu ersetzen, also

$$\int_\Omega f(x)dx \approx \frac{\mathrm{Vol}(\Omega)}{N}\sum_{i=1}^{N} f(x_i).$$

Hier bezeichnen die x_i Punkte die im Integrationsbereich Ω liegen. Um diese Punkte zu bestimmen stehen verschiedene Methoden zur Verfügung, eine davon ist die Monte Carlo Methode, bei der diese zufällig gewählt werden. Eine andere Möglichkeit besteht darin Folgen mit bestimmten Verteilungs-Eigenschaften zu verwenden, in diesem Fall spricht man von der Quasi-Monte Carlo Methode. Ein großer Vorteil dieser Vorgehensweise besteht darin, dass Punkte zu einem späteren Zeitpunkt eingefügt werden können ohne die ganze Folge neu berechnen zu müssen. Außerdem erlaubt die Approximation eines Integrals mit der Quasi-Monte Carlo Methode eine Zerlegung des Fehlers in einen Term, der nur von dem Integranden $f(x)$ abhängt und einen, der nur von der Folge bestimmt wird. Das ist ein Resultat aus der Formel von Koksma-Hlawka, siehe auch Satz 1.2.1. Ein wichtiger Anwendungsbereich solcher Folgen ist die Finanz- und Versicherungsmathematik, wo hochdimensionale Integrale berechnet werden mssen und gerade diese Methoden sehr erfolgreich sind.

In diesem Zusammenhang sind Folgen mit kleiner Diskrepanz, das heißt Folgen deren Diskrepanz von der Ordnung $(\log N)^s$ ist, wobei s die Dimension der Folge bezeichnet, von besonderer Bedeutung. Ein Beispiel für solche Folgen sind die van der Corput Folgen, die 1935 von J. G. van der Corput eingeführt wurden. Eine Verallgemeinerung dieser Folgen geht auf H. Faure zurück, sie werden permutierte oder verallgemeinerte van der Corput Folgen genannt. Ziel der vorliegenden Arbeit ist es, ein Resultat von M. Drmota, G. Larcher und F. Pillichshammer (siehe [5]) zu verallgemeinern. Dort wurde ein zentraler Grenzwertsatz für die Diskrepanz von van der Corput Folgen in der Basis 2 gezeigt. Diese Ergebnisse sollen nun auf allgemeine Basen erweitert werden, außerdem wird versucht ähnliche Aussagen für verallgemeinerte van der Corput Folgen zu beweisen.

Im ersten Kapitel werden die verschiedenen Arten von Diskrepanzen definiert und bekannte Resultate zusammengefasst. Ein kurzer Abschnitt ist der Ungleichung von Koksma-Hlawka gewidmet, die wir anfangs als einen der Motivationsgründe für die Beschäftigung mit Diskrepanzen genannt haben. Des Weiteren werden untere Schranken für die Diskrepanz behandelt. Abschließend wird ein kurzer Überblick über (t, s)-Folgen gegeben. Die van der Corput Folge ist eine $(0, 1)$-Folge, auch die verallgemeinerte van der Corput Folge ist eine $(0, 1)$-Folge, allerdings kommt hier eine leicht modifizierte Definition der (t, s)-Folgen zum Einsatz, siehe dazu [10].

Zu Beginn des zweiten Kaptiel werden spter bentigte Definitionen und Resultate im Zusammenhang mit den verallgemeinerten van der Corput folgen zusammengefasst. Anschlieend wird das zugrundeliegende Wahrscheinlichkeitsmodell behandelt, das die Interpretation der Diskrepanz als eine Summe schwach abhängiger Zufallsvariablen, auf die stckweise lineare Funktionen angewendet werden, ermöglicht. Im nächsten Abschnitt wird ein zentraler Grenzwertsatz für diese Art von Summen bewiesen, wobei vorausgesetzt wird, dass ihre Varianz zumindest mit der Ordnung N^{α} für $\alpha > 2/3$ wchst. Zum Abschluss wird eine Formel für den Erwartungswert und die Varianz der Diskrepanz bewiesen.

Im dritten Kapitel werden die vorhandenen Ergebnisse auf die verallgemeinerte van der Corput Folge in den Basen 2, 3, und 4 angewendet. Insbesondere erhlt man einen zentralen Grenzwertsatz für die extreme und die Stern-Diskrepanz in den angegebenen Fällen.

Das vierte Kapitel beinhaltet eine Erweiterung der Resultate auf allgemeine Basen. Es wird ein zentraler Grenzwertsatz für die extreme und die Stern-Diskrepanz der van der Corput Folge bewiesen. Ähnliches wird für eine Klasse von verallgemeinerten van der Corput Folgen gezeigt. Abschließend werden anhand von NUT-Folgen einige der Probleme bei der allgemeinen Lösung dieses Problems besprochen. Diese Folgen können als Spezialisierung der verallgemeinerten van der Corput Folgen gesehen werden, allerdings werden in ihrer Definition Matrizen anstelle von Permutationen verwendet.

Abstract

The importance of sequences with low discrepancies arises from various applications, for example in integration and optimization theory. In numerical integration one tries to do the following approximation

$$\int_\Omega f(x)dx \approx \frac{\mathrm{Vol}(\Omega)}{N}\sum_{i=1}^{N} f(x_i),$$

where x_i are points lying in Ω. There are different techniques to obtain the point sets $\{x_1, \ldots, x_n\}$. For example, using the Monte Carlo Method they are chosen randomly. Another possibility is to use certain deterministic sequences with given distribution properties, in this case one speaks of Quasi-Monte Carlo Methods. A big advantage of the latter is the possibility to add additional points without recalculating the whole series. Furthermore the Quasi-Monte Carlo integration allows a decomposition of its error term in the product of a term only depending on the sequence and one only depending on the function $f(x)$. This is accomplished by the formula of Koksma-Hlawka stated in Theorem 1.2.1. An important application of Quasi-Monte Carlo integration can be found in financial and actuary mathematics. There one needs to evaluate high-dimensional integrals. With the help of Quasi-Monte Carlo methods it is frequently possible to obtain good results for those integrals.

In this context low discrepancy sequences are of special interest. This are sequences with a discrepancy of order $(\log N)^s$ where s is the dimension of the sequence. The van der Corput sequences are one example for low discrepancy sequences. They were first introduced by J. G. van der Corput in 1935. A generalization of these sequences is due to H. Faure, they are called permuted or generalized van der Corput sequences. In [5] M. Drmota, G. Larcher and F. Pillichshammer proved a central limit theorem for the discrepancy of the van der Corput sequence in base 2. In this work we want to extend these results to generalized van der Corput sequences and arbitrary bases.

The first chapter summarizes known results for the discrepancy. The various kinds of discrepancies are defined. Further we introduce the Koksma-Hlawka inequality as it provides a motive for the study of the discrepancy. In the next section we state some lower bounds and finally we give a short overview of (t, s)-sequences. The generalized van der Corput sequences are $(0, 1)$-sequences in the broad sense see [10].

The second chapter provides all necessary definitions and theorems concerning the generalized van der Corput sequences. Further a probability model is introduced which provides the means to view the discrepancy as the sum of linear functions of weakly dependent random variables. In the second section a central limit theorem for sums of this type is proved under the condition that the variance tends to infinity with order at least N^{α} for some $\alpha > 2/3$. Finally the expectation and the variance of the discrepancy are calculated. While this can achieved easily for the expectation the computation of the variance is more involved as certain error terms have to be treated.

In the third chapter the results obtained so far are applied to generalized van der Corput sequences in special bases. For the bases 2, 3 and 4 a central limit theorem for the star and the extreme discrepancy is proved.

The fourth chapter provides a central limit theorem for the star and the extreme discrepancy of the van der Corput sequence and a special class of generalized van der Corput sequences in arbitrary bases. Finally NUT-sequences are mentioned as an example for the difficulties one faces when trying to obtain general results. In some context these sequences can be viewed as specialization of generalized van der Corput sequences but for their definition matrices are used instead of permutations.

Chapter 1

The Discrepancy of Sequences

This section is meant to give a short overview of known results for the discrepancy of arbitrary sequences and point sets. First we mention necessary definitions and basic results concerning the discrepancy. Then we will state the Koksma-Hlawka inequality and give some lower bounds. Finally we introduce a class of sequences with certain distribution properties. They are called (t, s)-sequences and the van der Corput sequence belongs to them.

1.1 The Discrepancy Function

In order to obtain good results for the approximation of integrals using Quasi-Monte Carlo Methods, it is essential that the nodes are well distributed on the unit interval. This means their distribution is close to a uniform distribution. Therefore one needs a function providing the means to decide whether a sequence is well distributed or not. This is achieved by the discrepancy. There are various types of discrepancies. We will describe several here.

Let $I = [0, 1)^s$ be the s-dimensional unit cube, $B \subseteq I$ and $P = \{x_1, \ldots, x_N\}$ a multiset of points lying in I then

$$A(B, N, P) := |\{i | 1 \leq i \leq N, x_i \in B\}|$$

is the number of elements of P lying in B. We can also define this quantity for sequences. Let $S = (x_i)_{i \geq 1}$ be a sequence in I, then

$$A(B, N, S) := |\{i | 1 \leq i \leq N, x_i \in B\}|$$

is the number of elements x_i, $1 \leq i \leq N$ of S lying in B. This allows us to give a definition of the discrepancy but first we will state what we mean by uniform distribution in this context.

Definition 1.1.1. *Let $I = [0,1)^s$ be the s-dimensional unit cube, the sequence $S = (x_i)_{i\geq 1}$ of points lying in I is called uniformly distributed in I if*

$$\lim_{N\to\infty} \frac{1}{N} A(B,N,S) = \lambda(B)$$

holds for every subinterval B of I. Here $\lambda(B)$ denotes the Lebesgue-measure.

Definition 1.1.2. *If $\mathbb{B}$ is a non-empty family of Lebesgue-measurable subsets of I, then the discrepancy of the point set P is given by*

$$D(\mathbb{B},N,P) := \sup_{B\in\mathbb{B}} |A(B,N,P) - \lambda(B)N|$$

where $|P| = N$.

In many cases this definition is too general. Two of the most common specializations are the star discrepancy and the extreme discrepancy. These two specializations impose restrictions on the sets B.

Definition 1.1.3. *If the family $\mathbb{B}$ consists of all subintervals of I of the form $\prod_{i=1}^{s}[0,u_i)$, $u_i \in [0,1)$, then the discrepancy is called star discrepancy.*

Definition 1.1.4. *If the family $\mathbb{B}$ consists of all subintervals of I of the form $\prod_{i=1}^{s}[v_i,u_i)$, $v_i, u_i \in [0,1)$, then the discrepancy is called extreme discrepancy.*

Using these definitions we omit the family of subsets in the notation for the star and the extreme discrepancy and write $D^*(N,P)$ and $D(N,P)$ respectively. The next proposition provides some relation between this two discrepancies.

Proposition 1.1.1. *Let $P \subseteq I$ be a finite subset of cardinality N, then*

$$D^*(N,P) \leq D(N,P) \leq 2^s D^*(N,P).$$

For a proof of Proposition 1.1.1 see [18].

Remark. It is also possible to look at the star discrepancy in a slightly different manner. Taking the function

$$g(u) = A([0,u), N, P) - \lambda([0,u))N,$$

we can interpret it as a supremum norm. From this point of view it is only natural to consider the L^p-norms of this function as well. This leads to the notion of the L^p-discrepancy, which is defined as

$$D^{(p)}(N,P) = \left(\int_I |A([0,u), N, P) - \lambda([0,u))N|^p \, du \right)^{\frac{1}{p}}.$$

So far, we only defined the discrepancy for finite sets. Now we want to extend this definitions to sequences as well. We will use the first N elements of the sequence to obtain the set P, this will achieve our objective.

Definition 1.1.5. *Let $S = (x_i)_{i \geq 1}$ be a sequence of points lying in I, then*

$$D(N,S) = D(N,P)$$

with $P = \{x_1, \ldots, x_N\}$.

In the beginning of this section, we stated the discrepancy function allows us to decide, whether a set of points is well distributed or not. This is a result of the following proposition. A proof of it can be found in [6].

Proposition 1.1.2. *Let S be a sequence of points lying in I then the following three properties are equivalent:*

i. *S is uniformly distributed in I*

ii. $\lim_{N\to\infty} \frac{D^*(N,S)}{N} = 0$

iii. $\lim_{N\to\infty} \frac{D(N,S)}{N} = 0$

Finally we want to give formulas for the calculation of the extreme and the star discrepancy in the case where $s = 1$. Proofs are, for example, given in [18].

Theorem 1.1.1. *Let* $0 \leq x_1 \leq \ldots \leq x_N \leq 1$ *then*

- $D^*(N; x_1, \ldots, x_N) = \frac{1}{2} + \max_{1 \leq n \leq N} \left| x_n N - \frac{2n-1}{2} \right|$ *and*
- $D(N; x_1, \ldots, x_N) = 1 + \max_{1 \leq n \leq N} (n - x_n N) - \min_{1 \leq n \leq N} (n - x_n N)$.

1.2 The Koksma-Hlawka Inequality

From Proposition 1.1.2 we know there exists a direct connection between the distribution of a sequence and its discrepancy. Furthermore this is not the only reason one should be interested in a profound study of this quantity. The discrepancy plays an important role in the error bound for numerical integration. The Koksma-Hlawka inequality states the order of convergence of the approximation of the integral towards its real value can be estimated in terms of the variation of the function and the star discrepancy. Before we can write down the result, we have to define the variation of a function.

Be f a function on I and J a subinterval of I, then by $\Delta(f; J)$ we define the alternating sum of the values of f at the vertices of J, meaning the function values at adjacent vertices have opposite signs. Using this function, we are now able to define the variation in the sense of Vitali.

Definition 1.2.1. *Let* $\mathbb{J}$ *be a partition of* I *into subintervals, then*

$$V^{(s)}(f) = \sup_{\mathbb{J}} \sum_{J \in \mathbb{J}} |\Delta(f; J)|$$

is called the variation in the sense of Vitali.

With the help of the previous definition, it is now possible to write down the variation in the sense of Hardy and Krause

Definition 1.2.2. *We write* $V^{(k)}(f; i_1, \ldots, i_k)$ *for the restriction of the variation in the sense of Vitali to the k-dimensional face* $\{(u_1, \ldots, u_s) \in I | u_j = 1,$ *for* $j \neq i_1, \ldots, i_k\}$, *then the variation in the sense of Hardy and Krause is given by*

$$V(f) = \sum_{k=1}^{s} \sum_{1 \leq i_1 < \ldots < i_k \leq s} V^{(k)}(f; i_1, \ldots, i_s).$$

A function f is said to have bounded variation, if $V(f)$ is finite. With this definition, it is now possible to formulate the Koksma-Hlawka inequality (see [12]).

Theorem 1.2.1 (Koksma-Hlawka). *If f has bounded variation in the sense of Hardy and Krause on I then*

$$\left|\frac{1}{N}\sum_{i=1}^{N} f(x_i) - \int_I f(u)du\right| \leq V(f)\frac{D_N^*(x_1,\ldots,x_N)}{N}$$

for any $x_1,\ldots,x_N \in I$.

1.3 Lower Bounds

The applications mentioned so far always use sequences with low discrepancy. In this context it is only natural to ask if there exist lower bounds and if there are sequences having a discrepancy close to them. This section gives a short overview of known lower bounds for the discrepancy. In the next section, we will state examples for sequences with a discrepancy having the same order as these bounds.

We will start with a bound for dimension $s = 1$. These results are a direct consequences of Theorem 1.1.1

- $D^*(N; x_1,\ldots,x_N) \geq \frac{1}{2}$ and
- $D(N; x_1,\ldots,x_N) \geq 1$

Taking the point set $x_n = \frac{n}{N}$, $1 \leq n \leq N$ the same theorem implies

$$D^*(N; x_1,\ldots,x_N) = D(N; x_1,\ldots,x_N) = 1.$$

For dimension one, we have therefore found a point set with very good distribution properties. The problem with this point set is one has to fix the number of points N in advance. For example, in numerical integration if one wants to get a better approximation by choosing a refined partition of the integration interval, then one has to recalculate all the points. It is often more convenient to use sequences instead of sets. The advantage of sequences is that one can add additional nodes later and can still use the nodes, that have already been computed. From this point of view it is clear one of the important properties of such a sequence is no matter how many points are chosen, they are always well distributed on the interval. It would be desirable to find sequences with discrepancies of the same order as the point set above. Unfortunately this is not possible. As we will conclude later, the best we can achieve is $O(\log N)$.

Schmidt [32] was the first to find a lower bound for the discrepancy of a given sequence.

Theorem 1.3.1. *For any infinite real sequence* $S = (x_n)_{n\geq 1}$*, we have*

$$D(N, S) \geq C_1 \log N$$

for infinitely many N*, with*

$$C_1 = \max_{a\geq 3} \frac{1}{8} \frac{a-2}{a \log a} = 0.04667\ldots.$$

Theorem 1.3.2. *For any infinite real sequence* $S = (x_n)_{n\geq 1}$*, we have*

$$D^*(N, S) \geq C_2 \log N$$

for infinitely many N*, with*

$$C_2 = \max_{a\geq 3} \frac{1}{16} \frac{a-2}{a \log a} = 0.02333\ldots.$$

Remark. The second theorem is a direct consequence of the first one by Proposition 1.1.1.

The two constants C_1 and C_2 are not the originals given by Schmidt. They were first improved by Kuipers and Niederreiter [15], however the values given here are due to Liardet [16].

So far, we have only been concerned with one dimensional sequences. It is only natural to ask if similar bounds exist for dimension $s > 1$. A well known conjecture states for every dimension s there exists a constant C_s so for any infinite sequence $S = (x_n)_{n\geq 1}$, $x_n \in \mathbb{R}^s$ we have

$$D(N, S) \geq C_s (\log N)^s$$

for infinitely many N.

So far the best known result for $s > 3$ is the following theorem found by Roth [31].

Theorem 1.3.3. *Let $s \geq 2$, then the discrepancy $D(N, P)$ of N points $P = \{x_1, \dots, x_N\}$ in the s-dimensional space $\mathbb{R}^s$ is bounded from below by*

$$D(N, P) \geq D^*(N, P) \geq D^{(2)}(N, P) \geq \frac{1}{2^{4s}} \frac{1}{((s-1)\log 2)^{\frac{s-1}{2}}} (\log N)^{\frac{s-1}{2}}.$$

For further information on bounds for the dimensions 2 and 3 and refinements of Roth's theorem we refer to [6].

1.4 (t, s)-Sequences

The results we have stated for lower bounds for the discrepancy suggest that the best possible sequences are those having a discrepancy of the order $O(\log N)$. This type of sequences are called low discrepancy sequences. We should now research if such sequences exist and if there are general terms to describe them. Although later we will only consider the one dimensional case, we now state all results for arbitrary dimension.

In this section, we will take a closer look at a special sort of low discrepancy sequences, the (t, s)-sequences. Here s again denotes the dimension and the parameter t describes the quality of the distribution of the sequence. Roughly speaking, the idea behind the (t, s)-sequences is to divide the unit interval in specific subintervals. These subintervals are called elementary intervals. Once this is completed it is possible to construct sequences in such a way that in every elementary interval we have the same number of elements.

Definition 1.4.1. *Let $b \geq 2$ be some chosen base and $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$, $1 \leq i \leq s$, then an s-dimensional elementary interval is an interval of the form*

$$E = \prod_{i=1}^{s} \left[a_i b^{-d_i}, (a_i + 1) b^{-d_i}\right).$$

Remark. It is easy to see, that $\lambda(E) = b^{-d}$ with $d = \sum_{i=1}^{s} d_i$.

Definition 1.4.2. *Let t and m be integers satisfying $0 \leq t \leq m$. We define a (t, m, s)-net in base b as a point set $P \in I$ with $|P| = b^m$, such that*

$$A(E, b^m, x_n) = b^t$$

for all elementary intervals E with $\lambda(E) = b^{t-m}$.

Remark. From the definition of the (t, m, s)-net it follows, that

$$\frac{A(E, N, P)}{N} - \lambda(E) = 0$$

for every (t, m, s)-net and every elementary interval E.

Definition 1.4.3. *Let $t \geq 0$ be an integer, a sequence $(x_n)_{n\geq 1}$ in I is called a (t, s)-sequence in b, if for all $k \geq 0$ and $m > t$ the point set*

$$P = \{x_n | kb^m < n \leq (k + 1)b^m\}$$

is a (t, m, s)-net in base b.

Remark. Every (t, m, s)-net in base b is also a (u, m, s)-net in base b for $t \leq u \leq m$. This follows from the fact that every elementary interval E with $\lambda(E) = b^{u-m}$ can be written as a union of b^{u-t} disjoint elementary intervals E_j with $\lambda(E_j) = b^{t-m}$, $j = 1, \ldots, b^{u-t}$. The same holds true for the sequences, so every (t, s)-sequence is also a (u, s)-sequence, for $t < u$.

The next theorem shows that (t, s)-sequences have the distribution properties of low discrepancy sequences. A proof can be found in [19]. Another result of this theorem is smaller values for t imply better uniform distribution.

Theorem 1.4.1. *The star discrepancy of a (t, m, s)-net $x_1, \ldots, x_N$ in b with $m > 0$ satisfies*

$$D^*(N; x_1, \ldots, x_N) \leq K_s(b) b^t (\log N)^{s-1} + O\left(b^t (\log N)^{s-2}\right),$$

with

$$K_s(b) = \left(\frac{b-1}{2 \log b}\right)^{s-1}$$

if $s = 2$ *or* $b = 2$ *and* $s = 3, 4$, *otherwise*

$$K_s(b) = \frac{1}{(s-1)!} \left(\frac{\lfloor b/2 \rfloor}{\log b} \right)^{s-1}.$$

The star discrepancy of a (t, s)*-sequence* $(x_n)_{n \geq 1}$ *in base* b *with* $m > 0$ *satisfies*

$$D^*(N; x_1, \ldots, x_N) \leq M_s(b) b^t (\log N)^s + O\left(b^t (\log N)^{s-1}\right),$$

with

$$M_s(b) = \frac{1}{s} \left(\frac{b-1}{2 \log b} \right)^s$$

if $s = 2$ *or* $b = 2$ *and* $s = 3, 4$, *otherwise*

$$M_s(b) = \frac{1}{(s)!} \frac{b-1}{2 \lfloor b/2 \rfloor} \left(\frac{\lfloor b/2 \rfloor}{\log b} \right)^s.$$

Further we have

$$\lim_{s \to \infty} M_s(b) = 0.$$

Now we want to give some examples for (t, s)-sequences. We will start with the van der Corput sequence, which is a $(0, 1)$-sequence.

Definition 1.4.4. *The radical inverse function of* $n \in \mathbb{N}$ *with* b*-adic expansion* $n = \sum_{r=0}^{\infty} a_r(n) b^r$ *is defined as*

$$rad_b(n) = \sum_{r=0}^{\infty} a_r(n) b^{-(r+1)}.$$

Definition 1.4.5. *The van der Corput sequence* $(x_n)_{n \geq 1}$ *in base* b *is defined as*

$$x_n = rad_b(n-1).$$

It is not hard to see, that the van der Corput sequence is a $(0, 1)$-sequence. Taking the digital expansion of some integer n, with $n = kb^m, \ldots, (k+1)b^m - 1$

$$n = \sum_{r=0}^{\infty} a_r(n) b^r$$

then the digits $a_r(n)$, $0 \leq r \leq m-1$ run through all possible combinations of $\{0, 1, \ldots, b-1\}^m$. We can now apply the radical inverse function. This function can also be viewed as the reflection of the digital expansion with respect to the decimal point. In doing this we obtain numbers in $[0, 1)$, whose m leading digits again attain all possible values from $\{0, 1, \ldots, b-1\}$, while the remaining digits stay fixed. So every interval of the form

$$[ab^{-m}, (a+1)b^{-m}), \quad a = 0, 1, \ldots, b^m - 1,$$

contains exactly one point x_n, with $n \in \{kb^m, \ldots, (k+1)b^m - 1\}$.

The van der Corput sequence can easily be generalized to a s-dimensional sequence. This is called the Halton sequence.

Definition 1.4.6. *For a given dimension s and integers $b_1, \ldots, b_s$, $b_i \geq 2$ the Halton sequence $(x_n)_{n\geq 1}$ is defined as*

$$x_n = (rad_{b_1}(n-1), \ldots, rad_{b_s}(n-1)) \in [0, 1)^s.$$

Remark. Note that the Halton sequence is mentioned here as a generalization of the van der Corput sequence, however it is not a (t, s)-sequence.

There have been several construction principals proposed for (t, s)-sequences with arbitrary dimension, see for example [20] or [21], [22], [23]. We want to define the Faure sequences as a special case. These sequences are $(0, s)$-sequences in base p, where p is the smallest prime number larger than s.

Definition 1.4.7. *Let s be the dimension, p the smallest prime number with $p \geq s$ and $y_j^{(0)}(n)$ the digits of the expansion of n in base p*

$$n = \sum_{j=0}^{\infty} y_j^{(0)}(n) p^j,$$

then the Faure sequences $(x_n)_{n\geq 1}$ is defined as

$$x_n^{(i)} = \sum_{j=0}^{\infty} y_j^{(i)}(n) p^{-(j+1)}, \quad 1 \leq i \leq s.$$

The $y_j^{(i)}(n)$ *are obtained by*

$$y^{(i)}(n) = C^i y^{(0)}(n) \pmod p$$

with

$$C^i = \left(\binom{k}{j} i^{k-j} \right)_{j,k \geq 0}.$$

We do not want to prove that the Faure sequence really is a $(0, s)$-sequence. Figure 1.1 shows the eight points of the Faure sequence in base 2 for $k = 3$ in the four different types of elementary intervals. This example is used to illustrate the construction principal of the Faure sequence.

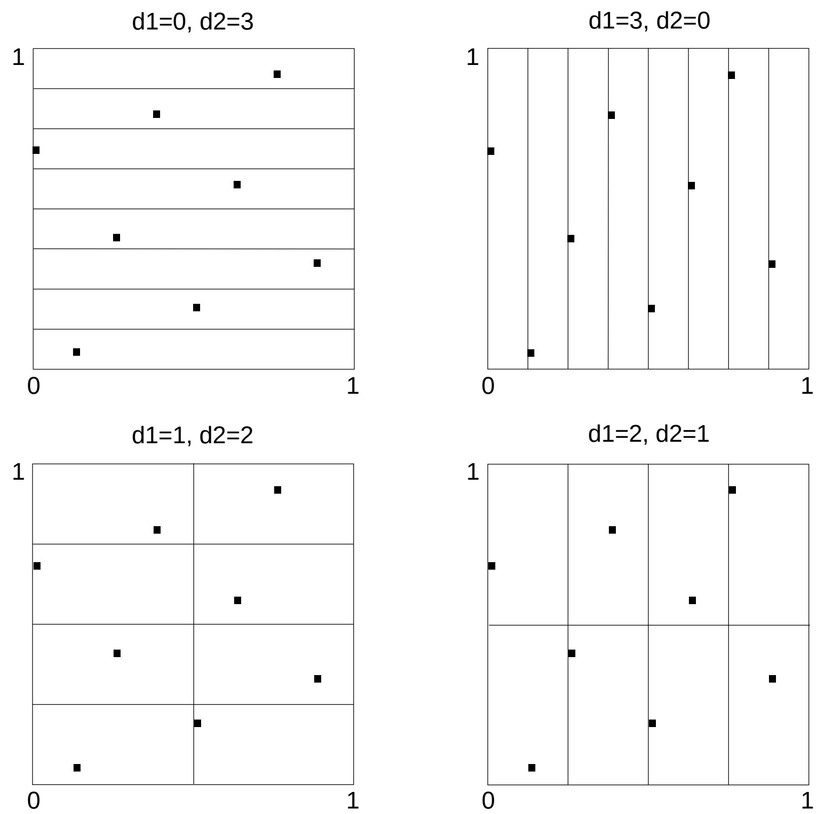

Figure 1.1: Eight points of the Faure sequence in base 2 for $k = 3$

Chapter 2

A Central Limit Theorem for Generalized van der Corput Sequences

2.1 Definitions and Basic Results

So far we have been concerned with the general notions of the discrepancy and weakly dependent random variables. Now we want to apply these results to special sequences, the generalized van der Corput sequences. These sequences are constructed similarly to the van der Corput sequence, but in addition a specific kind of scrambling is applied to the digits of the expansion of n. There exist special formulas for their discrepancy due to Faure [4]. In this section we will show by using these formulas one can interpret the discrepancy as a sum of weakly dependent random variables in a proper probability setting. Further, we will calculate the expectation and the variance of the discrepancy (according to this setting).

Definition 2.1.1. *Let $b \geq 2$ and $\Sigma = (\sigma_j)_{j\geq 0}$ a sequence of permutations σ_j of the set $\{0, 1, \ldots, b-1\}$. For $n \in \mathbb{N}$ we write*

$$n - 1 = \sum_{r=0}^{\infty} a_r(n) b^r.$$

Then the generalized van der Corput sequence $S_b^\Sigma = (S_b^\Sigma(n))_{n\geq 1}$ is given by

$$S_b^\Sigma(n) = \sum_{r=0}^{\infty} \frac{\sigma_r(a_r(n))}{b^{r+1}}.$$

Remark. If we only consider $b \geq 2$ to be a prime number and permutations, where $\sigma(i) = ik \pmod{b}$, $k < b$, then we can reformulate the above definition

$$S_b^{\Sigma}(n) = \sum_{r=0}^{\infty} \frac{s_{n,r}}{b^{r+1}}$$

with

$$s_{n,r} = \sum_{k=0}^{\infty} c_r^k a_k(n) \pmod{b}$$

where $c_r^k \in \{1, 2, \ldots, b-1\}$ for $k = r$ and $c_r^k = 0$ for $k \neq r$. This definition may seem complicated, but in this context this type of generalized van der Corput sequences can be viewed as special case of the NUT-sequences. These sequences are defined in the same manner but with matrices $C = (c_r^k)_{r \geq 0, k \geq 0}$, which are not diagonal but non-singular upper triangular (NUT) matrices.

We are considering generalized van der Corput sequences here because there exist formulas for the discrepancy of these sequences. They have been developed by H. Faure [7]. We will now define several functions, that play an important role in these formulas.

Definition 2.1.2. *Let σ be a permutation of $\{0, 1, \ldots, b-1\}$ and*

$$Z_b^{\sigma} = \left(\frac{\sigma(0)}{b}, \ldots, \frac{\sigma(b-1)}{b} \right)$$

and $1 \leq k \leq b$, then for $x \in [(k-1)/b, k/b[$ we set

$$\varphi_{b,h}^{\sigma}(x) = \begin{cases} A(h/b, k, Z_b^{\sigma}) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([h/b, 1[, k, Z_b^{\sigma}) & \text{if } \sigma(k-1) < h < b \end{cases} .$$

The function is extended to the reals by periodicity. We now define $\Psi_b^{\sigma,+}$, $\Psi_b^{\sigma,-}$ and Ψ_b^{σ} in the following way

$$\Psi_b^{\sigma,+} = \max_{0 \leq h \leq b-1} (\varphi_{b,h}^{\sigma}), \quad \Psi_b^{\sigma,-} = \max_{0 \leq h \leq b-1} (-\varphi_{b,h}^{\sigma}), \quad \Psi_b^{\sigma} = \Psi_b^{\sigma,+} + \Psi_b^{\sigma,-}.$$

The following proposition states some properties of the functions Ψ_b^{σ}, $\Psi_b^{\sigma,+}$ and $\Psi_b^{\sigma,-}$, a proof can be found in [7]

Proposition 2.1.1. *Let* $\Psi_b^{\sigma,+}$*,* $\Psi_b^{\sigma,-}$ *and* Ψ_b^{σ} *be defined as above then the following four properties hold true:*

i. *The coefficients of the functions* Ψ *are integers with an absolute value smaller or equal* $b-1$*. Further the functions are positive, continuous, piecewise linear and equal to* 0 *at* $x=0$ *and* $x=1$*. They are never constant, except when they are equal to zero.*

ii. *For* $x \in [0,1/b)$ *we have*

$$\Psi_b^{\sigma,+}(x) = (b-\sigma(0)-1)x,$$
$$\Psi_b^{\sigma,-}(x) = \sigma(0)x,$$
$$\Psi_b^{\sigma}(x) = (b-1)x$$

iii. *For* $x \in [(b-1)/b,1)$ *we have*

$$\Psi_b^{\sigma,+}(x) = \sigma(b-1)(1-x),$$
$$\Psi_b^{\sigma,-}(x) = (b-\sigma(b-1)-1)(1-x),$$
$$\Psi_b^{\sigma}(x) = (b-1)(1-x)$$

iv. *The function* Ψ_b^{σ} *is bounded from below by* $\min((b-1)x, 1-x)$ *on* $[0,1/2)$ *and by* $\min(x,(b-1)(1-x))$ *on* $[1/2,1)$*.*

The next lemma shows, that the set of all functions $\Psi_b^{\sigma,+}$ is the same as the set of the functions $\Psi_b^{\sigma,-}$.

Lemma 2.1.1. *For every permutation* σ *of* $\{0,1,...,b-1\}$ *there exists a permutation* $\widetilde{\sigma}$*, such that*

$$\Psi_b^{\sigma,-} = \Psi_b^{\widetilde{\sigma},+}$$

and $\widetilde{\sigma}$ *is given by*

$$\widetilde{\sigma}(i) = b-1-\sigma(i).$$

Proof. In order to verify this assumption we take advantage of the fact that there exists a bijection between the permutations σ of $\{0,\ldots,b-1\}$ and the vectors $\overline{\varphi}_b^{\sigma}$ defined by

$$\overline{\varphi}_b^{\sigma} = \left(\varphi_{b,h}^{\sigma}(x)\right)_{0\le h<b}.$$

The map is injective because two different permutations can never lead to the same vector. On the other hand there are only $b!$ different vectors $\overline{\varphi}_b^{\sigma}$, so the map has to be bijective.

We will now show for any vector $\overline{\varphi}_b^\sigma$ exists a permutation $\widetilde{\sigma}$ such that $-\overline{\varphi}_b^\sigma = \overline{\varphi}_b^{\widetilde{\sigma}}$. Taking some permutation $\sigma(i)$, $0 \le i \le b-1$ on $\{0,\ldots,b-1\}$, we define

$$\widetilde{\sigma}(i) := b - 1 - \sigma(i).$$

This is again a permutation on $\{0,\ldots,b-1\}$. As $A(h/b,k,Z_b^\sigma)$ is the number of indices i, $0 \le i \le k-1$ with $\sigma(i) < h$, we obtain

$$A(h/b,k,Z_b^\sigma) = A([\widetilde{h}/b,1[,k,Z_b^{\widetilde{\sigma}})$$

with $\widetilde{h} = b - h$. This is due to the fact that if $\sigma(i) < h$ then $\widetilde{\sigma}(i) = b - 1 - \sigma(i) > b - 1 - h$ and so $\widetilde{\sigma}(i) \ge \widetilde{h}$. The same arguments lead to

$$A([h/b,1[,k,Z_b^\sigma) = A(\widetilde{h}/b,k,Z_b^{\widetilde{\sigma}}).$$

Looking closer at the vector $-\overline{\varphi_b^\sigma}$, we get

$$\begin{aligned}
-\overline{\varphi}_b^\sigma &= \begin{cases} hx - A(h/b,k,Z_b^\sigma) & \text{if} \quad 1 \le h \le \sigma(k-1), \\ (A([h/b,1[;k;Z_b^\sigma) - (b-h)x & \text{if} \quad \sigma(k-1) < h < b \end{cases} \\
&= \begin{cases} A(\widetilde{h}/b,k,Z_b^{\widetilde{\sigma}}) - \widetilde{h}x & \text{if} \quad 1 \le \widetilde{h} \le \widetilde{\sigma}(k-1), \\ (b-\widetilde{h})x - A([\widetilde{h}/b,1[;k;Z_b^{\widetilde{\sigma}}) & \text{if} \quad \widetilde{\sigma}(k-1) < \widetilde{h} < b \end{cases} \\
&= \overline{\varphi}_b^{\widetilde{\sigma}}.
\end{aligned}$$

For the first component of the vector, we have $\varphi_{b,0}^\sigma(x) = \varphi_{b,0}^{\widetilde{\sigma}}(x) = 0$. □

If we set $\sigma = id$, we obtain the ordinary van der Corput sequence. In this case, it can easily be verified that the functions Ψ are of the following form

$$\Psi_b^{id,+} = \begin{cases} k(1-x) & \text{if} \quad x \in [k/b, k/(b-1)), \\ (b-k-1)x & \text{if} \quad x \in [k/(b-1),(k+1)/b) \end{cases}$$

and

$$\Psi_b^{id,-} = 0$$

so

$$\Psi_b^{id} = \Psi_b^{id,+}.$$

The following lemma gives a relation between the functions Ψ_b^σ and Ψ_b^{id}, a proof can be found in [7].

Lemma 2.1.2. *For all permutations σ, we have*

$$\Psi_b^{\sigma} \leq \Psi_b^{id}$$

In the following theorem we state a limit relation and an upper bound for special cases of the generalized van der Corput sequence. It was proven by Faure in [7].

Theorem 2.1.1. *Let Σ be the constant sequence $(\sigma)_{i\geq 0}$ and set*

$$\alpha_{b,\sigma} = \inf_{n\geq 1} \sup_{x\in\mathbb{R}} \left(\frac{1}{n} \sum_{j=1}^{n} \Psi_b^{\sigma} \left(\frac{x}{b^j} \right) \right)$$

then we have

$$\overline{\lim}_{N\to\infty} \frac{D(N, S_b^{\sigma})}{\log N} = \frac{\alpha_{b,\sigma}}{\log b}$$

and

$$D(N, S_b^{\sigma}) \leq \frac{\alpha_{b,\sigma}}{\log b} \log N + \max\left(2, \alpha_{b,\sigma} + 1 + \frac{1}{b}\right).$$

2.2 The Probability Model

In this section we will introduce the probabilistic settings we need. The next theorem provides the means to write the discrepancy of the generalized van der Corput sequence as the sum of weakly dependent random variables. It was found by H. Faure [7]. Together with Corollary 2.2.1 it is the starting point for the calculation of the expectation and the variance of the discrepancy.

Theorem 2.2.1. *Let $N \geq 1$ then we get*

$$\begin{aligned}
D^+(N, S_b^\Sigma) &= \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right), \\
D^-(N, S_b^\Sigma) &= \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right), \\
D(N, S_b^\Sigma) &= \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right), \\
D^*(N, S_b^\Sigma) &= \max(D^+(N, S_b^\Sigma), D^-(N, S_b^\Sigma)).
\end{aligned}$$

Corollary 2.2.1. *Let n and N be integers with $1 \leq N \leq b^n$ then on gets:*

$$\begin{aligned}
D^+(N, S_b^\Sigma) &= \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right) + \frac{N}{b^n} - N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j}, \\
D^-(N, S_b^\Sigma) &= \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right) + N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j}, \\
D(N, S_b^\Sigma) &= \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) + \frac{N}{b^n},
\end{aligned}$$

With the help of these formulas we are able to prove the next theorem. In the proof we essentially follow [5], where a similar result was shown for van der Corput sequences in base 2.

In the sequel we will use the following probability model: For a fixed n we assume N is uniformly distributed on $\{1, 2, \ldots, b^n\}$. We consider the fractional parts

$$\left\{\frac{N}{b^j}\right\} = \frac{N}{b^j} - \left\lfloor \frac{N}{b^j} \right\rfloor.$$

Then $\{\frac{N}{b^j}\}$ is approximately uniformly distributed on $[0, 1)$, in particular we can write $N = \lfloor b^n U \rfloor$ where U is uniformly distributed on $[0, 1)$. The following theorem demonstrates the discrepancy can be written as the sum of random variables. Therefore it is possible to calculate the expectation and the variance of the discrepancy (according to this probabilistic setting).

Theorem 2.2.2. *Suppose N is chosen uniformly on $\{1, 2, \ldots, b^n\}$ and is represented by $N = \lfloor b^n U \rfloor$ with U uniformly distributed on $[0, 1)$. Then the discrepancies D^+, D^- and D of the generalized van der Corput sequences can be written as*

$$\begin{aligned} D^+(N, S_b^\Sigma) &= \sum_{j=1}^{n} \Psi_b^{\sigma_{n-j},+}\left(Ub^j\right) + O(1), \\ D^-(N, S_b^\Sigma) &= \sum_{j=1}^{n} \Psi_b^{\sigma_{n-j},-}\left(Ub^j\right) + O(1), \\ D(N, S_b^\Sigma) &= \sum_{j=1}^{n} \Psi_b^{\sigma_{n-j}}\left(Ub^j\right) + O(1). \end{aligned}$$

Proof. We will verify the theorem for $D^+(N, S_b^\Sigma)$. The other cases can be treated similarly. From Corollary 2.2.1 we know, we can write the discrepancy of the generalized van der Corput sequence as

$$D^+(N, S_b^\Sigma) = \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right) + \frac{N}{b^n} - N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j}.$$

For the term $\frac{N}{b^n} - N\sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j}$ we obtain the following inequality

$$\begin{aligned} 0 \leq \frac{N}{b^n} - N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j} & \left|\frac{N}{b^n}\right| + \left|N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j}\right| \\ & \leq \left|\frac{N}{b^n}\right| + \left|\frac{N}{b^n}\right| = O(1). \end{aligned}$$

Hence we can write $D^+(N, S_b^\Sigma)$ as

$$D^+(N, S_b^\Sigma) = \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right) + O(1)$$

Let $N = N_0 + bN_1 + \cdots + b^{n-1}N_{n-1}$ be the b-adic expansion of N, then

$$U_j = \left\{\frac{N}{b^j}\right\} = \frac{N_{j-1}}{b} + \frac{N_{j-2}}{b^2} + \cdots + \frac{N_0}{b^j}$$

is close to a random variable that is uniformly distributed on $[0, 1)$, if we consider that we chose $1 \leq N \leq b^n$ at random. By adding digits $N_{-1}, N_{-2}, \ldots$, that are independent and uniformly distributed on $0, 1, \dots, b-1$, we can define

$$U_j' = U_j + \sum_{r=1}^{\infty} \frac{N_{-r}}{b^{r+j}}$$

then U_j' is uniformly distributed on $[0,1)$ and $|U_j - U_j'| \le b^{-j}$.

We can also obtain a similar result in another way. Let $\widetilde{N}$ be a random variable, that is uniformly distributed on $[0, b^n)$, then

$$\widetilde{U}_j = \left\{ \frac{\widetilde{N}}{b^j} \right\}$$

is uniformly distributed on $[0,1)$. On the other hand, we can set $\omega = \widetilde{N}/b^n$, so ω is uniformly distributed on $[0,1)$, and $V_j = \widetilde{U}_{n-j+1}$, then

$$V_j = \{\omega b^j\}.$$

As $\widetilde{U}_j$ and U_j' have the same distribution properties, we get

$$|V_j - U_{n-j+1}| \le b^{-(k-j+1)},$$

and so

$$\left| \Psi_b^{\sigma_{n-j},+}\left(\omega b^j\right) - \Psi_b^{\sigma_{n-j},+}\left(\frac{N}{b^{n-j+1}}\right) \right| \le c \frac{1}{b^{n-j+1}}.$$

Finally we obtain

$$D^+(N, S_b^\Sigma) = \sum_{j=1}^{n} \Psi_b^{\sigma_{n-j},+}(\omega b^j) + O(1).$$

□

Concerning the dependence of two variables $\{Ub^j\}$ and $\{Ub^{j+k}\}$ one can observe, they get more and more independent, as k gets larger. Eventually, a modification of the order b^{-k} makes them independent. This fact was used in [5].

2.3 A Central Limit Theorem

Suppose that U is a random variable that is uniformly distribution on $[0,1]$ and the random variables X_j, $j \ge 0$, are given by

$$X_j = \{b^j U\},$$

where $b \geq 1$ is an integer. Then the random variables X_j are uniformly distributed on $[0,1]$, too, but they are not independent. Nevertheless, they are just *weakly dependent* in the following sense.

Lemma 2.3.1. *Suppose that* $0 \leq j_1 < j_2 < \ldots < j_m$ *is a finite subsequence of non-negative integers. Then there exist random variables* $\tilde{X}_{j_1}, \ldots, \tilde{X}_{j_m}$ *which are uniformly distributed on* $[0,1]$ *and independent such that (for some absolute constant* $C > 0$*)*

$$|X_{j_i} - \tilde{X}_{j_i}| \leq C\, b^{-(j_{i+1}-j_i)}, \qquad 1 \leq i < m$$

and $\tilde{X}_{j_m} = X_{j_m}$.

Proof. Suppose that the b-ary expansion of some number $u \in [0,1)$ is given by

$$u = 0.u_1u_2u_3\ldots = \sum_{k\geq 1} u_k b^{-k}$$

with digits $u_k \in \{0,1,\ldots,b-1\}$. Then the digital expansion of $x_j = \{b^j u\}$ has the representation

$$x_j = 0.u_{j+1}u_{j+2}u_{j+3}\ldots$$

Now, if U is uniformly distributed on $[0,1)$, the random digits U_k of U are independent random variables which are uniformly distributed on the set $\{0,1,\ldots,b-1\}$. Furthermore, the random variables $X_j = \{b^j U\}$ just depend on the digits U_k of U for $k > j$. Thus, if we modify the digits U_{j+1}, U_{j+2} of $X_0 = U$ then this does not affect the distribution of $X_j = \{b^j U\}$. The proof of the lemma is just an successive application of this idea.

Let $0 \leq j_1 < j_2 < \ldots < j_m$ be a finite sequence of integers. Without loss of generality we can assume that $j_1 = 0$. We now replace the random variables $X_{j_0} = U$, X_{j_1}, …, X_{j_m} by slightly modified random variables. Recall that $U = \sum_{k\geq 1} U_k b^{-k}$ is represented by its (random) digits U_k. We set

$$\tilde{X}_{j_0} = \sum_{k=1}^{j_1} U_k b^{-k} + \sum_{k>j_1} \tilde{U}_k^{(0)} b^{-k},$$

$$\tilde{X}_{j_1} = \sum_{k=1}^{j_2-j_1} U_{k+j_1} b^{-k} + \sum_{k>j_2-j_1} \tilde{U}_k^{(1)} b^{-k},$$

$$
\begin{aligned}
&\vdots, \\
\tilde{X}_{j_{m-1}} &= \sum_{k=1}^{j_m - j_{m-1}} U_{k+j_{m-1}} b^{-k} + \sum_{k > j_m - j_{m-1}} \tilde{U}_k^{(m-1)} b^{-k}, \\
\tilde{X}_{j_m} &= X_{j_m},
\end{aligned}
$$

where $\tilde{U}_k^{(j_i)}$ are uniformly distributed on $\{0, 1, \dots, b-1\}$, independent from each other and also independent from the original (random) digits U_k.

Obviously, the sequence $\tilde{X}_{j_1}, \dots, \tilde{X}_{j_m}$ is independent, since the are given by independent digits. Furthermore we have

$$
|X_{j_i} - \tilde{X}_{j_i}| \le 2(b-1) \sum_{k > j_{i+1} - j_i} b^{-k} \le C b^{-(j_{i+1} - j_i)}.
$$

This completes the proof of the lemma □

Lemma 2.3.1 has a natural extension to the following situations.

Lemma 2.3.2. *Suppose that* $0 \le j_1^{(1)} < j_2^{(1)} < \dots < j_m^{(1)}$ *and* $0 \le j_1^{(2)} < j_2^{(2)} < \dots < j_m^{(2)}$ *are interlacing finite subsequences of non-negative integers, that is,*

$$
j_1^{(1)} < j_1^{(2)} < j_2^{(1)} < j_2^{(2)} < \dots < j_m^{(1)} < j_m^{(2)}.
$$

Then there exist random variables

$$
\tilde{X}_j \quad \textit{for} \quad j \in \bigcup_{i=1}^{m} \{ j_i^{(1)}, j_i^{(1)} + 1, \dots, j_i^{(2)} \}
$$

such that for all $i = 1, \dots, m$, *the two random vectors*

$$
B_i = \left(X_{j_i^{(1)}}, X_{j_i^{(1)}+1}, \dots, X_{j_i^{(2)}} \right) \quad \textit{and} \quad \tilde{B}_i = \left(\tilde{X}_{j_i^{(1)}}, \tilde{X}_{j_i^{(1)}+1}, \dots, \tilde{X}_{j_i^{(2)}} \right)
$$

have the same distribution, that the blocks $\tilde{B}_1, \dots, \tilde{B}_m$ *are independent, and that*

$$
|X_j - \tilde{X}_j| \le C \, b^{-(j_{i+1}^{(1)} - j_i^{(2)})}
$$

for $j \in \{ j_i^{(1)}, j_i^{(1)} + 1, \dots, j_i^{(2)} \}$ *and* $i = 1, \dots, m$.

Now suppose that $f_j : [0,1] \to \mathbb{R}$ are functions which are uniformly bounded and (uniformly) Lipschitz continuous:

$$|f_j(x) - f_j(y)| \le c\,|x-y|.$$

Our main goal is to describe the limiting distribution of the partial sums

$$S_N = \sum_{j=0}^{N-1} f_j(X_j), \tag{2.1}$$

where $X_j = \{b^j U\}$ from above. In particular we are interested in a central limit theorem.

Theorem 2.3.1. *Suppose that S_N is given by (2.1), where $X_j = \{b^j U\}$ and f_j are uniformly bounded and uniformly Lipschitz continuous. If the variance S_N satisfies $\mathbb{V}\mathrm{ar}\, S_N \ge N^\alpha$ for some $\alpha > 2/3$ then S_N satisfies a central limit theorem:*

$$\frac{S_N - \mathbb{E}\, S_N}{\sqrt{\mathbb{V}\mathrm{ar}\, S_N}} \to N(0,1).$$

Proof. The idea is to subdivide the sequence $Y_j = f_j(X_j)$, $j < N$, into blocks

$$B_i = (Y_{(L+S)i}, \ldots Y_{(L+S)i+L-1}) \quad \text{and} \quad C_i = (Y_{(L+S)i+L}, \ldots Y_{(L+S)(i+1)-1}),$$

where $L \ge 1$ and $S \ge 1$ with $S \le L$ will be defined later. For simplicity we assume that N is an integer multiple of $L+S$. By Lemma 2.3.2 we can modify the blocks B_i to $\tilde{B}_i$ such that B_i and $\tilde{B}_i$ have the same distribution but the sequence $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \ldots$ is independent. Furthermore the elements of B_i and $\tilde{B}_i$ differ at most by $C\, b^{-S}$. A similar procedure applies to the blocks C_i.

We now define

$$Z_i = \sum_{j=(L+S)i}^{(L+S)i+L-1} f_j(X_j)$$

and

$$R = \sum_{i<N/(L+S)} \sum_{j=(L+S)i+L}^{(L+S)(i+1)-1} f_j(X_j).$$

Then

$$S_N = \sum_{i=0}^{N/(L+S)-1} Z_i + R.$$

The principle idea is to replace the random variables Z_i by independent random variables $\tilde{Z}_i$ (with the help of Lemma 2.3.2). This will show a central limit theorem for the sum $\sum Z_i$. By a proper choice of L and S the remainder term R will be small enough and will not affect the limit distribution.

Recall that by the Cauchy-Schwarz inequality we have

$$\mathbb{V}\mathrm{ar}(X+Y) = \mathbb{V}\mathrm{ar}\,X + \mathbb{V}\mathrm{ar}\,Y + O\left(\sqrt{\mathbb{V}\mathrm{ar}\,X \cdot \mathbb{V}\mathrm{ar}\,Y}\right).$$

Assume now that S_N', S_N'' and S_N are random variables with $S_N = S_N' + S_N''$ and we know that S_N' satisfies a central limit theorem of the form

$$(S_N' - \mathbb{E}\,S_N')/\sqrt{\mathbb{V}\mathrm{ar}\,S_N'} \to N(0,1)$$

such that $\mathbb{V}\mathrm{ar}\,S_N' \to \infty$ as well as $\mathbb{V}\mathrm{ar}\,S_N''/\mathbb{V}\mathrm{ar}\,S_N' \to 0$ as $N \to \infty$. Then also S_N satisfies a central limit theorem.

We first consider the remainder term R and estimate its variance. Recall that

$$R = \sum_{i<N/(L+S)} R_i$$

with

$$R_i = \sum_{j=(L+S)i+L}^{(L+S)(i+1)-1} f_j(X_j).$$

By replacing X_j by $\tilde{X}_j$ in the blocks C_i then we get (according to Lemma 2.3.2)

$$R_i = \tilde{R}_i + O(Sb^{-L}),$$

where the $\tilde{R}_i$ are independent, and and consequently

$$R = \tilde{R} + O(NS/L\, b^{-L}),$$

where $\tilde{R}$ is the sum of independent random variables $\tilde{R}_i$. Thus the variance of R is bounded by

$$\mathbb{V}\mathrm{ar}\,R = O\left(\frac{N}{L}S^2 + \left(\frac{NS}{L}b^{-L}\right)^2\right).$$

Next we consider the sum $\sum_i Z_i$. By replacing X_j by $\tilde{X}_j$ in the blocks B_i we get (as above)

$$\sum_{i=0}^{N/(L+S)-1} Z_i = \sum_{i=0}^{N/(L+S)-1} \tilde{Z}_i, + O\left(Nb^{-S}\right).$$

where the random variables $\tilde{Z}_i$ are independent. Let $\tilde{\sigma}_i^2 = \mathbb{V}\text{ar}\,\tilde{Z}_i$ and let

$$\tilde{s}_N^2 = \sum_{i=0}^{N/(L+S)-1} \tilde{\sigma}_i^2$$

be the variance of $\sum_i \tilde{Z}_i$. If we assume the Lyapunov criterion

$$\lim_{N/(L+S)\to\infty} \frac{1}{\tilde{s}_N^{2+\delta}} \sum_{i=0}^{N/(L+S)-1} \mathbb{E}\left|\tilde{Z}_i - \mathbb{E}\,\tilde{Z}_i\right|^{2+\delta} = 0$$

for some $\delta > 0$ then the sum $\sum_i \tilde{Z}_i$ satisfies a central limit theorem (see [1]). Since $\tilde{Z}_i = O(L)$ we just have to assure

$$\lim_{N/(L+S)\to\infty} \frac{1}{\tilde{s}_N^{2+\delta}} N\,L^{1+\delta} = 0.$$

We assume now $\sigma_N^2 = \mathbb{V}\text{ar}\,S_N \geq N^\alpha$. Furthermore we set $L = N^\kappa$ and $S = N^\lambda$ for suitable positive constants (that we will choose appropriately in the sequel). Hence, $\mathbb{V}\text{ar}\,R = O(N^{1-\kappa+2\lambda})$. We observe that the term $O(Nb^{-S})$ is negligible. Therefore, if

$$N^{1-\kappa+2\lambda} = o(N^\alpha) \tag{2.2}$$

then the variance σ_N^2 dominates the behavior and we obtain

$$\tilde{s}_N^2 = \mathbb{V}\text{ar}\left(\sum_{i=0}^{N/(L+S)-1} \tilde{Z}_i\right) \geq \frac{1}{2} N^\alpha$$

Finally, if we have

$$\frac{1}{\tilde{s}_N^{2+\delta}} N\,L^{1+\delta} \leq N^{1+\kappa(1+\delta)-\alpha(1+\delta/2)} \to 0 \tag{2.3}$$

we are done.

This can be achieved by choosing $\alpha > \frac{2}{3}$, for example $\alpha = \frac{2}{3} + \varepsilon$, with $0 < \varepsilon \leq \frac{1}{3}$. Then we set $\kappa = \frac{1}{3} - \frac{\varepsilon}{2}$ and $\lambda = \frac{\varepsilon}{8}$. This leads to

$$1 - \kappa + 2\lambda = \frac{2}{3} + \frac{3\varepsilon}{4} < \frac{2}{3} + \varepsilon = \alpha.$$

Therefore condition (2.2) is satisfied. By selecting

$$\delta = \frac{4\kappa}{1-3\kappa} > 0$$

we get

$$1-\kappa = \frac{1+\kappa(1+\delta)}{1+\frac{\delta}{2}}.$$

Since

$$1-\kappa < 1-\kappa+2\delta < \alpha$$

we have

$$\frac{1+\kappa(1+\delta)}{1+\frac{\delta}{2}} < \alpha.$$

Equivalently this can be written as

$$1+\kappa(1+\delta)-\alpha(1+\delta/2) < 0 \qquad \square$$

and hence we obtain (2.3).

2.4 Expectation and Variance

This section is devoted to the calculation of the expectation and the variance of the discrepancy of the generalized van der Corput sequence. To cover the general situation these computations should be done for any $M \in \mathbb{N}$ but we will restrict ourselves to the case where $M = b^n$. Especially for the variance the calculations would get much more complicated. We will give a proof for arbitrary M for the expectation (see also [5]).

Theorem 2.4.1. *Let $M = a_1 b^{m_1} + a_2 b^{m_2} + \cdots + a_s b^{m_s}$ with $m_1 > m_2 > \cdots > m_s$ and $a_j \in \{1, \ldots, b-1\}$ be a positive integer and let $1 \leq N \leq M$ be uniformly distributed. Then the expectation of the discrepancy of the generalized van der Corput sequence can be written as*

$$\mathbb{E}(D(N, S_b^{\Sigma})) = \sum_{j=1}^{m_1} \left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx \right) + O(1).$$

Proof. First we assume $M = b^n$. According to Corollary 2.2.1 the discrepancy can be written as

$$D(N, S_b^{\Sigma}) = \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) + O(1).$$

So we get for the expectation of the discrepancy

$$\begin{aligned}\mathbb{E}(D(N, S_b^{\Sigma})) &= \frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)+O(1)\right)\\ &= \sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)+O(1).\end{aligned}$$

As $\Psi_b^{\sigma_{j-1}}(x)$ is periodic with period 1 we can write

$$\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = \frac{1}{b^j}\sum_{N=1}^{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)$$

We want to show that

$$\frac{1}{b^j}\sum_{N=1}^{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = \int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx + O\left(\frac{1}{b^j}\right).$$

If we write

$$\int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx = \sum_{N=1}^{b^j}\int_{(N-1)/b^j}^{N/b^j}\Psi_b^{\sigma_{j-1}}(x)dx$$

we can apply the mean value theorem and arrive at

$$\begin{aligned}&\left|\frac{1}{b^j}\sum_{N=1}^{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) - \sum_{N=1}^{b^j}\int_{(N-1)/b^j}^{N/b^j}\Psi_b^{\sigma_{j-1}}(x)dx\right|\\ &\leq \sum_{N=1}^{b^j}\frac{1}{b^j}\left|\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) - \Psi_b^{\sigma_{j-1}}(\xi_N)\right|,\end{aligned}$$

with $(N-1)/b^j \leq \xi_N \leq N/b^j$. As $\Psi_b^{\sigma_{j-1}}$ is a continuous and piecewise linear function, we obtain

$$\left|\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) - \Psi_b^{\sigma_{j-1}}(\xi_N)\right| \leq C\frac{1}{b^j}.$$

Summarizing all this results we arrive at

$$\mathbb{E}(D(N, S_b^{\Sigma})) = \sum_{j=1}^{n}\left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx + O\left(\frac{1}{b^j}\right)\right) + O(1).$$

The error terms sum up to $O(1)$, which proves the formula for the expectation.

We will now verify the theorem for arbitrary M. According to theorem 2.1.2 we can write the discrepancy as

$$D(N, S_b^{\Sigma}) = \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right).$$

In order to calculate the expectation of discrepancy for $N \leq M$ we have to evaluate the term

$$\frac{1}{M} \sum_{N=1}^{M} D(N, S_b^{\Sigma}) = \frac{1}{M} \sum_{N=1}^{M} \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right).$$

If we write $M = a_1 b^{m_1} + a_2 b^{m_2} + \cdots + a_s b^{m_s}$ with $m_1 > m_2 > \cdots > m_s$ and $a_j \in \{1, \ldots, b-1\}$ then

$$\begin{aligned} \sum_{N \leq M} D(N, S_b^{\Sigma}) = \sum_{N \leq a_1 b^{m_1}} D(N, S_b^{\Sigma}) + \sum_{N \leq a_2 b^{m_2}} D(N + a_1 b^{m_1}, S_b^{\Sigma}) + \cdots \\ + \sum_{N \leq a_s b^{m_s}} D(N + a_1 b^{m_1} + \cdots + a_{s-1} b^{m_{s-1}}, S_b^{\Sigma}). \end{aligned}$$

Using this decomposition of the sum we want to show

$$\begin{aligned} \sum_{N \leq M} D(N, S_b^{\Sigma}) = a_1 \sum_{N \leq b^{m_1}} D(N, S_b^{\Sigma}) + a_2 \sum_{N \leq b^{m_2}} D(N, S_b^{\Sigma}) + \cdots \\ + a_s \sum_{N \leq b^{m_s}} D(N, S_b^{\Sigma}) + O(M). \end{aligned} \tag{2.4}$$

Therefore we have to verify $D(N + a_1 b^{m_1} + \cdots + a_l b^{m_l}, S_b^{\Sigma}) = D(N, S_b^{\Sigma}) + O(l)$. We will use induction on l to show this, we start with $l = 1$.

$$\begin{aligned} \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N + ab^m}{b^j} \right) &= \sum_{j=1}^{m} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) + \sum_{j=m+1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N + ab^m}{b^j} \right) \\ &= \sum_{j=1}^{m} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) + \sum_{j=m+1}^{\infty} (b-1) \frac{N + ab^m}{b^j} + O(1) = \\ &= \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) + a \sum_{j=1}^{\infty} (b-1) b^{-j} + O(1) = \\ &= D(N, S_b^{\Sigma}) + O(1) \end{aligned}$$

If we now consider the general case, we obtain

$$\sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N + a_1 b^{m_1} + \cdots + a_l b^{m_l}}{b^j} \right) =$$

$$= \sum_{j=1}^{m_l} \Psi_b^{\sigma_{j-1}} \left(\frac{N + a_1 b^{m_1} + \cdots + a_{l-1} b^{m_{l-1}}}{b^j} \right)$$

$$+ \sum_{j=m_l+1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N + a_1 b^{m_1} + \cdots + a_l b^{m_l}}{b^j} \right) =$$

$$= \sum_{j=1}^{\infty} \Psi_b^{\sigma_{j-1}} \left(\frac{N + a_1 b^{m_1} + \cdots + a_{l-1} b^{m_{l-1}}}{b^j} \right) + O(1).$$

In the same way we can verify

$$\sum_{N \le ab^m} D(N, S_b^{\Sigma}) = a \sum_{N \le b^m} D(N, S_b^{\Sigma}) + O(ab^m)$$

With the help of 2.4 we are able to calculate the expectation. Inserting our results in the formula for the expectation, we obtain

$$\frac{1}{M} \sum_{N \le M} D(N, S_b^{\Sigma}) = \frac{1}{M} \sum_{l=1}^{s} a_l \sum_{N \le b^{m_l}} D(N, S_b^{\Sigma}) + O(1)$$

$$= \frac{1}{M} \sum_{l=1}^{s} a_l \left(b^{m_l} \sum_{j \le m_l} \int_0^1 \Psi_b^{\sigma_{j-1}}(x) dx + O\left(b^{m_l}\right) \right) + O(1) =$$

$$= \frac{1}{M} \sum_{l=1}^{s} a_l \left(b^{m_l} \sum_{j \le m_1} \int_0^1 \Psi_b^{\sigma_{j-1}}(x) dx + O\left((m_1 - m_l + 1) b^{m_l}\right) \right) + O(1) =$$

$$= \frac{1}{M} \sum_{l=1}^{s} a_l \left(b^{m_l} \sum_{j \le m_1} \int_0^1 \Psi_b^{\sigma_{j-1}}(x) dx \right) + O(1) =$$

$$= \sum_{j \le m_1} \int_0^1 \Psi_b^{\sigma_{j-1}}(x) dx + O(1). \qquad \square$$

Theorem 2.4.2. *Let $1 \leq N \leq b^n$ then the variance of the generalized van der Corput sequence can be written as*

$$\mathbb{V}(D(N, S_b^{\Sigma})) = \sum_{j=1}^{n} \mathbb{V}(\Psi_b^{\sigma_{j-1}}(x)) + 2\sum_{j=1}^{n}\sum_{l<j}\left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx - \int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx \int_0^1 \Psi_b^{\sigma_{l-1}}(x)dx\right) + O(1).$$

Proof. In order to verify the theorem we use the representation of the discrepancy as stated in Corollary 2.2.1

$$D(N, S_b^{\Sigma}) = \sum_{j=1}^{n} \Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) + \frac{N}{b^n}.$$

First, we need to calculate the terms $\mathbb{E}\left(D(N, S_b^{\Sigma})^2\right)$ and $\mathbb{E}\left(D(N, S_b^{\Sigma})\right)^2$.

$$\begin{aligned}
\mathbb{E}\left(D(N, S_b^{\Sigma})^2\right) &= \frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) + \frac{N}{b^n}\right)^2 = \\
&= \frac{1}{b^n}\sum_{N=1}^{b^n}\left(\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2 + \frac{2N}{b^n}\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) + \right. \\
&\quad \left. + \left(\frac{N}{b^n}\right)^2\right) = \\
&= \frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2 + \frac{2}{b^n}\sum_{N=1}^{b^n}\frac{N}{b^n}\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&\quad + \frac{2b^{2n}+3b^n+1}{6b^{2n}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left(D(N, S_b^{\Sigma})\right)^2 &= \left(\frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) + \frac{N}{b^n}\right)\right)^2 \\
&= \left(\frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right) + \frac{b^n+1}{2b^n}\right)^2 \\
&= \left(\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2 +
\end{aligned}$$

$$+\frac{b^n+1}{b^n}\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)+\frac{(b^n+1)^2}{4b^{2n}}$$

Combining these results we obtain a formula for the Variance of the discrepancy.

$$\mathbb{V}\left(D(N,S_b^{\Sigma})\right)=\mathbb{E}\left(D(N,S_b^{\Sigma})^2\right)-\mathbb{E}\left(D(N,S_b^{\Sigma})\right)^2=$$

$$=\frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2-\left(\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2+ \tag{2.5}$$

$$+\frac{2}{b^n}\sum_{N=1}^{b^n}\frac{N}{b^n}\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\frac{b^n+1}{b^n}\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)+ \tag{2.6}$$

$$+\frac{2b^{2n}+3b^n+1}{6b^{2n}}-\frac{(b^n+1)^2}{4b^{2n}} \tag{2.7}$$

We will show that the last two lines 2.6 and 2.7 are of order $O(1)$. For the last line 2.7 this is obvious. In order to prove the assumption for 2.6 we will do the following approximation

$$\frac{2}{b^n}\sum_{j=1}^{n}\sum_{N=1}^{b^n}\frac{N}{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=\sum_{j=1}^{n}\int_0^1\Psi_b^{\sigma_{j-1}}(x)dx+O(1)$$

In order to simplify the calculations, we only consider the term $\frac{1}{b^n}\sum_{N=1}^{b^n}\frac{N}{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)$:

$$\frac{1}{b^n}\sum_{N=1}^{b^n}\frac{N}{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=\frac{1}{b^n}\sum_{N=1}^{b^j}\sum_{k=0}^{b^{n-j}-1}\frac{kb^j+N}{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=$$

$$=\frac{1}{b^n}\sum_{N=1}^{b^j}\left[\frac{1}{2}\left(b^{n-j}-1\right)\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)+\frac{N}{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right]$$

This leads to

$$\frac{1}{2b^j}\sum_{N=1}^{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\frac{1}{2b^n}\sum_{N=1}^{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)+\frac{1}{b^n}\sum_{N=1}^{b^j}\frac{N}{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=$$
$$=\frac{1}{2}\left(\int_0^1\Psi_b^{\sigma_{j-1}}(x)dx+O\left(\frac{1}{b^j}\right)\right)-\frac{1}{2b^{n-j}}\left(\int_0^1\Psi_b^{\sigma_{j-1}}(x)dx+O\left(\frac{1}{b^j}\right)\right)+$$
$$+\frac{1}{b^{n-j}}\left(\int_0^1 x\Psi_b^{\sigma_{j-1}}(x)dx+O\left(\frac{1}{b^j}\right)\right)=$$
$$=\frac{1}{2}\int_0^1\Psi_b^{\sigma_{j-1}}(x)dx+O\left(\frac{1}{b^j}\right)+O\left(\frac{1}{b^{n-j}}\right).$$

It remains to show

$$\frac{1}{b^j}\sum_{N=1}^{b^j}\frac{N}{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=\int_0^1 x\Psi_b^{\sigma_{j-1}}(x)dx+O\left(\frac{1}{b^j}\right)$$

If we split the integral in a sum and then apply the mean value theorem, we get

$$\left|\frac{1}{b^j}\sum_{N=1}^{b^j}\frac{N}{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\sum_{N=1}^{b^j}\int_{(N-1)/b^j}^{N/b^j}x\Psi_b^{\sigma_{j-1}}(x)dx\right|\le$$
$$\le\frac{1}{b^j}\sum_{N=1}^{b^j}\left|\frac{N}{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\xi_N\Psi_b^{\sigma_{j-1}}(\xi_N)\right|.$$

We know $\Psi_b^{\sigma_{j-1}}$ is a bounded and piecewise linear function. With the help of the same arguments as in the case of the expectation we obtain

$$\left|\frac{N}{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\xi_N\Psi_b^{\sigma_{j-1}}(\xi_N)\right|\le$$
$$\le\frac{N}{b^j}\left|\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\Psi_b^{\sigma_{j-1}}(\xi_N)\right|\le$$
$$C\frac{N}{b^{2j}}.$$

According to the proof of Theorem 2.4.1 the second term in 2.6 can be written as

$$\frac{b^n+1}{b^n}\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=\sum_{j=1}^{n}\int_0^1\Psi_b^{\sigma_{j-1}}(x)dx+O(1)$$

Summarizing these results we arrive at

$$\frac{2}{b^n}\sum_{N=1}^{b^n}\frac{N}{b^n}\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)-\frac{b^n+1}{b^n}\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)=O(1).$$

Before we prove the remaining part of the theorem we introduce some new notation. Set

$$\overline{\Psi}_j(x):=\Psi_b^{\sigma_{j-1}}(x)-\frac{1}{b^n}\sum_{N=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right),$$

then according to Lemma 2.4.1 we have to show

$$\begin{aligned}&\frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\right)^2=\\&=\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)^2+2\sum_{j=1}^{n}\sum_{l<j}\frac{1}{b^n}\sum_{N=1}^{b^n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\overline{\Psi}_l\left(\frac{N}{b^l}\right)=\\&=\sum_{j=1}^{n}\mathbb{V}(\Psi_b^{\sigma_{j-1}}(x))+2\sum_{j=1}^{n}\sum_{l<j}\left(\int_0^1\Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx\right.\qquad(2.8)\\&\left.-\int_0^1\Psi_b^{\sigma_{j-1}}(x)dx\int_0^1\Psi_b^{\sigma_{l-1}}(x)dx\right)+O(1).\end{aligned}$$

First, we will verify

$$\frac{1}{b^j}\sum_{N=1}^{b^j}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\Psi_b^{\sigma_{l-1}}\left(\frac{N}{b^l}\right)=\int_0^1\Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx+O\left(\frac{1}{b^j}\right).\qquad(2.9)$$

Writing

$$\int_0^1\Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx=\sum_{N=1}^{b^j}\int_{(N-1)/b^j}^{N/b^j}\Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx$$

we obtain

$$\begin{aligned}&\left|\frac{1}{b^j}\sum_{N=1}^{b^j}\left(\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\Psi_b^{\sigma_{l-1}}\left(\frac{N}{b^l}\right)-b^j\int_{(N-1)/b^j}^{N/b^j}\Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx\right)\right|\leq\\&\leq\frac{1}{b^j}\sum_{N=1}^{b^j}\left|\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\Psi_b^{\sigma_{l-1}}\left(\frac{N}{b^l}\right)-\Psi_b^{\sigma_{j-1}}\left(\xi_N\right)\Psi_b^{\sigma_{l-1}}\left(\xi_N b^{j-l}\right)\right|\end{aligned}$$

with $(N-1)/b^j \leq \xi_N \leq N/b^j$. As the functions $\Psi_b^{\sigma_{l-1}}$ are positive we can define $c = \max(\Psi_b^{\sigma_{l-1}}(N/b^l), \Psi_b^{\sigma_{l-1}}(\xi_N b^{j-l})) > 0$. The case $c = 0$ would imply that $\Psi_b^{\sigma_{l-1}} == 0$ in $[(N-1)/b^j, N/b^j)$. Utilizing the same arguments as in the case of the expectation we get

$$\begin{aligned}
\left|\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\Psi_b^{\sigma_{l-1}}\left(\frac{N}{b^l}\right)\right. &- \left.\Psi_b^{\sigma_{j-1}}(\xi_N)\,\Psi_b^{\sigma_{l-1}}\left(\xi_N b^{j-l}\right)\right| \\
&\leq c\left|\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) - \Psi_b^{\sigma_{j-1}}(\xi_N)\right| \\
&\leq \tilde{c}\frac{1}{b^j}
\end{aligned}$$

We will split the proof of 2.8 in two parts. First, we want to show

$$\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)^2 = \sum_{j=1}^{n}\mathbb{V}(\Psi_b^{\sigma_{j-1}}(x))$$

Using property 2.9 we arrive at

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{1}{b^n}\sum_{N=1}^{b^n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)^2 = \sum_{j=1}^{n}\frac{1}{b^j}\sum_{N=1}^{b^j}\overline{\Psi}_j\left(\frac{N}{b^j}\right)^2 = \\
&= \sum_{j=1}^{n}\int_0^1 \overline{\Psi}_j(x)^2 dx + O(1) = \\
&= \sum_{j=1}^{n}\int_0^1\left(\Psi_b^{\sigma_{j-1}}(x) - \int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx + O\left(\frac{1}{b^j}\right)\right)^2 dx + O(1) = \\
&= \sum_{j=1}^{n}\left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x)^2 dx - \left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx\right)^2\right) + O(1).
\end{aligned}$$

In order to verify the second part, we will need the following two approximations of $\frac{1}{b^n}\sum_{N=1}^{b^n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\overline{\Psi}_l\left(\frac{N}{b^l}\right)$.

$$\begin{aligned}
&\frac{1}{b^n}\sum_{N=1}^{b^n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\overline{\Psi}_l\left(\frac{N}{b^l}\right) = \frac{1}{b^j}\sum_{N=1}^{b^j}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\overline{\Psi}_l\left(\frac{N}{b^l}\right) = \\
&= \frac{1}{b^l}\sum_{r=1}^{b^l}\overline{\Psi}_l\left(\frac{r}{b^l}\right)\frac{1}{b^{j-l}}\sum_{\substack{N=1\\ N\equiv r \pmod{b^l}}}^{b^j}\overline{\Psi}_j\left(\frac{N}{b^j}\right) = \qquad (2.10)
\end{aligned}$$

$$= \frac{1}{b^l} \sum_{r=1}^{b^l} \overline{\Psi}_l \left(\frac{r}{b^l} \right) \left(\int_0^1 \overline{\Psi}_j(x) dx + O\left(\frac{1}{b^{j-l}} \right) \right) = O\left(\frac{1}{b^{j-l}} \right)$$

$$\frac{1}{b^n} \sum_{N=1}^{b^n} \overline{\Psi}_j \left(\frac{N}{b^j} \right) \overline{\Psi}_l \left(\frac{N}{b^l} \right) = \int_0^1 \overline{\Psi}_j(x) \overline{\Psi}_l \left(x b^{j-l} \right) dx + O\left(\frac{1}{b^l} \right) =$$

$$= \int_0^1 \left(\Psi_b^{\sigma_{j-1}}(x) - \frac{1}{b^j} \sum_{N=1}^{b^j} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) \right)$$

$$\times \left(\Psi_b^{\sigma_{l-1}} \left(x b^{j-l} \right) - \frac{1}{b^l} \sum_{\widehat{N}=1}^{b^l} \Psi_b^{\sigma_{l-1}} \left(\frac{\widehat{N}}{b^l} \right) \right) dx + O\left(\frac{1}{b^l} \right) \tag{2.11}$$

$$= \int_0^1 \Psi_b^{\sigma_{j-1}}(x) \Psi_b^{\sigma_{l-1}} \left(x b^{j-l} \right) dx - \int_0^1 \Psi_b^{\sigma_{j-1}}(x) dx \int_0^1 \Psi_b^{\sigma_{l-1}}(x) dx + O\left(\frac{1}{b^l} \right)$$

We can observe the representation 2.11 is very close to the desired result. However, the error term $O(1/b^l)$ will not sum up to $O(1)$ but $O(n)$. The largest contributions are provided by the terms with small l. We know from 2.10 the sum $\frac{1}{b^n} \sum_{N=1}^{b^n} \overline{\Psi}_j \left(\frac{N}{b^j} \right) \overline{\Psi}_l \left(\frac{N}{b^l} \right)$ is of order $O(1/b^{j-l})$. Therefore we obtain

$$2 \sum_{j=1}^{n} \sum_{l<j} \frac{1}{b^n} \sum_{N=1}^{b^n} \overline{\Psi}_j \left(\frac{N}{b^j} \right) \overline{\Psi}_l \left(\frac{N}{b^l} \right) =$$

$$= 2 \sum_{j=1}^{n} \left[\sum_{l<j/2} \frac{1}{b^n} \sum_{N=1}^{b^n} \overline{\Psi}_j \left(\frac{N}{b^j} \right) \overline{\Psi}_l \left(\frac{N}{b^l} \right) \right.$$

$$\left. + \sum_{j/2 \le l < j} \frac{1}{b^n} \sum_{N=1}^{b^n} \overline{\Psi}_j \left(\frac{N}{b^j} \right) \overline{\Psi}_l \left(\frac{N}{b^l} \right) \right] =$$

$$= 2 \sum_{j=1}^{n} \left[\sum_{l<j/2} O\left(\frac{1}{b^{j-l}} \right) + \sum_{j/2 \le l < j} \left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x) \Psi_b^{\sigma_{l-1}} \left(x b^{j-l} \right) dx \right. \right.$$

$$\left. \left. - \int_0^1 \Psi_b^{\sigma_{j-1}}(x) dx \int_0^1 \Psi_b^{\sigma_{l-1}}(x) dx + O\left(\frac{1}{b^l} \right) \right) \right] =$$

$$= 2 \sum_{j=1}^{n} \sum_{j/2 \le l < j} \left(\int_0^1 \Psi_b^{\sigma_{j-1}}(x) \Psi_b^{\sigma_{l-1}} \left(x b^{j-l} \right) dx \right.$$

$$- \int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx \int_0^1 \Psi_b^{\sigma_{l-1}}(x)dx \Bigg) + O(1).$$

In Theorem 2.5.2 we will prove

$$\int_0^1 \Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}\left(xb^{j-l}\right)dx - \int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx \int_0^1 \Psi_b^{\sigma_{l-1}}(x)dx \qquad \square$$

is of order $O(1/b^{2(j-l)})$. Therefore we can add these terms for $1 \le l < j/2$ without augmenting the order of the error term. Hence we attain the result.

Lemma 2.4.1. *Let $\overline{\Psi}_j(x)$ be defined as*

$$\overline{\Psi}_j(x) = \Psi_b^{\sigma_{j-1}}(x) - \frac{1}{b^n}\sum_{N=1}^{b^n} \Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)$$

then we have

$$\frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2 - \left(\frac{1}{b^n}\sum_{N=1}^{b^n}\sum_{j=1}^{n}\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\right)^2 =$$

$$= \frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\right)^2.$$

Proof.

$$\frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\overline{\Psi}_j\left(\frac{N}{b^j}\right)\right)^2 =$$

$$= \frac{1}{b^n}\sum_{N=1}^{b^n}\left(\sum_{j=1}^{n}\left(\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) - \frac{1}{b^n}\sum_{\widehat{N}=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{\widehat{N}}{b^j}\right)\right)\right)^2 =$$

$$= \frac{1}{b^n}\sum_{N=1}^{b^n}\sum_{j,l=1}^{n}\left(\Psi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) - \frac{1}{b^n}\sum_{\widehat{N}=1}^{b^n}\Psi_b^{\sigma_{j-1}}\left(\frac{\widehat{N}}{b^j}\right)\right)\times$$

$$\times\left(\Psi_b^{\sigma_{l-1}}\left(\frac{N}{b^l}\right) - \frac{1}{b^n}\sum_{\widehat{N}=1}^{b^n}\Psi_b^{\sigma_{l-1}}\left(\frac{\widehat{N}}{b^l}\right)\right) =$$

$$= \frac{1}{b^n} \sum_{N=1}^{b^n} \sum_{j,l=1}^{n} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) \Psi_b^{\sigma_{l-1}} \left(\frac{N}{b^l} \right)$$
$$- \frac{1}{b^n} \sum_{N=1}^{b^n} \sum_{j,l=1}^{n} \Psi_b^{\sigma_{j-1}} \left(\frac{N}{b^j} \right) \frac{1}{b^n} \sum_{\widehat{N}=1}^{b^n} \Psi_b^{\sigma_{l-1}} \left(\frac{\widehat{N}}{b^l} \right) \qquad \square$$

2.5 Error term of the Variance

In the last section we showed, the variance of the discrepancy of the generalized van der Corput sequence can be written as a sum of variances plus some additional terms, see Theorem 2.4.2. We need some better understanding of the error terms, as it is crucial for the central limit theorem, to confirm that the variance tends to infinity. Therefore we will verify the terms $\int_0^1 \Psi_b^{\sigma_{j-1}}(x)\Psi_b^{\sigma_{l-1}}(xb^{j-l})dx - \int_0^1 \Psi_b^{\sigma_{j-1}}(x)dx \int_0^1 \Psi_b^{\sigma_{l-1}}(x)dx$ are either equal to zero or of the form $c_{|j-l|}^{j,l}/b^{2|j-l|}$, where $(c_k^{j,l})_k$ is a sequence, that is ultimately periodic.

First, we want to take a look at a more general situation. Let $g(x)$ be some periodic function with period one and $f(x)$ a continuous, piecewise linear function with $f(0) = 0$ and $f(1) = 0$. We want to calculate

$$R_k = \int_0^1 f(x)g(xb^k)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

As f is defined piecewise we can write

$$f(x) = f_i(x) \quad \text{for} \quad e_{i-1} \le x \le e_i, \quad i = 1, \dots, n$$

and

$$f_i(x) = k_i x + d_i$$

with $e_i \in [0, 1]$. Then R_k can be written as $R_k = c_k b^{-2k}$. There exist different formulas for the constant c_k depending on the form of the e_i, see Theorems van:theorem3, van:theorem1 and van:theorem7

The function f has some useful properties, which we will summarize in the following lemma.

Lemma 2.5.1. *The function f as defined above satisfies*

- $$\sum_{l=1}^{n}(e_l - e_{l-1})k_l = 0$$

- $$d_i = \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l$$

- $$d_1 = 0 \quad and \quad d_n = -k_n.$$

Proof. The first and the third property follow directly because $f(x)$ is equal to zero at 0 and 1, the second one uses that f is continuous. □

It is clear from the definitions, our functions Ψ fulfill all the requirements for the functions f and g. If we show that R_k is either equal to zero or of order $O(b^{-2k})$ the same will hold true for the error terms of the variance of the discrepancy. We will look at the two cases $R_k = 0$ and $R_k = O(b^{-2k})$ separately, the first one is a little bit easier to prove, here we will make the assumption, that $e_i = i/b$, $i = 0, \ldots, b$.

Theorem 2.5.1. *Let f and g be as defined above, $e_i = i/b$, $i = 0, \ldots, b$ and $k \in \mathbb{N}$, $k \geq 1$ then*

$$R_k = \int_0^1 f(x)g(xb^k)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx = 0.$$

In order to verify this theorem we will use the following two lemmas. In the first one we treat the integral of the product of the two functions f and g,

$$\int_0^1 f(x)g(xb^k)dx,$$

in the second one we will use a geometrical approach in order to obtain the same result for the product of the integrals

$$\int_0^1 f(x)dx \int_0^1 g(x)dx.$$

Lemma 2.5.2. *Let $e_i = i/b = a_i/b^k$ with $a_i = ib^{k-1}$, then we have*

$$\int_0^1 f(x)g(xb^k)dx = \sum_{i=1}^{b} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(\frac{jk_i}{b^k} + d_i\right)\int_0^1 g(y)dy.$$

Proof. The idea behind the proof is to split the integral in a sum, in such a way that in every integration interval the function g runs through a whole period, see Figure 2.1. In order to achieve this, we write

$$\int_0^1 f(x)g(xb^k)dx = \sum_{j=0}^{b^k-1} \int_{j/b^k}^{(j+1)/b^k} f(x)g(xb^k)dx.$$

The function g is periodic with period one and so we can replace $g(xb^k)$ by $g(xb^k - j)$ on $[j/b^k, (j+1)/b^k)$, this leads to

$$\int_0^1 f(x)g(xb^k)dx = \sum_{j=0}^{b^k-1} \int_{j/b^k}^{(j+1)/b^k} f(x)g(xb^k - j)dx.$$

As the function f is piecewise linear with $f(x) = k_i x + d_i$ for $x \in [a_{i-1}/b^k, a_i/b^k)$, we can write

$$\int_0^1 f(x)g(xb^k)dx = \sum_{i=1}^{b} \sum_{j=a_{i-1}}^{a_i-1} \int_{j/b^k}^{(j+1)/b^k} f_i(x)g(xb^k - j)dx.$$

We will now substitute x by $y = xb^k - j$. Then we will split the integral in two parts, one containing the term $yg(y)$ and one containing the term $g(y)$.

$$\begin{aligned}
\int_0^1 f(x)g(xb^k)dx &= \sum_{i=1}^{b} \sum_{j=a_{i-1}}^{a_i-1} \int_0^1 \frac{1}{b^k}\left(k_i\frac{y+j}{b^k} + d_i\right)g(y)dy \\
&= \sum_{i=1}^{b} (a_i - a_{i-1})\frac{k_i}{b^{2k}} \int_0^1 yg(y)dy \\
&+ \sum_{i=1}^{b} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(\frac{jk_i}{b^k} + d_i\right)\int_0^1 g(y)dy
\end{aligned}$$

From Lemma 2.5.1 we know

$$\sum_{l=1}^{n} (e_l - e_{l-1})k_l = 0.$$

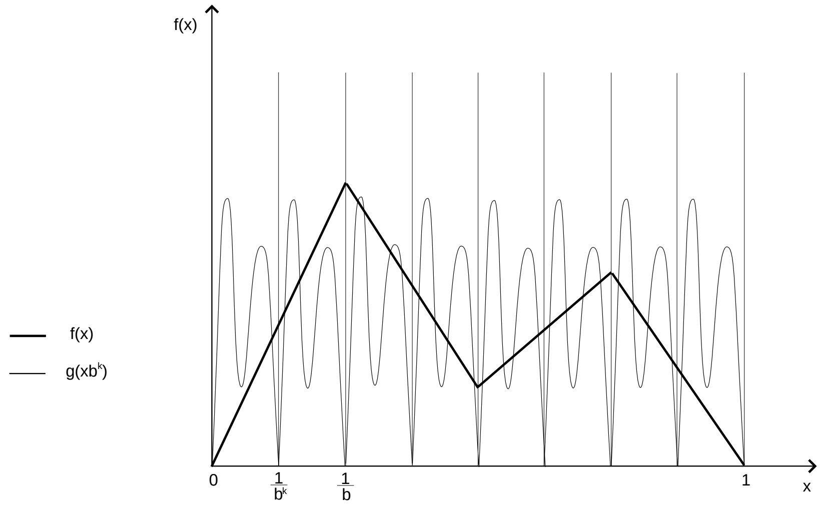

Figure 2.1: Splitting of the integral

In our case this is equivalent to

$$\sum_{i=1}^{b}\left(\frac{a_i}{b^k}-\frac{a_{i-1}}{b^k}\right)k_i=0,$$

which finally leads to the desired result

$$\int_0^1 f(x)g(xb^k)dx=\sum_{i=1}^{b}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\left(\frac{jk_i}{b^k}+d_i\right)\int_0^1 g(y)dy.$$

□

Now we will prove a similar lemma for the product of the integrals.

Lemma 2.5.3. *Let $e_i = a_i/b^k$ with $a_i = ib^{k-1}$, then we have*

$$\int_0^1 f(x)\int_0^1 g(x)dx=\sum_{i=1}^{b}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\left(\frac{jk_i}{b^k}+d_i\right)\int_0^1 g(y)dy.$$

Proof. We can write

$$\int_0^1 f(x)dx = \sum_{i=1}^{b} \sum_{j=a_{i-1}}^{a_i-1} \int_{j/b^k}^{(j+1)/b^k} f_i(x)dx$$

with $f_i(x) = k_i x + d_i$. As $f_i(x)$ is a linear function we obtain

$$\int_{j/b^k}^{(j+1)/b^k} f_i(x)dx = \frac{f_i(j/b^k) + f_i((j+1)/b^k)}{2b^k} = \frac{1}{b^k}\left(\frac{jk_i}{b^k} + d_i\right) + \frac{k_i}{2b^{2k}}.$$

According to Lemma 2.5.1 we have

$$\sum_{i=1}^{b} \sum_{j=a_{i-1}}^{a_i-1} \frac{k_i}{2b^{2k}} = \frac{1}{2b^k} \sum_{i=1}^{b} (e_i - e_{i-1})\, k_i = 0.$$

This proves the lemma. □

The proof of the theorem follows directly from the two lemmas.

In general the problem is a little bit more involved. The e_i's are not all of the form i/b, there are some in between too. We define new values a_i in such a way that $a_i/b^k = e_i$ if $e_i = j/b$ for some $0 \leq j \leq b$ and $a_i/b^k < e_i < (a_i+1)/b^k$ in all other cases. We further set $\mathfrak{L} = \{i | e_i \notin \frac{1}{b}\mathbb{Z}\}$, this leads to the following theorem.

Theorem 2.5.2. *Let f,g, e_i, a_i and $\mathfrak{L}$ be defined as above and $k \geq k_0$, then we have*

$$\begin{aligned} R_k = \sum_{i \in \mathfrak{L}} \Bigg\{ &- \left[\left(e_i - \frac{a_i}{b^k}\right)\frac{k_i}{b^k} + \left(\frac{a_i+1}{b^k} - e_i\right)\frac{k_{i+1}}{b^k}\right] \int_0^1 yg(y)dy \\ &+ \frac{k_i}{b^{2k}} \int_0^1 yg(y)\mathbb{1}_{[0,e_i b^k - a_i]}(y)dy + \frac{k_{i+1}}{b^{2k}} \int_0^1 yg(y)\mathbb{1}_{[e_i b^k - a_i, 1]}(y)dy \\ &+ (k_i - k_{i+1})\left(e_i - \frac{a_i}{b^k}\right)\frac{1}{b^k} \int_0^1 g(y)\mathbb{1}_{[e_i b^k - a_i, 1]}(y)dy \\ &+ \left[\frac{1}{2}\left(e_i - \frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k} - e_i\right)(k_{i+1} - k_i)\right] \int_0^1 g(y)dy \Bigg\} \end{aligned}$$

and there exists a sequence c_k that is ultimately periodic with $R_k = \frac{c_k}{b^{2k}}$.

Proof. We will first show the second property follows directly from the representation of R_k. The e_i are all rational numbers and therefore they can be written as

$$e_i = \frac{\epsilon_1}{b} + \frac{\epsilon_2}{b^2} + \frac{\epsilon_3}{b^3} + \cdots$$

where $\epsilon_1, \epsilon_2, \ldots$ is a ultimately periodic sequence with period d. Then the numbers a_i/b^k can be represented by

$$\frac{a_i}{b^k} = \frac{\epsilon_1}{b} + \cdots + \frac{\epsilon_k}{b^k}$$

and we obtain

$$\begin{aligned} e_i - \frac{a_i}{b^k} &= \frac{1}{b^k}\left(\frac{\epsilon_{k+1}}{b^1} + \frac{\epsilon_{k+2}}{b^2} + \cdots\right) \\ &= \frac{z_k^{(i)}}{b^k}. \end{aligned}$$

Further we can write

$$\frac{a_i + 1}{b^k} - e_i = \frac{1 - z_k^{(i)}}{b^k}.$$

Replacing these terms in the formula of R_k we arrive at

$$\begin{aligned} R_k = \frac{1}{b^{2k}} \sum_{i \in \mathfrak{L}} \Bigg\{ &- \left[z_k^{(i)} k_i + \left(1 - z_k^{(i)}\right) k_{i+1}\right] \int_0^1 y g(y) dy \\ &+ k_i \int_0^{z_k^{(i)}} y g(y) dy + k_{i+1} \int_{z_k^{(i)}}^1 y g(y) dy + (k_i - k_{i+1}) z_k^{(i)} \int_{z_k^{(i)}}^1 g(y) dy \\ &+ \left[\frac{1}{2} z_k^{(i)} \left(1 - z_k^{(i)}\right) (k_{i+1} - k_i)\right] \int_0^1 g(y) dy \Bigg\}. \end{aligned}$$

This verifies the second property. The proof of the first property we will split in several lemmas similar to the proof of Theorem 2.5.1. □

Compared to Theorem 2.5.1 the situation is now more complicated because the local extrema of the function f do not coincide with the period of the function g. Before we look at the terms $\int_0^1 f(x) g(b^k x)$ and $\int_0^1 f(x) \int_0^1 g(x)$ we will prove the following three lemmas.

Lemma 2.5.4. *Let $f_i(x) = k_i x + d_i$ then we have for every a_i*

$$f_{i+1}\left(\frac{a_i}{b^k}\right) = f_i\left(\frac{a_i}{b^k}\right) + \left(e_i - \frac{a_i}{b^k}\right) k_i + \left(\frac{a_i}{b^k} - e_i\right) k_{i+1}.$$

Proof. Using Lemma 2.5.1 we get

$$\begin{aligned}
\frac{a_i}{b^k}k_{i+1} + d_{i+1} &= \frac{a_i}{b^k}k_{i+1} + \sum_{l=1}^{i}(k_l - k_{l+1})e_l \\
&= e_i k_i + \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l + \left(\frac{a_i}{b^k} - e_i\right) k_{i+1} \\
&= \frac{a_i}{b^k}k_i + \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l + \left(e_i - \frac{a_i}{b^k}\right) k_i + \left(\frac{a_i}{b^k} - e_i\right) k_{i+1}
\end{aligned}$$

With the help of the same lemma, we arrive at

$$\frac{a_i}{b^k}k_{i+1} + d_{i+1} = \frac{a_i}{b^k}k_i + d_i + \left(e_i - \frac{a_i}{b^k}\right) k_i + \left(\frac{a_i}{b^k} - e_i\right) k_{i+1}$$

and obtain the desired result. □

Lemma 2.5.5. *Let $\mathfrak{L}$, a_i, e_i and k_i be defined as above, then we have*

$$\begin{aligned}
\sum_{(i-1)\in\mathfrak{L}}(a_i &- (a_{i-1}+1))\frac{k_i}{b^k} + \sum_{(i-1)\notin\mathfrak{L}}(a_i - a_{i-1})\frac{k_i}{b^k} = \\
&- \sum_{(i-1)\in\mathfrak{L}}\left[\left(e_{i-1} - \frac{a_{i-1}}{b^k}\right) k_{i-1} + \left(\frac{a_{i-1}+1}{b^k} - e_{i-1}\right) k_i\right]
\end{aligned}$$

Proof. We will show

$$\begin{aligned}
\sum_{(i-1)\in\mathfrak{L}}(a_i - (a_{i-1}+1))\frac{k_i}{b^k} &+ \left(e_{i-1} - \frac{a_{i-1}}{b^k}\right) k_{i-1} + \left(\frac{a_{i-1}+1}{b^k} - e_{i-1}\right) k_i \\
&+ \sum_{(i-1)\notin\mathfrak{L}}(a_i - a_{i-1})\frac{k_i}{b^k} = 0
\end{aligned}$$

This follows directly from the first property of Lemma 2.5.1. □

Lemma 2.5.6. *For f_i, a_i and e_i defined as above we have*

$$f_i\left(\frac{a_i}{b^k}\right)\frac{1}{b^k} - f_i\left(\frac{a_i}{b^k}\right)\left(e_i - \frac{a_i}{b^k}\right) - f_{i+1}(e_i)\left(\frac{a_i+1}{b^k} - e_i\right) =$$

$$= -k_i\left(e_i - \frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k} - e_i\right).$$

Proof. Obviously we have

$$f_i\left(\frac{a_i}{b^k}\right)\frac{1}{b^k} - f_i\left(\frac{a_i}{b^k}\right)\left(e_i - \frac{a_i}{b^k}\right) - f_{i+1}\left(e_i\right)\left(\frac{a_i+1}{b^k} - e_i\right)$$
$$= -\left[f_{i+1}\left(e_i\right) - f_i\left(\frac{a_i}{b^k}\right)\right]\left(\frac{a_i+1}{b^k} - e_i\right).$$

According to Lemma 2.5.1

$$d_{i+1} - d_i = (k_i - k_{i+1})e_i,$$

this leads to

$$f_i\left(\frac{a_i}{b^k}\right)\frac{1}{b^k} - f_i\left(\frac{a_i}{b^k}\right)\left(e_i - \frac{a_i}{b^k}\right) - f_{i+1}\left(\frac{a_i+1}{b^k}\right)\left(\frac{a_i+1}{b^k} - e_i\right) =$$
$$= -\left(\left(e_i - \frac{a_i}{b^k}\right)k_i\right)\left(\frac{a_i+1}{b^k} - e_i\right).$$

□

Now we are able to verify the formulas we need in order to prove the theorem. We will start with the integral of the product of the two functions.

Lemma 2.5.7. *For the integral of the product of the functions $f(x)$ and $g(b^k x)$ we have*

$$\int_0^1 f(x)g(b^k x)dx =$$
$$= \sum_{(i-1)\notin\mathfrak{L}}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy$$
$$+ \sum_{(i-1)\in\mathfrak{L}}\sum_{j=a_{i-1}+1}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy$$
$$+ \sum_{i\in\mathfrak{L}}\left[\frac{1}{b^k}\int_0^1\left(\frac{a_i}{b^k}k_i+d_i\right)g(y)dy + \frac{1}{b^k}\int_0^1\frac{y}{b^k}k_i g(y)\mathbb{1}_{[0,e_i b^k-a_i]}(y)dy\right.$$
$$\left.+ \frac{1}{b^k}\int_0^1\left(\frac{y}{b^k}k_{i+1}+\left(e_i-\frac{a_i}{b^k}\right)(k_i-k_{i+1})\right)g(y)\mathbb{1}_{[e_i b^k-a_i,1]}(y)dy\right].$$

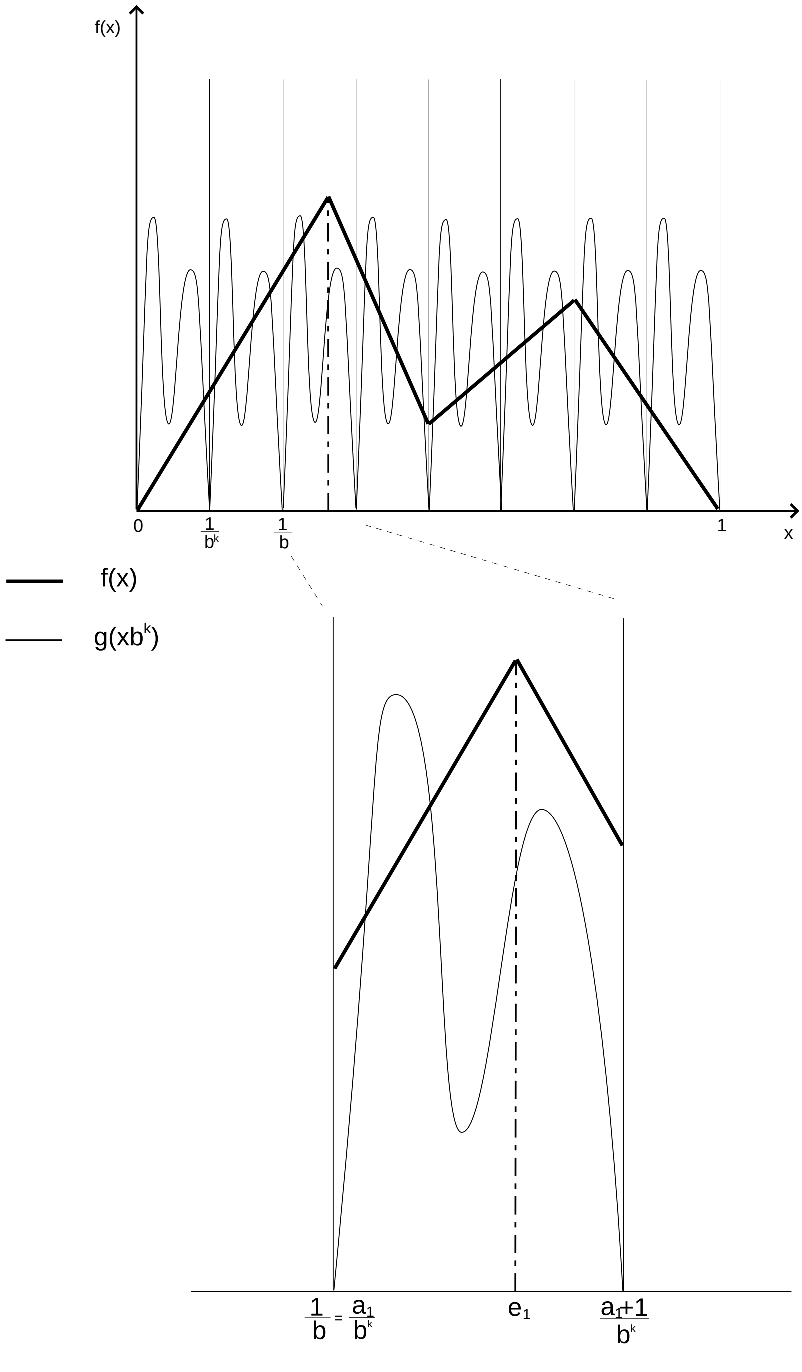

Figure 2.2: Splitting of the integral

Proof. We will try to apply the same strategy we used in Lemma 2.5.2. We want to split the integral into subintegrals in such a way, the function g runs through a whole period on each integration interval. The problem we are facing now is, the function f does not only have changes from one f_i to the next at points a_i/b^k, see Figure 2.2 . This leads to the following construction principal for the integral $\int_0^1 f(x)g(b^k x)dx$

$$\begin{aligned}\int_0^1 f(x)g(b^k x)dx &= \sum_{(i-1)\notin\mathfrak{L}}\sum_{j=a_{i-1}}^{a_i-1}\int_{j/b^k}^{(j+1)/b^k} f_i(x)g(b^k x-j)dx \\ &+ \sum_{(i-1)\in\mathfrak{L}}\sum_{j=a_{i-1}+1}^{a_i-1}\int_{j/b^k}^{(j+1)/b^k} f_i(x)g(b^k x-j)dx \\ &+ \sum_{i\in\mathfrak{L}}\left(\int_{a_i/b^k}^{e_i} f_i(x)g(b^k x-a_i)dx\right. \\ &+ \left.\int_{e_i}^{(a_i+1)/b^k} f_{i+1}(x)g(b^k x-a_i)dx\right)\end{aligned}$$

Substituting $y = xb^k - j$ and applying Lemma 2.5.4 we arrive at

$$\begin{aligned}&\int_0^1 f(x)g(b^k x)dx = \\ &= \sum_{(i-1)\notin\mathfrak{L}}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy \\ &+ \sum_{(i-1)\in\mathfrak{L}}\sum_{j=a_{i-1}+1}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy \\ &+ \sum_{i\in\mathfrak{L}}\left[\frac{1}{b^k}\int_0^1\left(\frac{a_i}{b^k}k_i+d_i\right)g(y)dy+\frac{1}{b^k}\int_0^1\frac{y}{b^k}k_i g(y)\mathbb{1}_{[0,e_ib^k-a_i]}(y)dy\right. \\ &+ \left.\frac{1}{b^k}\int_0^1\left(\frac{y}{b^k}k_{i+1}+\left(e_i-\frac{a_i}{b^k}\right)(k_i-k_{i+1})\right)g(y)\mathbb{1}_{[e_ib^k-a_i,1]}(y)dy\right].\end{aligned}$$

This is the desired result. □

For the formula of the product of the integrals of the two functions, we will again decompose the function f in triangles and rectangles. Similar to the previous lemma, we now have the problem, not all elements have the same length $1/b^k$.

Lemma 2.5.8. *For the product of the integral of the functions $f(x)$ and $g(x)$, we have*

$$\int_0^1 f(x)dx \int_0^1 g(x)dx =$$

$$= \Bigg[\sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(k_i\frac{j}{b^k}+d_i\right) + \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k}\left(k_i\frac{j}{b^k}+d_i\right)$$

$$+\sum_{i\in\mathfrak{L}}\left(\left(k_i\frac{a_i}{b^k}+d_i\right)\left(e_i-\frac{a_i}{b^k}\right)+(k_{i+1}e_i+d_{i+1})\left(\frac{a_i+1}{b^k}-e_i\right)\right)$$

$$+\sum_{(i-1)\in\mathfrak{L}}(a_i-a_{i-1}-1)\frac{k_i}{2b^{2k}}+\sum_{(i-1)\notin\mathfrak{L}}(a_i-a_{i-1})\frac{k_i}{2b^{2k}}$$

$$+\sum_{i\in\mathfrak{L}}\left(\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)^2+\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)^2\right)\Bigg]\int_0^1 g(y)dy.$$

Proof. Decomposing $f(x)$ in triangles and rectangles according to Figure 2.3 one gets

$$\int_0^1 f(x)dx \int_0^1 g(x)dx = \tag{2.12}$$

$$= \Bigg[\sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i>0}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k} f_i\left(\frac{j}{b^k}\right) + \sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i<0}} \sum_{j=a_{i-1}+1}^{a_i} \frac{1}{b^k} f_i\left(\frac{j}{b^k}\right) \tag{2.13}$$

$$+ \sum_{\substack{(i-1)\in\mathfrak{L}\\ k_i>0}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k} f_i\left(\frac{j}{b^k}\right) + \sum_{\substack{(i-1)\in\mathfrak{L}\\ k_i<0}} \sum_{j=a_{i-1}+2}^{a_i} \frac{1}{b^k} f_i\left(\frac{j}{b^k}\right) \tag{2.14}$$

$$+\sum_{i\in\mathfrak{L}}\left(f_i\left(\frac{a_i}{b^k}\right)\left(e_i-\frac{a_i}{b^k}\right)+f_{i+1}\left(e_i\right)\left(\frac{a_i+1}{b^k}-e_i\right)\right) \tag{2.15}$$

$$+\sum_{(i-1)\notin\mathfrak{L}}(a_i-a_{i-1})\frac{|k_i|}{2b^{2k}}+\sum_{(i-1)\in\mathfrak{L}}(a_i-a_{i-1}-1)\frac{|k_i|}{2b^{2k}} \tag{2.16}$$

$$+\sum_{i\in\mathfrak{L}}\left(\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)^2+\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)^2\right)\Bigg]\int_0^1 g(y)dy. \tag{2.17}$$

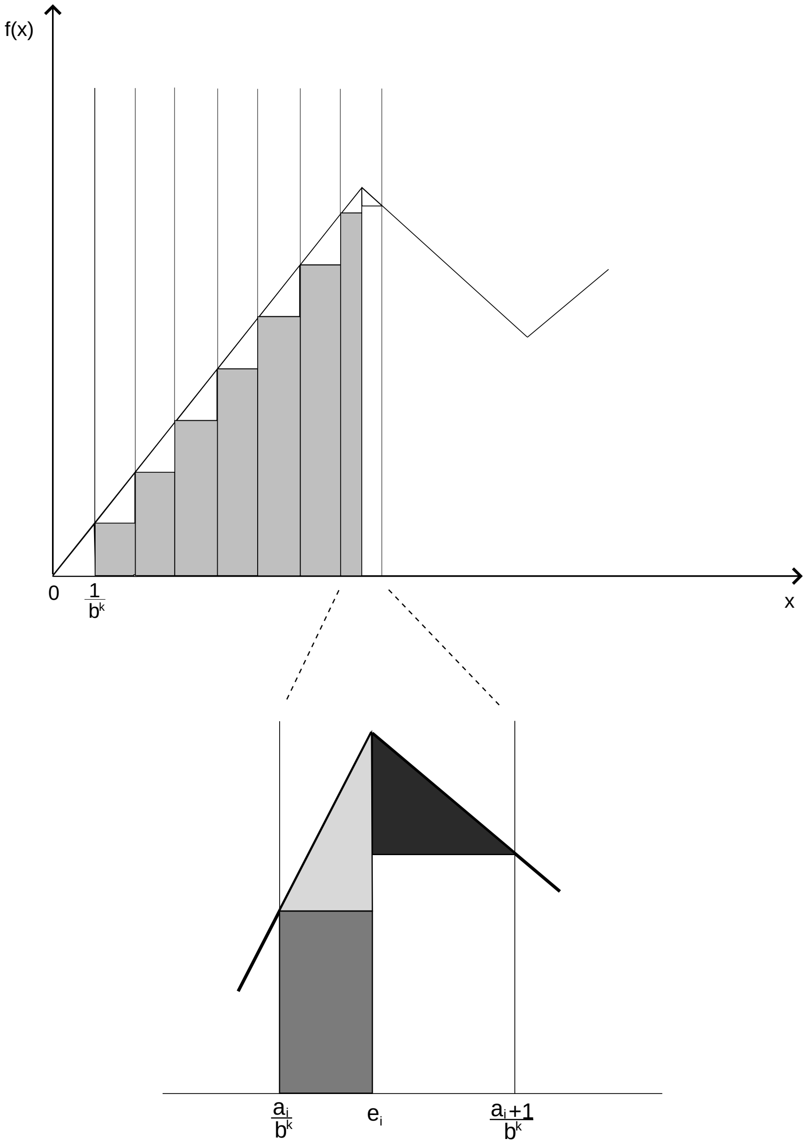

Figure 2.3: Decomposition of the integral in triangles and rectangles

The two sums of 2.13 differ only in their range of summation. If we add for $k_i < 0$ the terms for a_{i-1} and sum only up to $a_i - 1$, we get

$$\sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i>0}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k} f_i\left(\frac{j}{b^k}\right) + \sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i<0}} \sum_{j=a_{i-1}+1}^{a_i} \frac{1}{b^k} f_i\left(\frac{j}{b^k}\right) =$$

$$= \sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(k_i \frac{j}{b^k} + d_i\right) + \sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i<0}} (a_i - a_{i-1}) \frac{k_i}{b^{2k}}.$$

By combining the last sum

$$\sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i<0}} (a_i - a_{i-1}) \frac{k_i}{b^{2k}}$$

with the first sum in line 2.16 we arrive at

$$\sum_{(i-1)\notin\mathfrak{L}} (a_i - a_{i-1}) \frac{|k_i|}{2b^{2k}} + \sum_{\substack{(i-1)\notin\mathfrak{L}\\ k_i<0}} (a_i - a_{i-1}) \frac{k_i}{b^{2k}} = \sum_{(i-1)\notin\mathfrak{L}} (a_i - a_{i-1}) \frac{k_i}{2b^{2k}}.$$

A similar result holds true for line 2.14. So one finally obtains the result

$$\begin{aligned}
&\int_0^1 f(x)dx \int_0^1 g(x)dx = \\
&= \Bigg[\sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(k_i \frac{j}{b^k} + d_i\right) + \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k}\left(k_i \frac{j}{b^k} + d_i\right) \\
&+ \sum_{i\in\mathfrak{L}} \left(\left(k_i \frac{a_i}{b^k} + d_i\right)\left(e_i - \frac{a_i}{b^k}\right) + (k_{i+1} e_i + d_{i+1})\left(\frac{a_i+1}{b^k} - e_i\right) \right) \\
&+ \sum_{(i-1)\in\mathfrak{L}} (a_i - a_{i-1} - 1) \frac{k_i}{2b^{2k}} + \sum_{(i-1)\notin\mathfrak{L}} (a_i - a_{i-1}) \frac{k_i}{2b^{2k}} \\
&+ \sum_{i\in\mathfrak{L}} \left(\frac{k_i}{2}\left(e_i - \frac{a_i}{b^k}\right)^2 + \frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k} - e_i\right)^2 \right) \Bigg] \int_0^1 g(y)dy. \qquad \square
\end{aligned}$$

Now we can prove Theorem 2.5.2.

Proof. According to Lemma 2.5.8 and 2.5.7 we have

$$\int_0^1 f(x)g(xb^k)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx = \tag{2.18}$$

$$= \sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k} \int_0^1 \left[k_i \left(\frac{y+j}{b^k}\right) + d_i\right] g(y)dy \tag{2.19}$$

$$+ \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k} \int_0^1 \left[k_i \left(\frac{y+j}{b^k}\right) + d_i\right] g(y)dy \tag{2.20}$$

$$+ \sum_{i\in\mathfrak{L}} \left[\frac{1}{b^k} \int_0^1 \left(\frac{a_i}{b^k} k_i + d_i\right) g(y)dy + \frac{1}{b^k} \int_0^1 \frac{y}{b^k} k_i g(y) \mathbb{1}_{[0, e_i b^k - a_i]}(y)dy \right. \tag{2.21}$$

$$\left. + \frac{1}{b^k} \int_0^1 \left(\frac{y}{b^k} k_{i+1} + \left(e_i - \frac{a_i}{b^k}\right)(k_i - k_{i+1})\right) g(y) \mathbb{1}_{[e_i b^k - a_i, 1]}(y)dy\right] \tag{2.22}$$

$$- \left[\sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(k_i \frac{j}{b^k} + d_i\right) + \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k}\left(k_i \frac{j}{b^k} + d_i\right)\right. \tag{2.23}$$

$$+ \sum_{i\in\mathfrak{L}} \left(\left(k_i \frac{a_i}{b^k} + d_i\right)\left(e_i - \frac{a_i}{b^k}\right) + \left(k_{i+1} e_i b^k + d_{i+1}\right)\left(\frac{a_i+1}{b^k} - e_i\right)\right) \tag{2.24}$$

$$+ \sum_{(i-1)\in\mathfrak{L}} (a_i - a_{i-1} - 1)\frac{k_i}{2b^{2k}} + \sum_{(i-1)\notin\mathfrak{L}} (a_i - a_{i-1})\frac{k_i}{2b^{2k}} \tag{2.25}$$

$$\left. + \sum_{i\in\mathfrak{L}} \left(\frac{k_i}{2}\left(e_i - \frac{a_i}{b^k}\right)^2 + \frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k} - e_i\right)^2\right)\right] \int_0^1 g(y)dy. \tag{2.26}$$

Combining lines 2.19, 2.20 and 2.23 and applying Lemma 2.5.5 we get

$$\left[\sum_{(i-1)\notin\mathfrak{L}} (a_i - a_{i-1})\frac{k_i}{b^{2k}} + \sum_{(i-1)\in\mathfrak{L}} (a_i - (a_{i-1}+1))\frac{k_i}{b^{2k}}\right] \int_0^1 yg(y)dy =$$

$$= - \sum_{(i-1)\in\mathfrak{L}} \left[\left(e_{i-1} - \frac{a_{i-1}}{b^k}\right)\frac{k_{i-1}}{b^k} + \left(\frac{a_{i-1}+1}{b^k} - e_{i-1}\right)\frac{k_i}{b^k}\right] \int_0^1 yg(y)dy.$$

The same formula holds true for line 2.25. By using Lemma 2.5.6 we obtain for the first term of line 2.21 together with line 2.24

$$-k_i \left(e_i - \frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k} - e_i\right) \int_0^1 g(y)dy.$$

If we combine this result with line 2.25 and 2.26, we get

$$\Bigg[-k_i\left(e_i-\frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k}-e_i\right)+\frac{k_i}{2b^k}\left(e_i-\frac{a_i}{b^k}\right)+\frac{k_{i+1}}{2b^k}\left(\frac{a_i+1}{b^k}-e_i\right)$$
$$-\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)^2-\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)^2\Bigg]\int_0^1 g(y)dy=$$
$$=\Bigg[k_i\left(e_i-\frac{a_i}{b^k}\right)\left(-\frac{a_i}{b^k}-\frac{1}{b^k}+e_i+\frac{1}{2b^k}-\frac{e_i}{2}+\frac{a_i}{2b^k}\right)$$
$$+\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)\left(\frac{1}{b^k}-\frac{a_i}{b^k}-\frac{1}{b^k}+e_i\right)\Bigg]\int_0^1 g(y)dy=$$
$$=\left[\frac{k_{i+1}}{2}\left(e_i-\frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k}-e_i\right)-\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k}-e_i\right)\right]\int_0^1 g(y)dy.$$

So we finally arrive at

$$R_k(x)=\sum_{i\in\mathfrak{L}}\Bigg\{-\left[\left(e_i-\frac{a_i}{b^k}\right)\frac{k_i}{b^k}+\left(\frac{a_i+1}{b^k}-e_i\right)\frac{k_{i+1}}{b^k}\right]\int_0^1 yg(y)dy$$
$$+\frac{k_i}{b^{2k}}\int_0^1 yg(y)\mathbb{1}_{[0,e_ib^k-a_i]}(y)dy+\frac{k_{i+1}}{b^{2k}}\int_0^1 yg(y)\mathbb{1}_{[e_ib^k-a_i,1]}(y)dy$$
$$+(k_i-k_{i+1})\left(e_i-\frac{a_i}{b^k}\right)\frac{1}{b^k}\int_0^1 g(y)\mathbb{1}_{[e_ib^k-a_i,1]}(y)dy$$
$$+\left[\frac{1}{2}\left(e_i-\frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k}-e_i\right)(k_{i+1}-k_i)\right]\int_0^1 g(y)dy\Bigg\}$$
$$=O(b^{-2k}).$$

The last equality is due to the fact that $\left(e_i-\frac{a_i}{b^k}\right)<\frac{1}{b^k}$ and $\left(\frac{a_i+1}{b^k}-e_i\right)<\frac{1}{b^k}$.□

In Theorem 2.5.2 we stated that $k\geq k_0$, though we never needed this requirement explicitly in the proof. We made the assumption, that it is always possible to write $a_i/b^k<e_i<(a_i+1)/b^k$ for exact one e_i. This is only possible if k is large enough. We will now discuss what happens if we have $a_i/b^k<e_i^{(1)}<e_i^{(2)}<\cdots<e_i^{(n_i)}<(a_i+1)/b^k$. The definition of the set $\mathfrak{L}$ is in principal the same as before but now $e_i\in[0,1)^{n_i}$. Further, we will write $e_i^{(1)}=e_i$, $k_i^{(1)}=k_i$, $e_i^{(n_i+1)}=e_{i+1}$ and $k_i^{(n_i+1)}=k_{i+1}$. If all n_i are equal to 1 we are in the situation of Theorem 2.5.2. We want to prove the lemmas we need in the new context first, before we reformulate the theorem.

Lemma 2.5.9. *Let $f_i(x) = k_i x + d_i$ then we have for every a_i*

$$f_{i+1}\left(\frac{a_i}{b^k}\right) = f_i\left(\frac{a_i}{b^k}\right) + \sum_{j=1}^{n}\left(e_i^{(j)} - \frac{a_i}{b^k}\right)k_i^{(j)} + \left(\frac{a_i}{b^k} - e_i^{(j)}\right)k_i^{(j+1)}.$$

Proof. Using Lemma 2.5.1 we get

$$\begin{aligned}
\frac{a_i}{b^k}k_{i+1} + d_{i+1} &= \frac{a_i}{b^k}k_{i+1} + \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l + \sum_{j=1}^{n}\left(k_i^{(j)} - k_i^{(j+1)}\right)e_i^{(j)} \\
&= e_i^{(n)}k_i^{(n)} + \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l + \sum_{j=1}^{n-1}\left(k_i^{(j)} - k_i^{(j+1)}\right)e_i^{(j)} + \left(\frac{a_i}{b^k} - e_i^{(n)}\right)k_{i+1} \\
&= \frac{a_i}{b^k}k_i^{(n)} + \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l + \sum_{j=1}^{n-1}\left(k_i^{(j)} - k_i^{(j+1)}\right)e_i^{(j)} \\
&+ \left(e_i^{(n)} - \frac{a_i}{b^k}\right)k_i^{(n)} + \left(\frac{a_i}{b^k} - e_i^{(n)}\right)k_{i+1} \\
&= \frac{a_i}{b^k}k_i + \sum_{l=1}^{i-1}(k_l - k_{l+1})e_l + \sum_{j=1}^{n}\left(e_i^{(j)} - \frac{a_i}{b^k}\right)k_i^{(j)} + \left(\frac{a_i}{b^k} - e_i^{(j)}\right)k_i^{(j+1)}.
\end{aligned}$$

With the help of the same lemma, we arrive at

$$\frac{a_i}{b^k}k_{i+1} + d_{i+1} = \frac{a_i}{b^k}k_i + d_i + \sum_{j=1}^{n}\left(e_i^{(j)} - \frac{a_i}{b^k}\right)k_i^{(j)} + \left(\frac{a_i}{b^k} - e_i^{(j)}\right)k_i^{(j+1)}$$

and obtain the desired result. □

Lemma 2.5.10. *Let $\mathfrak{L}$, a_i, e_i and k_i be defined as above, then we have*

$$\begin{aligned}
&\sum_{(i-1)\in\mathfrak{L}}(a_i - (a_{i-1}+1))\frac{k_i}{b^k} + \sum_{(i-1)\notin\mathfrak{L}}(a_i - a_{i-1})\frac{k_i}{b^k} = \\
&-\sum_{i\in\mathfrak{L}}\left[\left(e_i - \frac{a_i}{b^k}\right)k_i + \left(\frac{a_i+1}{b^k} - e_i\right)k_{i+1} + \sum_{j=1}^{n_i-1}\left(e_i^{(j+1)} - e_i^{(j)}\right)k_i^{j+1}\right].
\end{aligned}$$

Proof. As in the case of Lemma 2.5.5, this follows directly from the first property of Lemma 2.5.1. □

Lemma 2.5.11. *For f_i, a_i and e_i defined as above we have*

$$f_i\left(\frac{a_i}{b^k}\right)\frac{1}{b^k} - f_i\left(\frac{a_i}{b^k}\right)\left(e_i - \frac{a_i}{b^k}\right) - f_{i+1}\left(e_i^{(n_i)}\right)\left(\frac{a_i+1}{b^k} - e_i^{(n_i)}\right)$$
$$+ \sum_{j=1}^{n_i-1} f_i^{(j+1)}\left(e_i^{(j)}\right)\left(e_i^{(j+1)} - e_i(j)\right) =$$
$$= -\left\{\sum_{j=1}^{n_i-1}\left[\left(k_i^{(1)}\left(e_i^{(1)} - \frac{a_i}{b^k}\right) + \sum_{l=1}^{j-1} k_i^{(l+1)}\left(e_i^{(l+1)} - e_i^{(l)}\right)\right)\left(e_i^{(j+1)} - e_i^{(j)}\right)\right]\right.$$
$$\left. + \left[k_i^{(1)}\left(e_i^{(1)} - \frac{a_i}{b^k}\right) + \sum_{l=1}^{n_i-1} k_i^{(l+1)}\left(e_i^{(l+1)} - e_i^{(l)}\right)\right]\left(\frac{a_i+1}{b^k} - e_i^{(n_i)}\right)\right\}.$$

Proof. First we observe that we can write

$$\frac{1}{b^k} = \left(e_i - \frac{a_i}{b^k}\right) + \sum_{j=1}^{n_i-1}\left(e_i^{(j+1)} - e_i^{(j)}\right) + \left(\frac{a_i+1}{b^k} - e_i^{(n_i)}\right).$$

So we get

$$f_i\left(\frac{a_i}{b^k}\right)\frac{1}{b^k} - f_i\left(\frac{a_i}{b^k}\right)\left(e_i - \frac{a_i}{b^k}\right) - f_{i+1}\left(e_i^{(n_i)}\right)\left(\frac{a_i+1}{b^k} - e_i^{(n_i)}\right)$$
$$+ \sum_{j=1}^{n_i-1} f_i^{(j+1)}\left(e_i^{(j)}\right)\left(e_i^{(j+1)} - e_i(j)\right) =$$
$$= -\left\{\sum_{j=1}^{n_i-1}\left[\left(f_i^{(j+1)}\left(e_i^{(j)}\right) - f_i\left(\frac{a_i}{b^k}\right)\right)\left(e_i^{(j+1)} - e_i(j)\right)\right]\right.$$
$$\left. + \left(f_{i+1}\left(e_i^{(n_i)}\right) - f_i\left(\frac{a_i}{b^k}\right)\right)\left(\frac{a_i+1}{b^k} - e_i^{(n_i)}\right)\right\}.$$

If we set $f_i(x) = f_i^{(1)}(x) = k_i^{(1)}x + d_i^{(1)}$, we have according to Lemma 2.5.1

$$d_i^{(j+1)} - d_i^{(1)} = \sum_{l=1}^{j}(k_i^{(l)} - k_i^{(l+1)})e_i^{(l)}.$$

Using this result, the terms $f_i^{(j+1)}(e_i^{(j)}) - f_i\left(\frac{a_i}{b^k}\right)$ can be written as

$$k_i^{(1)}\left(e_i^{(1)} - \frac{a_i}{b^k}\right) + \sum_{l=1}^{j-1} k_i^{(l+1)}\left(e_i^{(l+1)} - e_i^{(l)}\right).$$

The same we can apply to the term $f_{i+1}\left(e_i^{(n_i)}\right) - f_i\left(\frac{a_i}{b^k}\right)$ which leads to the desired result. $\square$

Now we can verify the modified formula for the integral of the product of the functions f and g.

Lemma 2.5.12. *For the integral of the product of the functions $f(x)$ and $g(b^k x)$ we have*

$$\int_0^1 f(x)g(b^k x)dx =$$

$$= \sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k} \int_0^1 \left[k_i\left(\frac{y+j}{b^k}\right) + d_i\right] g(y)dy$$

$$+ \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k} \int_0^1 \left[k_i\left(\frac{y+j}{b^k}\right) + d_i\right] g(y)dy$$

$$+ \sum_{i\in\mathfrak{L}} \left\{ \frac{1}{b^k} \int_0^1 \left(\frac{a_i}{b^k}k_i + d_i\right) g(y)dy + \frac{1}{b^k} \int_0^1 \frac{y}{b^k} k_i g(y) \mathbb{1}_{[0, e_i b^k - a_i]}(y)dy\right.$$

$$+ \frac{1}{b^k} \sum_{j=1}^{n_i-1} \left[\int_0^1 \left(\frac{y}{b^k} k_i^{(j+1)} + \sum_{l=1}^{j} \left[\left(e_i^{(l)} - \frac{a_i}{b^k}\right)\left(k_i^{(l)} - k_i^{(l+1)}\right)\right]\right) \times\right.$$

$$\left. \times\, g(y)\mathbb{1}_{[e_i^{(j)} b^k - a_i, e_i^{(j+1)} b^k - a_i]}(y)dy \right] + \frac{1}{b^k} \int_0^1 \left(\frac{y}{b^k} k_{i+1} + \right.$$

$$\left.\left. + \sum_{l=1}^{n_i} \left[\left(e_i^{(l)} - \frac{a_i}{b^k}\right)\left(k_i^{(l)} - k_i^{(l+1)}\right)\right]\right) g(y)\mathbb{1}_{[e_i^{(n_i)} b^k - a_i, 1]} dy \right\}.$$

Proof. We will use the same principals as in the proof of Lemma 2.5.7, but now we have to deal with the extra $e_i^{(j)}$.

$$\int_0^1 f(x)g(b^k x)dx = \sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \int_{j/b^k}^{(j+1)/b^k} f_i(x)g(b^k x - j)dx$$

$$+ \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \int_{j/b^k}^{(j+1)/b^k} f_i(x)g(b^k x - j)dx$$

$$+\sum_{i\in\mathfrak{L}}\left(\int_{a_i/b^k}^{e_i} f_i(x)g(b^kx-a_i)dx+\sum_{j=1}^{n_i-1}\left(\int_{e_i^{(j)}}^{e_i^{(j+1)}} f_i^{(j)}(x)g(b^kx-a_i)dx\right)\right.$$
$$\left.+\int_{e_i}^{(a_i+1)/b^k} f_{i+1}(x)g(b^kx-a_i)dx\right)$$

Substituting $y=xb^k-j$ and applying Lemma 2.5.9 we arrive at

$$\int_0^1 f(x)g(b^kx)dx=$$
$$=\sum_{(i-1)\notin\mathfrak{L}}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy$$
$$+\sum_{(i-1)\in\mathfrak{L}}\sum_{j=a_{i-1}+1}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy$$
$$+\sum_{i\in\mathfrak{L}}\left\{\frac{1}{b^k}\int_0^1\left(\frac{a_i}{b^k}k_i+d_i\right)g(y)dy+\frac{1}{b^k}\int_0^1\frac{y}{b^k}k_ig(y)\mathbb{1}_{[0,e_ib^k-a_i]}(y)dy\right.$$
$$+\frac{1}{b^k}\sum_{j=1}^{n_i-1}\left[\int_0^1\left(\frac{y}{b^k}k_i^{(j+1)}+\sum_{l=1}^{j}\left[\left(e_i^{(l)}-\frac{a_i}{b^k}\right)\left(k_i^{(l)}-k_i^{(l+1)}\right)\right]\right)\times\right.$$
$$\left.\times\, g(y)\mathbb{1}_{[e_i^{(j)}b^k-a_i,e_i^{(j+1)}b^k-a_i]}(y)dy\right]+\frac{1}{b^k}\int_0^1\left(\frac{y}{b^k}k_{i+1}+\right.$$
$$\left.\left.+\sum_{l=1}^{n_i}\left[\left(e_i^{(l)}-\frac{a_i}{b^k}\right)\left(k_i^{(l)}-k_i^{(l+1)}\right)\right]\right)g(y)\mathbb{1}_{[e_i^{(n_i)}b^k-a_i,1]}dy\right\}.$$

This is the desired result. □

We also obtain a new formula for the product of the integrals of f and g, but in this case the proof is essentially the same as in the first case, we only add some more terms.

Lemma 2.5.13. *For the product of the integral of the functions $f(x)$ and $g(x)$, we have*

$$\int_0^1 f(x)dx\int_0^1 g(x)dx=$$

$$= \left[\sum_{(i-1)\notin\mathfrak{L}} \sum_{j=a_{i-1}}^{a_i-1} \frac{1}{b^k}\left(k_i\frac{j}{b^k}+d_i\right) + \sum_{(i-1)\in\mathfrak{L}} \sum_{j=a_{i-1}+1}^{a_i-1} \frac{1}{b^k}\left(k_i\frac{j}{b^k}+d_i\right) \right.$$

$$+\sum_{i\in\mathfrak{L}}\left(f_i\left(\frac{a_i}{b^k}\right)\left(e_i-\frac{a_i}{b^k}\right)+f_{i+1}\left(e_i^{(n_i)}\right)\left(\frac{a_i+1}{b^k}-e_i^{(n_i)}\right)+\right.$$

$$\left.+\sum_{j=1}^{n_i-1} f_i^{(j+1)}\left(e_i^{(j)}\right)\left(e_i^{(j+1)}-e_i^{(j)}\right)\right)+\sum_{(i-1)\in\mathfrak{L}}(a_i-a_{i-1}-1)\frac{k_i}{2b^{2k}}+$$

$$+\sum_{(i-1)\notin\mathfrak{L}}(a_i-a_{i-1})\frac{k_i}{2b^{2k}}+\sum_{i\in\mathfrak{L}}\left(\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)^2+\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)^2+\right.$$

$$\left.\left.+\sum_{j=1}^{n_i-1}\frac{k_i^{(j+1)}}{2}\left(e_i^{(j+1)}-e_i^{(j)}\right)^2\right)\right]\int_0^1 g(y)dy.$$

Finally we are able to state the theorem.

Theorem 2.5.3. *Let f,g, e_i, a_i and $\mathfrak{L}$ be defined as above, then we have*

$$R_k = \sum_{i\in\mathfrak{L}}\left\{-\left[\left(e_i-\frac{a_i}{b^k}\right)\frac{k_i}{b^k}+\left(\frac{a_i+1}{b^k}-e_i\right)\frac{k_{i+1}}{b^k}\right.\right.$$

$$\left.+\sum_{j=1}^{n_i-1}\left(e_i^{(j+1)}-e_i^{(j)}\right)\frac{k_i^{j+1}}{b^k}\right]\int_0^1 yg(y)dy+\frac{1}{b^k}\int_0^1\frac{y}{b^k}k_i g(y)\mathbb{1}_{[0,e_ib^k-a_i]}(y)dy$$

$$+\frac{1}{b^k}\sum_{j=1}^{n_i-1}\left[\int_0^1\left(\frac{y}{b^k}k_i^{(j+1)}+\sum_{l=1}^{j}\left[\left(e_i^{(l)}-\frac{a_i}{b^k}\right)\left(k_i^{(l)}-k_i^{(l+1)}\right)\right]\right)\times\right.$$

$$\left.\times\, g(y)\mathbb{1}_{[e_i^{(j)}b^k-a_i,e_i^{(j+1)}b^k-a_i]}(y)dy\right]+\frac{1}{b^k}\int_0^1\left(\frac{y}{b^k}k_{i+1}+\right.$$

$$\left.+\sum_{l=1}^{n_i}\left[\left(e_i^{(l)}-\frac{a_i}{b^k}\right)\left(k_i^{(l)}-k_i^{(l+1)}\right)\right]\right)g(y)\mathbb{1}_{[e_i^{(n_i)}b^k-a_i,1]}dy$$

$$\left\{-\left[\sum_{j=1}^{n_i-1}\left[\left(k_i^{(1)}\left(e_i^{(1)}-\frac{a_i}{b^k}\right)+\sum_{l=1}^{j-1}k_i^{(l+1)}\left(e_i^{(l+1)}-e_i^{(l)}\right)\right)\left(e_i^{(j+1)}-e_i^{(j)}\right)\right]\right.\right.$$

$$\left.+\left[k_i^{(1)}\left(e_i^{(1)}-\frac{a_i}{b^k}\right)+\sum_{l=1}^{n_i-1}k_i^{(l+1)}\left(e_i^{(l+1)}-e_i^{(l)}\right)\right]\left(\frac{a_i+1}{b^k}-e_i^{(n_i)}\right)\right]$$

$$+\sum_{i\in\mathfrak{L}}\left[\left(e_i-\frac{a_i}{b^k}\right)\frac{k_i}{b^k}+\left(\frac{a_i+1}{b^k}-e_i\right)\frac{k_{i+1}}{b^k}+\sum_{j=1}^{n_i-1}\left(e_i^{(j+1)}-e_i^{(j)}\right)\frac{k_i^{j+1}}{b^k}\right]$$
$$-\left(\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)^2+\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)^2\right.$$
$$\left.\left.-\sum_{j=1}^{n_i-1}\frac{k_i^{(j+1)}}{2}\left(e_i^{(j+1)}-e_i^{(j)}\right)^2\right)\right\}\int_0^1 g(y)dy\Bigg\}$$

and there exists a sequence c_k that is ultimately periodic with $R_k = \frac{c_k}{b^2k}$.

Proof. Similar to the corresponding theorem, we will first show the second property. For each a_i/b^k there exists a representation

$$\frac{a_i}{b^k}=\frac{\epsilon_1}{b}+\cdots+\frac{\epsilon_k}{b^k}$$

and for each $e_i^{(j)}$ we have

$$e_i^{(j)}=\frac{\epsilon_1}{b}+\cdots+\frac{\epsilon_k}{b^k}+\frac{\epsilon_{k+1}^{(j)}}{b^{k+1}}+\frac{\epsilon_{k+2}^{(j)}}{b^{k+2}}+\cdots.$$

In addition to the differences $e_i-\frac{a_i}{b^k}$ and $\frac{a_i+1}{b^k}$ we need to consider

$$e_i^{(j+1)}-e_i^{(j)}=\frac{1}{b^k}\left(\frac{\epsilon_{k+1}^{(j+1)}-\epsilon_{k+1}^{(j)}}{b}+\frac{\epsilon_{k+2}^{(j+1)}-\epsilon_{k+2}^{(j)}}{b^2}+\cdots\right)=$$
$$=\frac{z_k^{(i,j)}}{b^k}.$$

With the same arguments as before we are able to prove the periodicity.

According to Lemma 2.5.12 and 2.5.13 we have

$$\int_0^1 f(x)g(xb^k)dx-\int_0^1 f(x)dx\int_0^1 g(x)dx=$$
$$=\sum_{(i-1)\notin\mathfrak{L}}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy \tag{2.1}$$
$$+\sum_{(i-1)\in\mathfrak{L}}\sum_{j=a_{i-1}+1}^{a_i-1}\frac{1}{b^k}\int_0^1\left[k_i\left(\frac{y+j}{b^k}\right)+d_i\right]g(y)dy \tag{2.2}$$

$$+\sum_{i\in\mathfrak{L}}\left\{\frac{1}{b^k}\int_0^1\left(\frac{a_i}{b^k}k_i+d_i\right)g(y)dy+\frac{1}{b^k}\int_0^1\frac{y}{b^k}k_ig(y)\mathbb{1}_{[0,e_ib^k-a_i]}(y)dy\right. \tag{2.3}$$

$$+\frac{1}{b^k}\sum_{j=1}^{n_i-1}\left[\int_0^1\left(\frac{y}{b^k}k_i^{(j+1)}+\sum_{l=1}^{j}\left[\left(e_i^{(l)}-\frac{a_i}{b^k}\right)\left(k_i^{(l)}-k_i^{(l+1)}\right)\right]\right)\times\right. \tag{2.4}$$

$$\left.\times\, g(y)\mathbb{1}_{[e_i^{(j)}b^k-a_i,e_i^{(j+1)}b^k-a_i]}(y)dy\right]+\frac{1}{b^k}\int_0^1\left(\frac{y}{b^k}k_{i+1}+\right. \tag{2.5}$$

$$\left.\left.+\sum_{l=1}^{n_i}\left[\left(e_i^{(l)}-\frac{a_i}{b^k}\right)\left(k_i^{(l)}-k_i^{(l+1)}\right)\right]\right)g(y)\mathbb{1}_{[e_i^{(n_i)}b^k-a_i,1]}dy\right\} \tag{2.6}$$

$$-\left[\sum_{(i-1)\notin\mathfrak{L}}\sum_{j=a_{i-1}}^{a_i-1}\frac{1}{b^k}\left(k_i\frac{j}{b^k}+d_i\right)+\sum_{(i-1)\in\mathfrak{L}}\sum_{j=a_{i-1}+1}^{a_i-1}\frac{1}{b^k}\left(k_i\frac{j}{b^k}+d_i\right)\right. \tag{2.7}$$

$$+\sum_{i\in\mathfrak{L}}\left(f_i\left(\frac{a_i}{b^k}\right)\left(e_i-\frac{a_i}{b^k}\right)+f_{i+1}\left(e_i^{(n_i)}\right)\left(\frac{a_i+1}{b^k}-e_i^{(n_i)}\right)+\right. \tag{2.8}$$

$$\left.+\sum_{j=1}^{n_i-1}f_i^{(j+1)}\left(e_i^{(j)}\right)\left(e_i^{(j+1)}-e_i^{(j)}\right)\right)+\sum_{(i-1)\in\mathfrak{L}}(a_i-a_{i-1}-1)\frac{k_i}{2b^{2k}}+ \tag{2.9}$$

$$+\sum_{(i-1)\notin\mathfrak{L}}(a_i-a_{i-1})\frac{k_i}{2b^{2k}}+\sum_{i\in\mathfrak{L}}\left(\frac{k_i}{2}\left(e_i-\frac{a_i}{b^k}\right)^2+\frac{k_{i+1}}{2}\left(\frac{a_i+1}{b^k}-e_i\right)^2+\right. \tag{2.10}$$

$$\left.\left.+\sum_{j=1}^{n_i-1}\frac{k_i^{(j+1)}}{2}\left(e_i^{(j+1)}-e_i^{(j)}\right)^2\right)\right]\int_0^1 g(y)dy. \tag{2.11}$$

We can now combine lines (1), (2) and (7), as we did in the previous case and use Lemma 2.5.10

$$\sum_{(i-1)\in\mathfrak{L}}(a_i-(a_{i-1}+1))\frac{k_i}{b^{2k}}+\sum_{(i-1)\notin\mathfrak{L}}(a_i-a_{i-1})\frac{k_i}{b^{2k}}=$$
$$-\sum_{i\in\mathfrak{L}}\left[\left(e_i-\frac{a_i}{b^k}\right)\frac{k_i}{b^k}+\left(\frac{a_i+1}{b^k}-e_i\right)\frac{k_{i+1}}{b^k}+\sum_{j=1}^{n_i-1}\left(e_i^{(j+1)}-e_i^{(j)}\right)\frac{k_i^{j+1}}{b^k}\right].$$

The same holds true when applied to the last term of line (9) and the first of line (10). Another simplification of the formula can be made by using the first term of line (3), lines (8) and (9) and Lemma 2.5.11.

$$- \left\{ \sum_{j=1}^{n_i-1} \left[\left(k_i^{(1)} \left(e_i^{(1)} - \frac{a_i}{b^k} \right) + \sum_{l=1}^{j-1} k_i^{(l+1)} \left(e_i^{(l+1)} - e_i^{(l)} \right) \right) \left(e_i^{(j+1)} - e_i^{(j)} \right) \right] \right.$$

$$\left. + \left[k_i^{(1)} \left(e_i^{(1)} - \frac{a_i}{b^k} \right) + \sum_{l=1}^{n_i-1} k_i^{(l+1)} \left(e_i^{(l+1)} - e_i^{(l)} \right) \right] \left(\frac{a_i+1}{b^k} - e_i^{(n_i)} \right) \right\}$$

Summarizing all this we get

$$R_k = \sum_{i \in \mathfrak{L}} \left\{ - \left[\left(e_i - \frac{a_i}{b^k} \right) \frac{k_i}{b^k} + \left(\frac{a_i+1}{b^k} - e_i \right) \frac{k_{i+1}}{b^k} \right. \right.$$

$$\left. + \sum_{j=1}^{n_i-1} \left(e_i^{(j+1)} - e_i^{(j)} \right) \frac{k_i^{j+1}}{b^k} \right] \int_0^1 y g(y) dy + \frac{1}{b^k} \int_0^1 \frac{y}{b^k} k_i g(y) \mathbb{1}_{[0, e_i b^k - a_i]}(y) dy$$

$$+ \frac{1}{b^k} \sum_{j=1}^{n_i-1} \left[\int_0^1 \left(\frac{y}{b^k} k_i^{(j+1)} + \sum_{l=1}^{j} \left[\left(e_i^{(l)} - \frac{a_i}{b^k} \right) \left(k_i^{(l)} - k_i^{(l+1)} \right) \right] \right) \times \right.$$

$$\left. \times g(y) \mathbb{1}_{[e_i^{(j)} b^k - a_i, e_i^{(j+1)} b^k - a_i]}(y) dy \right] + \frac{1}{b^k} \int_0^1 \left(\frac{y}{b^k} k_{i+1} + \right.$$

$$\left. + \sum_{l=1}^{n_i} \left[\left(e_i^{(l)} - \frac{a_i}{b^k} \right) \left(k_i^{(l)} - k_i^{(l+1)} \right) \right] \right) g(y) \mathbb{1}_{[e_i^{(n_i)} b^k - a_i, 1]} dy$$

$$\left\{ - \left[\sum_{j=1}^{n_i-1} \left[\left(k_i^{(1)} \left(e_i^{(1)} - \frac{a_i}{b^k} \right) + \sum_{l=1}^{j-1} k_i^{(l+1)} \left(e_i^{(l+1)} - e_i^{(l)} \right) \right) \left(e_i^{(j+1)} - e_i^{(j)} \right) \right] \right. \right.$$

$$\left. + \left[k_i^{(1)} \left(e_i^{(1)} - \frac{a_i}{b^k} \right) + \sum_{l=1}^{n_i-1} k_i^{(l+1)} \left(e_i^{(l+1)} - e_i^{(l)} \right) \right] \left(\frac{a_i+1}{b^k} - e_i^{(n_i)} \right) \right]$$

$$+ \sum_{i \in \mathfrak{L}} \left[\left(e_i - \frac{a_i}{b^k} \right) \frac{k_i}{b^k} + \left(\frac{a_i+1}{b^k} - e_i \right) \frac{k_{i+1}}{b^k} + \sum_{j=1}^{n_i-1} \left(e_i^{(j+1)} - e_i^{(j)} \right) \frac{k_i^{j+1}}{b^k} \right]$$

$$- \left(\frac{k_i}{2} \left(e_i - \frac{a_i}{b^k} \right)^2 + \frac{k_{i+1}}{2} \left(\frac{a_i+1}{b^k} - e_i \right)^2 \right.$$

$$\left. \left. \left. - \sum_{j=1}^{n_i-1} \frac{k_i^{(j+1)}}{2} \left(e_i^{(j+1)} - e_i^{(j)} \right)^2 \right) \right\} \int_0^1 g(y) dy \right\}. \qquad \square$$

Chapter 3

Special Cases

In order to verify a central limit theorem for the discrepancy of the generalized van der Corput sequences we need to prove the order of divergence satisfies the conditions given in Theorem 2.3.1. During the last sections, we have found formulas for the expectation and the variance of the discrepancy. So far it was not necessary to impose any restrictions on the base b, all the results hold for arbitrary b. If we want to calculate the expected value and the variance and show that the later one tends to infinity then we face the problem where the functions $\Psi_b^{\pi_i,+}$ are not given explicitly, but as the maximum of b different functions. Thus the difficult part is not computing the values of the expectation and the variance but finding the necessary functions. We will first treat some special cases, as it is not an easy task to get a picture of the general situation.

3.1 Base 2

This case has already been treated by M. Drmota, G. Larcher and F. Pillichshammer in [5], where they showed a central limit theorem for the star and the extreme discrepancy of the van der Corput sequence in base 2.

Theorem 3.1.1. *Let D_N^* denote the star discrepancy of the first N points of the van der Corput sequence. Then we have for $1 \leq N \leq 2^n$ and every real y*

$$\frac{1}{2^n}\#\left\{N < 2^n : D_N^* \leq \frac{1}{4}n + y\frac{1}{4\sqrt{3}}\sqrt{n}\right\} = \Phi(y) + o(1),$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{\frac{t^2}{2}}\,dt$$

denotes the normal distribution function, that is, the star discrepancy satisfies a central limit theorem.

This result holds also true for the extreme discrepancy, as for ordinary van der Corput sequences it coincides with the star discrepancy. In addition, we will show the two permutations π_1 and π_2 on $\{0,1\}$ yield the same function $\Psi_2^{\pi_i}$, so the expected value and the variance of the extreme discrepancy are the same for all possible sequences Σ. This means that the above Theorem, if formulated for the extreme discrepancy, is not only valid for ordinary van der Corput sequences but also for the generalized ones.

We will now use the formulas we developed in Sections 2.4 and 2.5 in order to obtain the result. First we observe, that there exist two permutations $\pi_1 = id$ and $\pi_2 = (01)$, for the corresponding $\Psi_2^{\pi_i,\cdot}$ we have

$$\Psi_2^{\pi_1,+}(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1/2, \\ 1-x & \text{if} \quad 1/2 \le x < 1 \end{cases},$$

$$\Psi_2^{\pi_2,+}(x) = 0,$$

$$\Psi_2^{\pi_1,-}(x) = \Psi_2^{\pi_2,+}(x) \qquad \text{and} \qquad \Psi_2^{\pi_2,-} = \Psi_2^{\pi_1,-}.$$

So we get

$$\Psi_2^{\pi_1} = \Psi_2^{\pi_2} = \|x\|,$$

where $\|x\|$ denotes the distance to the nearest integer. It is a consequence of Theorem 2.2.1, that the extreme discrepancy is the same for every possible sequence Σ as all the functions $\Psi_2^{\pi_i}$ are equal. Further, we know if the sequence Σ is constant, that is $\Sigma = (\sigma_i)_{i\ge1}$ with $\sigma_i = \pi_1$ for all i or $\sigma_i = \pi_2$ for all i, then according to Theorem 2.2.1 the star and the extreme discrepancy are identical and we can calculate the expectation and the variance

$$\mathbb{E}(D(N, S_b^\Sigma)) = \mathbb{E}(D^*(N, S_b^\Sigma)) = n\int_0^1 \|x\|\,dx = \frac{n}{4} + O(1),$$

$$\mathbb{V}(D(N, S_b^\Sigma)) = \mathbb{V}(D^*(N, S_b^\Sigma)) = n\left(\int_0^1 \|x\|^2\,dx - \left(\int_0^1 \|x\|\,dx\right)^2\right) = \frac{n}{48} + O(1).$$

Note for the variance we don't have mixed terms as the only local extremum is attained at 1/2.

The general case for star discrepancy still remains to be discussed. If we denote by M_1 the number of indices i, $1 \leq i \leq n$, so that $\sigma_i = \pi_1$ with $\Sigma = (\sigma_i)_{i\geq 1}$, we get

$$\mathbb{E}(D^*(N, S_b^\Sigma)) = \frac{\max(M_1, n - M_1)}{4} + O(1),$$

$$\mathbb{V}(D^*(N, S_b^\Sigma)) = \frac{\max(M_1, n - M_1)}{48} + O(1).$$

Summarizing these results we obtain a central limit theorem for the generalized van der Corput sequences in base 2.

Theorem 3.1.2. *Let $D(N, S_b^\Sigma)$ and $D^*(N, S_b^\Sigma)$ denote the extreme discrepancy and the star discrepancy of the first N points of the generalized van der Corput sequence in base 2. Then we have for $1 \leq N \leq 2^n$ and every real y*

$$\frac{1}{2^n}\#\left\{N < 2^n : D(N, S_b^\Sigma) \leq \frac{1}{4}n + y\frac{1}{4\sqrt{3}}\sqrt{n}\right\} = \Phi(y) + o(1)$$

and

$$\frac{1}{2^n}\#\left\{N < 2^n : D^*(N, S_b^\Sigma) \leq \frac{1}{4}\max(M_1, n - M_1) + \right. \\ \left. + y\frac{1}{4\sqrt{3}}\sqrt{\max(M_1, n - M_1)}\right\} = \Phi(y) + o(1).$$

Proof. This Theorem is a consequence of the calculations done in this section and Theorem 2.3.1 □

3.2 Base 3

For base 2 we were able to make small improvements to the known results from [5]. However this was a simple case and did not provide a good insight into the application of the formulas. We will now look at the generalized van der Corput sequences in base 3. This base is still small and does not give a full picture of the general situation. As the complexity of the problem grows with $b!$, large bases are quite difficult to handle. In fact it is possible for base 3 to calculate the expectation and the variance of the extreme discrepancy for all possible sequences Σ without any restrictions. For the star discrepancy this is more complicated and we will only be able to give bounds for the variance.

First we need to calculate the functions $\Psi_3^{\pi_i,+}$. As there are six different permutations π_i of $(0,1,2)$, we get six different functions. These functions can be found in Table 3.1. Whenever it is possible in order to simplify notation we will write Ψ_i^+ instead of $\Psi_3^{\pi_i,+}$, as long as it does not cause confusions. Figure 3.1 provides a graphical illustration.

$\pi_1 = id$	$\Psi_1^+(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/3 \\ 1-x & \text{if } 1/3 \leq x < 1/2 \\ x & \text{if } 1/2 \leq x < 2/3 \\ 2(1-x) & \text{if } 2/3 \leq x < 1 \end{cases}$
$\pi_2 = (12)(3)$	$\Psi_2^+(x) = \begin{cases} x & \text{if } 0 \leq x < 2/3 \\ 2(1-x) & \text{if } 2/3 \leq x < 1 \end{cases}$
$\pi_3 = (13)(2)$	$\Psi_3^+(x) = 0$
$\pi_4 = (23)(1)$	$\Psi_4^+(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/3 \\ 1-x & \text{if } 1/3 \leq x < 1 \end{cases}$
$\pi_5 = (123)$	$\Psi_5^+(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 \\ 2x-1 & \text{if } 1/2 \leq x < 2/3 \\ 1-x & \text{if } 2/3 \leq x < 1 \end{cases}$
$\pi_6 = (132)$	$\Psi_6^+(x) = \begin{cases} x & \text{if } 0 \leq x < 1/3 \\ 1-2x & \text{if } 1/3 \leq x < 1/2 \\ 0 & \text{if } 1/2 \leq x < 1 \end{cases}$

Table 3.1: The functions $\Psi_3^{\pi_i,+}$

For the computation of the expectation and the variance of the extreme and the star discrepancy we need the functions Ψ_i^-. These are basically the

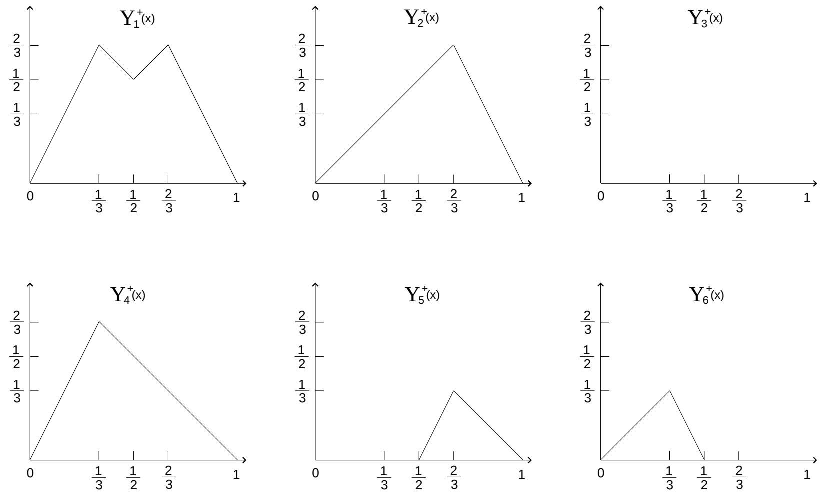

Figure 3.1: Graphs of the functions $\Psi_i^+(x)$

$\Psi_1^- = \Psi_3^+$	$\Psi_1 = \Psi_1^+ + \Psi_3^+$
$\Psi_2^- = \Psi_6^+$	$\Psi_2 = \Psi_2^+ + \Psi_6^+$
$\Psi_3^- = \Psi_1^+$	$\Psi_3 = \Psi_3^+ + \Psi_1^+$
$\Psi_4^- = \Psi_5^+$	$\Psi_4 = \Psi_4^+ + \Psi_5^+$
$\Psi_5^- = \Psi_4^+$	$\Psi_5 = \Psi_5^+ + \Psi_4^+$
$\Psi_6^- = \Psi_2^+$	$\Psi_6 = \Psi_6^+ + \Psi_2^+$

Table 3.2: Relations between the functions Ψ_i^+, Ψ_i^- and Ψ_i

same as the functions Ψ_i^+. One only needs to permute the indices i, so for each Ψ_i^- there exists a j with $\Psi_i^- = \Psi_j^+$. These equalities are given by Table 3.2.

First we will try to calculate the expectation of the extreme discrepancy for an arbitrary sequence Σ. For this purpose we need the functions $\Psi_i = \Psi_i^+ + \Psi_i^-$, this can also be found in 3.2. If we take a look at Figure 3.1, we can find certain symmetries suggesting there exist some kind of similarities among the Ψ_i's. In fact one can easily check

$$\Psi_i = \Psi_1^+ \quad \text{for all} \quad i.$$

Therefore the discrepancy has the same expectation for all possible sequences Σ, that is

$$\mathbb{E}(D(N, S_3^\Sigma)) = \frac{5n}{12} + O(1).$$

We will now calculate the variance of the extreme discrepancy. From

Theorem 2.4.2 and Theorem 2.5.2 we know, the variance of the discrepancy consists of the sum of the variances of the Ψ's and some error terms of order 9^{-k}. It is sufficient to compute the variance and the error term for Ψ_1^+, as all functions Ψ_i are equal to this function. We get

$$\begin{aligned} F_{11}^{+}\frac{1}{9^k} &= \left(\int_0^1 \Psi_1^+(x)\Psi_1^+(x3^k)dx - \int_0^1 \Psi_1^+(x)dx \int_0^1 \Psi_1^+(x)dx\right) \\ &= -\frac{11}{432}\frac{1}{9^k} \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}\left(\Psi_1^+(U)\right) &= \mathbb{E}\left(\Psi_1^+(U)^2\right) - \left(\mathbb{E}\left(\Psi_1^+(U)\right)\right)^2 = \\ &= \frac{17}{432}. \end{aligned}$$

As for the expectation, the result for the variance of the extreme discrepancy does not depend on the choice of the sequence Σ. Inserting the values for Ψ_1^+ in the formula of the variance we get

$$\begin{aligned} \mathbb{V}\left(D(N, S_3^{\Sigma})\right) &= \frac{17n}{432} - \frac{22}{432}\sum_{j=1}^{n}\sum_{l<j}\frac{1}{9^{j-l}} + O(1) = \\ &= \frac{17n}{432} - \frac{11}{432}\left(\frac{n}{4} - \frac{9}{32} + \frac{1}{32}\frac{1}{9^{n-1}}\right) + O(1) \\ &= \frac{19n}{576} + O(1). \end{aligned}$$

After these preparations we are able to state a central limit theorem for the extreme discrepancy of the generalized van der Corput sequence in base 3.

Theorem 3.2.1. *Let $D(N, S_b^{\Sigma})$ denote the extreme discrepancy of the first N points of the generalized van der Corput sequence in base 3. Then we have for $1 \le N \le 3^n$ and every real y*

$$\begin{aligned} &\frac{1}{3^n}\#\left\{N < 3^n : D(N, S_b^{\Sigma}) \le \frac{5}{12}n + y\sqrt{\frac{19n}{576}}\right\} \\ &\qquad = \Phi(y) + o(1). \end{aligned}$$

Proof. The proof is straightforward using the previous results and Theorem 2.3.1. ☐

In the case of the star discrepancy the problem is different as $D^*(N,\Sigma) = \max(D^+(N,S_3^\Sigma), D^-(N,S_3^\Sigma))$. In order to calculate the expected value it is sufficient to know the expectations of the Ψ_i^+ and the distribution of the π_i's in the sequence. For each sequence Σ exists a sequence $\widetilde{\Sigma}$, such that $D^-(N,S_3^\Sigma) = D^+(N,S_3^{\widetilde{\Sigma}})$. The values of the expected value of the Ψ_i^+ are given by 3.3.

$$\mathbb{E}(\Psi_1^+(U)) = \frac{5}{12}$$
$$\mathbb{E}(\Psi_2^+(U)) = \frac{1}{3}$$
$$\mathbb{E}(\Psi_3^+(U)) = 0$$
$$\mathbb{E}(\Psi_4^+(U)) = \frac{1}{3}$$
$$\mathbb{E}(\Psi_5^+(U)) = \frac{1}{12}$$
$$\mathbb{E}(\Psi_6^+(U)) = \frac{1}{12}$$

Table 3.3: Expectation of Ψ_i^+

We need to define numbers $\kappa_i^+(N)$ before we can write down the formula for the expectation. These figures give the number of occurrences of the permutation π_i among the first n elements of Σ

$$\kappa_i^+(N) = |\{j : 1 \le j \le N, \sigma_j = \pi_i\}|.$$

This leads to the following formula

$$\mathbb{E}(D^+(N,S_3^\Sigma)) = \sum_{i=1}^{6} \kappa_i^+(N)\mathbb{E}\left(\Psi_i^+(U)\right) + O(1).$$

If we assume, for example, n is a multiple of 6 and that the π_i are evenly distributed among the first n terms, we get for the expectation of the star discrepancy

$$\mathbb{E}(D^*(N,S_3^\Sigma)) = \frac{5n}{24} + O(1).$$

In this case we have $D^+(N, S_3^\Sigma) = D^-(N, S_3^\Sigma) = D^*(N, S_3^\Sigma)$.

The calculation of the variance of the star discrepancy is more complicated and we need to make additional assumptions about the sequence Σ. As we know from Table 3.2 each Ψ_j^- corresponds to a Ψ_i^+. Therefore it is sufficient to calculate the variance and the error term for the Ψ_i^+. We will start with the computation of the F_{jl}^+ for all possible combinations of j and l. Using

$$F_{jl}^+ \frac{1}{9^k} = \int_0^1 \Psi_j^+(x)\Psi_l^+(xb^k)dx - \int_0^1 \Psi_j^+(x)dx \int_0^1 \Psi_l^+(x)dx$$

we get

$$\begin{aligned}
F_{2l}^+ = F_{3l}^+ = F_{4l}^+ = F_{l3}^+ &= 0 \\
F_{11}^+ = F_{51}^+ = F_{61}^+ &= -\frac{11}{432} \\
F_{12}^+ = F_{52}^+ = F_{62}^+ &= -\frac{10}{432} \\
F_{14}^+ = F_{54}^+ = F_{64}^+ &= -\frac{10}{432} \\
F_{15}^+ = F_{55}^+ = F_{65}^+ &= -\frac{1}{432} \\
F_{16}^+ = F_{56}^+ = F_{66}^+ &= -\frac{1}{432}.
\end{aligned} \tag{3.1}$$

With the help of these terms, we obtain the following results for the variance:

$$\begin{aligned}
\mathbb{V}\left(\Psi_1^+\right) &= \frac{17}{432} \\
\mathbb{V}\left(\Psi_2^+\right) &= \frac{16}{432} \\
\mathbb{V}\left(\Psi_3^+\right) &= 0 \\
\mathbb{V}\left(\Psi_4^+\right) &= \frac{16}{432} \\
\mathbb{V}\left(\Psi_5^+\right) &= \frac{5}{432} \\
\mathbb{V}\left(\Psi_6^+\right) &= \frac{5}{432}.
\end{aligned}$$

If we want to calculate the variance of the star discrepancy, we are facing the problem, it does not only depend on the number of occurrences of the single permutations π_i, but also on their exact position in the sequence Σ.

$$\mathbb{V}(D^+(N,S_3^\Sigma)) = \sum_{i=1}^{n} \mathbb{V}\left(\Psi_3^{\sigma_{i-1},+}\right) + 2\sum_{j=1}^{n}\sum_{l<j} F_{\sigma_{j-1},\sigma_{l-1}} \frac{1}{9^{j-l}}$$

Therefore it is not possible to get a general result for all possible sequences. However we can give upper and lower bounds for the sum of the error terms. If we apply the maximum and the minimum value of the F_{ij}, we obtain the following two inequalities for the error term

$$2\sum_{j=1}^{n}\sum_{l<j} F_{\sigma_{j-1},\sigma_{l-1}} \frac{1}{9^{j-l}} \leq 0,$$

$$2\sum_{j=1}^{n}\sum_{l<j} F_{\sigma_{j-1},\sigma_{l-1}} \frac{1}{9^{j-l}} \geq -\frac{11}{432}\left(\frac{1}{4}n - \frac{9}{32} + \frac{1}{32}\frac{1}{9^{n-1}}\right).$$

Using these results, we can show the variance always tends to infinity with order n and therefore fulfills a central limit theorem.

Theorem 3.2.2. *Let $D^*(N,S_b^\Sigma)$ denote the star discrepancy of the first N points of the generalized van der Corput sequence in base 3. Then we have for $1 \leq N \leq 3^n$ and every real y*

$$\frac{1}{3^n}\#\left\{N < 3^n : D^*(N,S_b^\Sigma) \leq \frac{5n}{24} + y\sqrt{\mathbb{V}(D^*(N,S_b^\Sigma))}\right\}$$
$$= \Phi(y) + o(1)$$

and the Variance tends to infinity with order n.

Proof. First, we need to define a new sequence $\widehat{\Sigma}$. This sequence consists of all elements of Σ but the σ_i with $\sigma_i = \pi_3$. This means we removed all elements from the sequence, where the corresponding functions Ψ_i^+, Ψ_i^- and Ψ_i are identical to zero. We define $\widehat{N}$ as the number of remaining elements

$$\widehat{N} = |\{i, 1 \leq i \leq N : \sigma_i \neq \pi_3\}|.$$

This enables us to take $5/432$ as a lower bound for the variances and we get

$$\mathbb{V}(D^+(N,S_3^{\Sigma})) \geq \mathbb{V}(D^+(\widehat{N},S_3^{\widehat{\Sigma}})) \geq \frac{5n}{432} - \frac{11}{432}\left(\frac{1}{4}n - \frac{9}{32} + \frac{1}{32}\frac{1}{9^{n-1}}\right) + O(1).$$

In order to prove the theorem we need to show the variance of the star discrepancy tends to infinity. This means either $D^+(N,S_3^{\Sigma})$ or $D^-(N,S_3^{\Sigma})$ must be infinite. Using the last inequality, we obtain $\mathbb{V}(D^+(N,S_3^{\Sigma})$ tends to infinity if $\widetilde{n}$ tends to infinity. Therefore the sequence Σ must contain infinitely many elements $\sigma_i \neq \pi_3$. If we assume there are only finitely many elements $\sigma_i \neq \pi_3$ we can use $D^-(N,S_3^{\Sigma})$ instead of $D^+(N,S_3^{\Sigma})$. We have already shown there exists a sequence $\widetilde{\Sigma}$ with $D^-(N,S_3^{\Sigma}) = D^+(N,S_3^{\widetilde{\Sigma}})$. However, the sequence $\widetilde{\Sigma}$ contains infinitely many elements $\widetilde{\sigma}_i \neq \pi_3$. Hence $D^-(N,S_3^{\Sigma})$ tends to infinity.

If we now apply Theorem 2.3.1 we obtain the desired result. □

3.3 Base 4

We have seen, that for base 3 the extreme discrepancy can be treated much easier than the star discrepancy. For base 4 we will only consider the former one. There exist 24 different permutations, see Table 3.4. The corresponding functions $\Psi_4^{\pi_i,+}$ can be found in Table 3.5. Once again, if it is possible without causing confusion, we will write Ψ_i^+ instead of $\Psi_4^{\pi_i,+}$. Using some basic calculus one can easily show all functions Ψ_i are either equal to Ψ_1^+ or Ψ_3^+. We will denote the set of permutations π_i with $\Psi_i = \Psi_1^+$ by Π_1, so

$$\Pi_1 = \{\pi_i | \Psi_i = \Psi_1^+\}.$$

In the same way we define Π_3. If we want to calculate the expectation and the variance for the discrepancy of a given sequence Σ, it is sufficient to compute these values for Ψ_1^+ and Ψ_3^+. Additionally we need the terms F_{11}, F_{13}, F_{31} and F_{33} for the variance.

Similarly to Section 3.2, we will denote by κ_1 and κ_3 respectively the number of occurrences of the elements of Π_1 and Π_3 among the first n elements of the series Σ. With the help of this notation and

$$\mathbb{E}\left(\Psi_1(U)\right) = \frac{7}{12} \quad \text{and} \quad \mathbb{E}\left(\Psi_3(U)\right) = \frac{1}{2}$$

we can write the expectation of the discrepancy of an arbitrary sequence Σ as

$$\mathbb{E}\left(D\left(N,S_4^{\Sigma}\right)\right) = \kappa_1\frac{7}{12} + \kappa_3\frac{1}{2} + O(1).$$

$\pi_1 = id$	$\pi_9 = (021)(3)$	$\pi_{17} = (02)(13)$
$\pi_2 = (0)(1)(23)$	$\pi_{10} = (0321)$	$\pi_{18} = (0312)$
$\pi_3 = (0)(12)(3)$	$\pi_{11} = (0231)$	$\pi_{19} = (0123)$
$\pi_4 = (0)(132)$	$\pi_{12} = (031)(2)$	$\pi_{20} = (013)(2)$
$\pi_5 = (0)(123)$	$\pi_{13} = (012)(3)$	$\pi_{21} = (023)(1)$
$\pi_6 = (0)(13)(2)$	$\pi_{14} = (0132)$	$\pi_{22} = (03)(1)(2)$
$\pi_7 = (01)(2)(3)$	$\pi_{15} = (02)(1)(3)$	$\pi_{23} = (0213)$
$\pi_8 = (01)(23)$	$\pi_{16} = (032)(1)$	$\pi_{24} = (03)(12)$

Table 3.4: Permutations of $(0, 1, 2, 3)$

Concerning the variance it is more difficult to write down a formula. For arbitrary sequences it does not only depend on the number of occurrences of the different permutations but also on their position within the sequence. We did not face this problem for base three as we only had to consider one function Ψ. For base four we have to handle two, Ψ_1 and Ψ_3. The variances of these are given by

$$\mathbb{V}\left(\Psi_1(U)\right) = \frac{31}{432} \quad \text{and} \quad \mathbb{V}\left(\Psi_3(U)\right) = \frac{1}{24}$$

and the error terms by

$$\begin{aligned} F_{11} &= -\frac{37}{432} \\ F_{13} &= -\frac{25}{432} \\ F_{33} &= F_{31} = 0. \end{aligned}$$

For any sequence Σ the following two inequalities hold true

$$\mathbb{V}\left(D\left(N, S_4^\Sigma\right)\right) \leq \frac{31n}{432} - \frac{25}{432}\left(\frac{n}{15} - \frac{16}{225} + \frac{1}{225}\frac{1}{16^{n-1}}\right) + O(1)$$

and

$$\mathbb{V}\left(D\left(N, S_4^\Sigma\right)\right) \geq \frac{18n}{432} - \frac{37}{432}\left(\frac{n}{15} - \frac{16}{225} + \frac{1}{225}\frac{1}{16^{n-1}}\right) + O(1),$$

therefore, the variance of the extreme discrepancy will always tend to infinity and we obtain the following theorem.

$$\Psi_1^+(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/4 \\ 1-x & \text{if } 1/4 \le x < 1/3 \\ 2x & \text{if } 1/3 \le x < 1/2 \\ 2(1-x) & \text{if } 1/2 \le x < 2/3 \\ x & \text{if } 2/3 \le x < 3/4 \\ 3(1-x) & \text{if} 3/4 \le x < 1 \end{cases}$$

$$\Psi_2^+(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/4 \\ 1-x & \text{if } 1/4 \le x < 1/3 \\ 2x & \text{if } 1/3 \le x < 1/2 \\ 2(1-x) & \text{if } 1/2 \le x < 1 \end{cases}$$

$$\Psi_3^+(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/4 \\ 1-x & \text{if } 1/4 \le x < 1/2 \\ x & \text{if } 1/2 \le x < 3/4 \\ 3(1-x) & \text{if} 3/4 \le x < 1 \end{cases}$$

$$\Psi_4^+(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/4 \\ 1-x & \text{if } 1/4 \le x < 1 \end{cases}$$

$$\Psi_5^+(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/4 \\ 1-x & \text{if } 1/4 \le x < 2/3 \\ 2x-1 & \text{if } 2/3 \le x < 3/4 \\ 2(1-x) & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_6^+(x) = \begin{cases} 3x & \text{if } 0 \le x < 1/4 \\ 1-x & \text{if } 1/4 \le x < 1 \end{cases}$$

$$\Psi_7^+(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2(1-x) & \text{if } 1/2 \le x < 2/3 \\ x & \text{if } 2/3 \le x < 3/4 \\ 3(1-x) & \text{if} 3/4 \le x < 1 \end{cases}$$

$$\Psi_8^+(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2(1-x) & \text{if } 1/2 \le x < 1 \end{cases}$$

$$\Psi_9^+(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/4 \\ 1-2x & \text{if } 1/4 \le x < 1/3 \\ x & \text{if } 1/3 \le x < 3/4 \\ 3(1-x) & \text{if} 3/4 \le x < 1 \end{cases}$$

$$\Psi_{10}^+(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/4 \\ 1-2x & \text{if } 1/4 \le x < 1/3 \\ x & \text{if } 1/3 \le x < 1/2 \\ 2-3x & \text{if } 1/2 \le x < 2/3 \\ 0 & \text{if } 2/3 \le x < 1 \end{cases}$$

$$\Psi_{11}^+(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/4 \\ 1-2x & \text{if } 1/4 \le x < 1/2 \\ 2x-1 & \text{if } 1/2 \le x < 3/4 \\ 2(1-x) & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{12}^+(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/4 \\ 1-2x & \text{if } 1/4 \le x < 1/2 \\ 0 & \text{if } 1/2 \le x < 1 \end{cases}$$

$$\Psi_{13}^+(x) = \begin{cases} x & \text{if } 0 \le x < 3/4 \\ 3(1-x) & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{14}^+(x) = \begin{cases} x & \text{if } 0 \le x < 1/2 \\ (1-x) & \text{if } 1/2 \le x < 1 \end{cases}$$

$$\Psi_{15}^+(x) = \begin{cases} x & \text{if } 0 \le x < 3/4 \\ 3(1-x) & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{16}^+(x) = \begin{cases} x & \text{if } 0 \le x < 1/2 \\ 2-3x & \text{if } 1/2 \le x < 2/3 \\ 0 & \text{if } 2/3 \le x < 1 \end{cases}$$

$$\Psi_{17}^+(x) = \begin{cases} x & \text{if } 0 \le x < 1/4 \\ 1-3x & \text{if } 1/4 \le x < 1/3 \\ 0 & \text{if } 1/3 \le x < 2/3 \\ 3x-2 & \text{if } 2/3 \le x < 3/4 \\ 1-x & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{18}^+(x) = \begin{cases} x & \text{if } 0 \le x < 1/4 \\ 1-3x & \text{if } 1/4 \le x < 1/3 \\ 0 & \text{if } 1/3 \le x < 1 \end{cases}$$

$$\Psi_{19}^+(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/3 \\ 3x-1 & \text{if } 1/3 \le x < 1/2 \\ 1-x & \text{if } 1/2 \le x < 2/3 \\ 2x-1 & \text{if } 2/3 \le x < 3/4 \\ 2(1-x) & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{20}^+(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/3 \\ 3x-1 & \text{if } 1/3 \le x < 1/2 \\ 1-x & \text{if } 1/2 \le x < 1 \end{cases}$$

$$\Psi_{21}^+(x) = \begin{cases} 0 & \text{if } 0 \le x < 1/2 \\ 2x-1 & \text{if } 1/2 \le x < 3/4 \\ 2(1-x) & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{22}^+(x) = 0$$

$$\Psi_{23}^+(x) = \begin{cases} 0 & \text{if } 0 \le x < 2/3 \\ 3x-2 & \text{if } 2/3 \le x < 3/4 \\ 1-x & \text{if } 3/4 \le x < 1 \end{cases}$$

$$\Psi_{24}^+(x) = 0$$

Table 3.5: The functions $\Psi_4^{\pi_i,+}$

Theorem 3.3.1. *Let $D(N, S_b^{\Sigma})$ denote the extreme discrepancy of the first N points of the generalized van der Corput sequence in base 4. Then we have for $1 \leq N \leq 4^n$ and every real y*

$$\frac{1}{4^n}\#\left\{N < 4^n : D(N, S_b^{\Sigma}) \leq \mathbb{E}(D(N, S_b^{\Sigma}))+y\sqrt{\mathbb{V}(D(N, S_b^{\Sigma}))}\right\} = \Phi(y)+o(1)$$

and the Variance tends to infinity with order n.

Proof. This theorem is an immediate consequence of the results given in this section and Theorem 2.3.1. □

Chapter 4

Higher Bases

As the number of permutations increases rapidly with growing bases, it gets more and more difficult to compute all the functions $\Psi_b^{\pi_i,+}$, $\Psi_b^{\pi_i,-}$ and $\Psi_b^{\pi_i}$. The functions for bases 2,3 and 4 showed certain regularities suggesting we should expect the same behavior for larger bases. In particular many of the functions $\Psi_b^{\pi_i}$ were identical. This simplified the calculation of the extreme discrepancy. Although one can find groups of permutations connected by certain patterns and yielding similar functions, it is not an easy task to develop general formulas for arbitrary bases. The results for bases 2,3 and 4 seem to indicate, that the following three properties hold true for all possible choices of b:

i. The gradient only changes at a local extremum, so when it changes, it always changes the sign too.

ii. There are only local extrema at points $\frac{i}{b-j}$, $i = 1, \ldots, b-j-1$, $j = 1, \ldots, b-2$.

iii. There is a certain number of permutations with $\Psi_b^{\pi_i} = \Psi_b^{\pi_i,+}$ and all the other $\Psi_b^{\pi_i}$ are equal to one of these functions.

However, base 5 provides a counter-example to (i) and (iii). If we take the permutation $\pi_{12} = (0)(3)(142)$, we get

$$\Psi_5^{\pi_{12},+}(x) = \begin{cases} 4x & \text{if} \quad 0 \le x < 1/5 \\ 1-x & \text{if} \quad 1/5 \le x < 1/3 \\ 2x & \text{if} \quad 1/3 \le x < 2/5 \\ 2-3x & \text{if} \quad 2/5 \le x < 1/2 \\ 1-x & \text{if} \quad 1/2 \le x < 1 \end{cases}$$

as a contradiction to (i). Using the same permutation, one can prove, that (iii) does not hold true either. If we calculate $\Psi_5^{\pi_{12}}$ with the help of

$$\Psi_5^{\pi_{12},-}(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1/2 \\ 4x-2 & \text{if} \quad 1/2 \le x < 3/5 \\ 1-x & \text{if} \quad 3/5 \le x < 2/3 \\ 2x-1 & \text{if} \quad 2/3 \le x < 4/5 \\ 3-3x & \text{if} \quad 4/5 \le x < 1 \end{cases}$$

we get

$$\Psi_5^{\pi_{12}}(x) = \begin{cases} 4x & \text{if} \quad 0 \le x < 1/5 \\ 1-x & \text{if} \quad 1/5 \le x < 1/3 \\ 2x & \text{if} \quad 1/3 \le x < 2/5 \\ 2-3x & \text{if} \quad 2/5 \le x < 1/2 \\ 3x-1 & \text{if} \quad 1/2 \le x < 3/5 \\ 2-2x & \text{if} \quad 3/5 \le x < 2/3 \\ x & \text{if} \quad 2/3 \le x < 4/5 \\ 4-4x & \text{if} \quad 4/5 \le x < 1 \end{cases}.$$

This is not equal to any of the functions $\Psi_5^{\pi_i,+}$. We were neither able to find a counter-example nor a proof for the second item. In our opinion, the difficulty of finding and proving general properties of the functions $\Psi_b^{\pi_i}$ is mainly caused by two reasons. On the one hand, we must take the maximum in order to obtain the $\Psi_b^{\pi_i,+}$, this makes it quite hard to write down these functions explicitly for arbitrary bases. On the other hand, the permutations we are considering, get more and more complicated with larger bases. For example, for base 100 it is not possible to compare all 100! functions $\Psi_{100}^{\pi_i}$. This means we have to restrict ourselves to the study of relatively small bases which do not show all the irregularities of larger ones. In addition we know from Theorems van:theorem1 and van:theorem7 the error terms R_k in the formula for the variance of the discrepancy have the representation $R_k = c_k/b^{2k}$ where c_k is an ultimately periodic sequence. Therefore we cannot expect to find a general formula for the variance, especially as the terms for small k at the beginning of the sequence have the largest impact.

4.1 The van der Corput Sequence

If we only consider the identical permutation we obtain the ordinary van der Corput sequence in arbitrary base. For this sequence the functions $\Psi_b^{\pi_i,+}$, $\Psi_b^{\pi_i,-}$ and $\Psi_b^{\pi_i}$ are not difficult to determine. We have

$$\Psi_b^{id,+}(x) = \begin{cases} k(1-x) & \text{if} \quad x \in [k/b, k/(b-1)), \\ (b-k-1)x & \text{if} \quad x \in [k/(b-1), (k+1)/b) \end{cases} \tag{4.1}$$

and

$$\Psi_b^{id,-} = 0$$

so

$$\Psi_b^{id} = \Psi_b^{id,+}.$$

In this case the extreme and the star discrepancy coincide. It is sufficient to calculate the former one. We will start with the computation of the expected value and the variance of Ψ_b^{id}, given by

$$\begin{aligned}\mathbb{E}\left(\Psi_b^{id}(U)\right) &= \sum_{k=0}^{b-1}\left(\int_{k/b}^{k/(b-1)} k(1-x)dx + \int_{k/(b-1)}^{(k+1)/b} (b-k-1)xdx\right)\\ &= \frac{1}{6}b - \frac{1}{12}\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}\left(\Psi_b^{id}(U)\right) &= \mathbb{E}\left(\Psi_b^{id}(U)^2\right) - \mathbb{E}\left(\Psi_b^{id}(U)\right)^2\\ &= \frac{1}{720}\frac{4b^4 - 8b^3 + 7b^2 - 3b + 8}{b(b-1)}.\end{aligned}$$

The expectation of the discrepancy of the van der Corput sequence can be obtained by simply multiplying the expected value of the function $\Psi_b^{id}(x)$ by n

$$\mathbb{E}\left(D\left(N, S_b^{\Sigma}\right)\right) = \left(\frac{1}{6}b - \frac{1}{12}\right)n.$$

For the variance it is also necessary to calculate the error therm we obtained in Theorem 2.5.2

$$\begin{aligned}R_k = \sum_{i\in\mathfrak{L}} \Bigg\{ &-\left[\left(e_i - \frac{a_i}{b^k}\right)\frac{k_i}{b^k} + \left(\frac{a_i+1}{b^k} - e_i\right)\frac{k_{i+1}}{b^k}\right]\int_0^1 yg(y)dy\\ &+ \frac{k_i}{b^{2k}}\int_0^1 yg(y)\mathbb{1}_{[0,e_ib^k-a_i]}(y)dy + \frac{k_{i+1}}{b^{2k}}\int_0^1 yg(y)\mathbb{1}_{[e_ib^k-a_i,1]}(y)dy\\ &+ (k_i - k_{i+1})\left(e_i - \frac{a_i}{b^k}\right)\frac{1}{b^k}\int_0^1 g(y)\mathbb{1}_{[e_ib^k-a_i,1]}(y)dy\\ &+ \left[\frac{1}{2}\left(e_i - \frac{a_i}{b^k}\right)\left(\frac{a_i+1}{b^k} - e_i\right)(k_{i+1} - k_i)\right]\int_0^1 g(y)dy\Bigg\}..\end{aligned} \tag{4.2}$$

The points e_i are equal to $l/b-1$, $l = 1, \ldots, b-2$, the corresponding k_i and k_{i+1} are given by $-l$ and $b-l-1$ respectively. For the a_i we have

$$a_i = \left\lfloor \frac{lb^k}{b-1} \right\rfloor = \frac{l(b^k-1)}{b-1}.$$

If we insert these values in (4.2) we get

$$
\begin{aligned}
R_k = \sum_{l=1}^{b-2} \Bigg\{ -\Bigg[-\left(\frac{l}{b-1} - \frac{l(b^k-1)}{(b-1)b^k} \right) \frac{l}{b^k} + \left(\left(\frac{l(b^k-1)}{b-1} + 1 \right) \frac{1}{b^k} \right. \\
\left. - \frac{l}{b-1} \right) \times \frac{b-l-1}{b^k} \Bigg] \int_0^1 y \Psi_b^{id}(y) dy - \frac{l}{b^{2k}} \int_0^1 y \Psi_b^{id}(y) \mathbb{1}_{[0, l/(b-1)]}(y) dy + \\
+ \frac{b-l-1}{b^{2k}} \int_0^1 y \Psi_b^{id}(y) \mathbb{1}_{[l/(b-1),1]}(y) dy - \frac{b-1}{b^k} \left(\frac{l}{b-1} - \frac{l(b^k-1)}{(b-1)b^k} \right) \times \\
\times \int_0^1 \Psi_b^{id}(y) \mathbb{1}_{[l/(b-1),1]}(y) dy + \frac{l(b-l-1)}{2b^{2k}(b-1)} \int_0^1 \Psi_b^{id}(y) dy \Bigg\} = \\
= \sum_{l=1}^{b-2} \Bigg\{ -\frac{b-l-1}{b^{2k}} \int_0^{l/(b-1)} y \Psi_b^{id}(y) dy + \frac{l}{b^{2k}} \int_{l/(b-1)}^1 y \Psi_b^{id}(y) dy \\
- \frac{l}{b^{2k}} \int_{l/(b-1)}^1 \Psi_b^{id}(y) dy + \frac{l(b-l-1)}{2b^{2k}(b-1)} \int_0^1 \Psi_b^{id}(y) dy \Bigg\} = \\
= -\frac{1}{720} \frac{2b^6 - 19b^5 + 51b^4 - 46b^3 + 12b^2 - 12b + 8}{b^{2(k-1)}(b-1)}.
\end{aligned}
$$

With the help of R_k we are able to compute the value of the variance of the discrepancy of the van der Corput sequence. According to Theorem 2.4.2 and Theorem 2.5.2 we can write

$$
\mathbb{V}(D(N, S_b^{\Sigma})) = \sum_{j=1}^{n} \mathbb{V}(\Psi_b^{\sigma_{j-1}}(x)) + 2 \sum_{j=1}^{n} \sum_{l<j} R_{j-l}.
$$

Hence we obtain the following result

$$
\begin{aligned}
\mathbb{V}\left(D\left(N, S_b^{\Sigma}\right)\right) = \\
= \frac{1}{720} \left[n \frac{4b^7 - 10b^6 + 10b^5 + 14b^4 - 77b^3 + 127b^2 - 68b + 8}{b^2(b-1)^2(b+1)} \right] + O(1).
\end{aligned}
$$

As the real valued roots of the first polynomial $4b^7 - 10b^6 + 10b^5 + 14b^4 - 77b^3 + 127b^2 - 68b + 8$ are all smaller than 2 and the ones of the second polynomial $2b^6 - 7b^5 - 9b^4 + 78b^3 - 124b^2 + 60b - 8$ are at the most, equal to 2, the variance of the discrepancy will always tend to infinity. We will summarize these results in the following theorem.

Theorem 4.1.1. *Let $D(N, S_b^\Sigma)$ denote the extreme discrepancy of the first N points of the van der Corput sequence in base b. Then we have for $1 \leq N \leq b^n$ and every real y*

$$\frac{1}{b^n}\#\left\{N < b^n : D(N, S_b^\Sigma) \leq \mathbb{E}(D(N, S_b^\Sigma)) + y\sqrt{\mathbb{V}(D(N, S_b^\Sigma))}\right\} = \Phi(y) + o(1)$$

where

$$\mathbb{E}\left(D\left(N, S_b^\Sigma\right)\right) = \left(\frac{1}{6}b - \frac{1}{12}\right)n$$

and

$$\mathbb{V}\left(D\left(N, S_b^\Sigma\right)\right) = = \frac{1}{720}\left[n\frac{4b^7 - 10b^6 + 10b^5 + 14b^4 - 77b^3 + 127b^2 - 68b + 8}{b^2(b-1)^2(b+1)}\right] + O(1).$$

Proof. The proof is straightforward using the previous results and Theorem 2.3.1. □

4.2 The Generalized van der Corput Sequence

In case of the generalized van der Corput sequences the formulas for the functions Ψ_b^π are more difficult to handle because, so far, we are not able to write them down explicitly for an arbitrary permutation. Currently the best possible approach seems to be to find classes of permutations, that can be treated similarly. As a starting point we will use the permutation $\pi = (0b-2)(1b-3)\cdots((b-3)/2(b-1)/2)(b-1)$, with

$$Z_b^\pi = \left(\frac{b-2}{b}, \frac{b-3}{b}, \ldots, \frac{0}{b}, \frac{b-1}{b}\right).$$

For $0 \leq i < b-1$ and $x \in [i/b, (i+1)/b[$ we obtain the following formula for the functions $\varphi^{\pi}_{b,h}(x)$

$$\varphi^{\pi}_{b,h}(x) = \begin{cases} -x & h = 1 \\ -2x & h = 2 \\ \vdots & \vdots \\ -(b-i-2)x & h = b-i-2 \\ (i+1)x - i & h = b-i-1 \\ ix - i + 1 & h = b-i \\ \vdots & \vdots \\ 2x - 1 & h = b-2 \\ x & h = b-1 \end{cases} .$$

If we take the maximum of these h functions, we get

$$\Psi^{\pi,+}_b(x) = \begin{cases} x & \text{if} \quad 0 \leq x < \frac{b-1}{b} \\ (b-1)(1-x) & \text{if} \quad \frac{b-1}{b} \leq x < 1 \end{cases}$$

and

$$\Psi^{\pi,-}_b(x) = \begin{cases} k-(k+1)x & \text{if} \quad k/b \leq x < k/(b-1) \\ (b-k-2)x & \text{if} \quad k/(b-1) \leq x < (k+1)/b \\ 0 & \text{if} \quad (b-1)/b \leq x < 1 \end{cases} .$$

The resulting function Ψ^{π}_b is given by

$$\Psi^{\pi}_b = \begin{cases} k(1-x) & \text{if} \quad x \in [k/b, k/(b-1)), \\ (b-k-1)x & \text{if} \quad x \in [k/(b-1), (k+1)/b) \end{cases} \tag{4.3}$$

This is the same function as for the ordinary van der Corput sequences. This means we also get the same results for the expectation and the variance of the discrepancy.

We are now interested in finding permutations similar to the one we just treated and we can apply the results we already obtained with only a few changes. One permutation we can write down immediately with the help of Lemma 2.1.1. Using it we can find a permutation $\widetilde{\pi}$, such that $\Psi^{\pi,-}_b = \Psi^{\pi,+}_b$. In this case it is given by $\pi = (0b-1b-2\cdots 1)$.

For a more general result we look at the permutations defined by

$$Z^{\pi_l}_b = \left(\frac{b-l}{b}, \frac{b-l-1}{b}, \ldots, 0, \frac{b-l+1}{b}, \frac{b-l+2}{b}, \ldots, \frac{b-1}{b}\right), 1 \leq l \leq b.$$

The two extreme cases $l = 1$ and $l = b$ correspond to the ordinary van der Corput sequence and the permutation $\widetilde{\pi}$ with $\Psi^{id,-}_b = \Psi^{\widetilde{\pi},+}_b$ respectively.

The array $Z_b^{\pi_l}$ can be divided into two parts, the first one descending from $\frac{b-l}{b}$ to 0, the second one ascending from $\frac{b-l+1}{b}$ to $\frac{b-1}{b}$. As the later is equal to the ordinary van der Corput sequence, it is not surprising the part of the function $\Psi_b^{\pi_l,+}$ and $\Psi_b^{\pi_l,-}$ with $x \geq \frac{b-l+1}{b}$ is also equal to the corresponding functions for the ordinary van der Corput sequence. This is due to the fact, on the interval $[k/b,(k+1)/b[$ the functions only depend on the value of $\pi_l(k)$ and the elements before $\pi_l(k)/b$ in $Z_b^{\pi_l}$ but not on their position in the array. Therefore we obtain

$$\Psi_b^{\pi_l,+}(x) = \begin{cases} (l-1)x & \text{if} \quad 0 \leq x < \frac{b-l+1}{b} \\ k(1-x) & \text{if} \quad \frac{k}{b} \leq x < \frac{k}{b-1} \\ (b-k-1)x & \text{if} \quad \frac{k}{b-1} \leq x < \frac{k+1}{b} \end{cases}$$

and

$$\Psi_b^{\pi_l,-}(x) = \begin{cases} k-(k+l-1)x & \text{if} \quad \frac{k}{b} \leq x < \frac{k}{b-1} \\ (b-k-l)x & \text{if} \quad \frac{k}{b-1} \leq x < \frac{k+1}{b} \\ 0 & \text{if} \quad \frac{b-l+1}{b} \leq x < 1 \end{cases}.$$

By adding these two functions we get $\Psi_b^{\pi_l} = \Psi_b^{id}$. Hence we get the same expectation and variance as for the ordinary van der Corput sequence.

We have now found b different permutations all leading to the same function Ψ_b^{id}. The extreme discrepancy of any sequence consisting of these permutations will have the same distribution properties as the discrepancy of the ordinary van der Corput sequence.

Theorem 4.2.1. *Let Σ be a sequence consisting of permutations π_l defined by*

$$Z_b^{\pi_l} = \left(\frac{b-l}{b}, \frac{b-l-1}{b}, \ldots, 0, \frac{b-l+1}{b}, \frac{b-l+2}{b}, \ldots, \frac{b-1}{b}\right), 1 \leq l \leq b,$$

then all the functions $\Psi_b^{\sigma_i}$ are equal to Ψ_b^{id}. Further, the discrepancy of every sequence satisfies for $1 \leq N \leq b^n$ and every real y

$$\frac{1}{b^n}\#\left\{N < b^n : D(N, S_b^{\Sigma}) \leq \mathbb{E}(D(N, S_b^{\Sigma})) + y\sqrt{\mathbb{V}(D(N, S_b^{\Sigma}))}\right\} = \Phi(y) + o(1)$$

where

$$\mathbb{E}\left(D\left(N, S_b^{\Sigma}\right)\right) = \left(\frac{1}{6}b - \frac{1}{12}\right)n,$$

and

$$\mathbb{V}\left(D\left(N, S_b^{\Sigma}\right)\right)$$
$$= \frac{1}{720}\left[n\frac{4b^7 - 10b^6 + 10b^5 + 14b^4 - 77b^3 + 127b^2 - 68b + 8}{b^2(b-1)^2(b+1)}\right] + O(1).$$

Proof. This theorem is an immediate consequence of the results given in this section and Theorem 2.3.1. □

Chapter 5

Extensions and Open Problems

5.1 NUT-Sequences

We already mentioned in Section 2.1 there exists some relationship between the generalized van der Corput sequences and the NUT-sequences. Using theses similarities we are able to extent some of our results to these sequences as well. First we will state an explicit definition of the NUT sequences.

Definition 5.1.1. *Let b be prime, $N-1 = \sum_{r=0}^{\infty} a_r(N) b^r$ and $C = \left(c_r^k\right)_{r\geq 0, k\geq 0}$ an infinite nonsingular upper triangular (NUT) matrix with entries $c_r^k \in \{0,1,\ldots,b-1\}$, then we can define the sequence $X_b^C = (x_N)_{N\geq 1}$, where*

$$x_N = \sum_{r=0}^{\infty} \frac{x_{N,r}}{b^{r+1}}$$

and

$$x_{N,r} = \sum_{k=r}^{\infty} c_r^k a_k(N) \pmod{b}.$$

These sequences are called NUT-sequences.

If we now denote by $\delta_r(i)$ the permutation given by $\delta_r(i) = c_r^r i \pmod{b}$ we can define

$$\sigma_r(i) = \delta_r(i) + \sum_{k=r+1}^{\infty} c_r^k a_k(N) \pmod{b}.$$

This is again a permutation of $\{0, 1, \ldots, b-1\}$. In order to simplify notations we will write

$$\theta_r(N) = \sum_{k=r+1}^{\infty} c_r^k a_k(N) \pmod b$$

and

$$\delta_r(i) \uplus \theta_r(N) := \delta_r(i) + \theta_r(N) \pmod b.$$

Similar to the generalized van der Corput sequences there exist formulas for the extreme and the star discrepancy of NUT-sequences. Using the same definitions for the functions φ_b^σ, $\Psi_b^{\sigma,+}$, $\Psi_b^{\sigma,-}$ and Ψ_b^σ as before, we are able to write down the following theorem and corollary, which has been found by H. Faure [8].

Theorem 5.1.1. *For all NUT matrices C and integers $N \geq 1$ we have*

$$\begin{aligned}
D^+\left(N, X_b^C\right) &= \sum_{j=1}^{\infty} \Psi_b^{\delta_{j-1} \uplus \theta_{j-1}(N),+}\left(\frac{N}{b^j}\right) \\
D^-\left(N, X_b^C\right) &= \sum_{j=1}^{\infty} \Psi_b^{\delta_{j-1} \uplus \theta_{j-1}(N),-}\left(\frac{N}{b^j}\right) \\
D\left(N, X_b^C\right) &= \sum_{j=1}^{\infty} \Psi_b^{\delta_{j-1}}\left(\frac{N}{b^j}\right) \\
D^*\left(N, X_b^C\right) &= \max\left(D^+\left(N, X_b^C\right), D^-\left(N, X_b^C\right)\right).
\end{aligned}$$

Corollary 5.1.1. *For all integers n and N with $1 \leq N \leq b^n$ we have*

$$\begin{aligned}
D^+\left(N, X_b^C\right) &= \sum_{j=1}^{n} \Psi_b^{\delta_{j-1} \uplus \theta_{j-1}(N),+}\left(\frac{N}{b^j}\right) + \frac{N}{b^n} \\
D^-\left(N, X_b^C\right) &= \sum_{j=1}^{n} \Psi_b^{\delta_{j-1} \uplus \theta_{j-1}(N),-}\left(\frac{N}{b^j}\right) \\
D\left(N, X_b^C\right) &= \sum_{j=1}^{n} \Psi_b^{\delta_{j-1}}\left(\frac{N}{b^j}\right) + \frac{N}{b^n}.
\end{aligned}$$

These formulas for the discrepancy are the same or even a little more simple than the ones for the generalized van der Corput sequences, so all our results will hold true for NUT-sequences too. Especially we can use the formulas we developed for the calculation of the expectation and variance of the discrepancies. In the case of the extreme discrepancy the formulas for the NUT-sequences are exactly the same as the ones for the generalized van der Corput sequences. The only difference is they are restricted to a smaller class of permutations, namely all the permutations that can be obtained by multiplying the base b with an integer $1 \le l \le b-1$ modulo b.

The difficulty of studying the general behavior of discrepancies of the NUT-sequences arises from the fact that the permutations depend on N via the b-adic expansion of $N-1$ and on the generator matrix C via its entries above the diagonal. The only quantity, which can be treated comparatively easy is the extreme discrepancy as it does not show these dependencies. We need to calculate the expectation and the variance for the functions Ψ_b^σ where σ is one of the b different permutations obtained by multiplying b with an integer $1 \le l \le b-1$ modulo b. Although compared to the situation of the generalized van der Corput sequences this is a very reduced set of permutations, we did not manage to find a general solution to the problem. We will use this case to illustrate the problem one faces when dealing with permutations for arbitrary bases.

Let b be prime and $1 \le l \le b-1$ then the permutation given by $\sigma_l(i) = li \pmod{b}$ leads to

$$\left(0, l, 2l, \dots, \left\lfloor \frac{b}{l} \right\rfloor l, \left(\left\lfloor \frac{b}{l} \right\rfloor + 1\right) l - b, \dots, \left\lfloor \frac{2b}{l} \right\rfloor l - b, \dots \right.$$
$$\left. \dots, \left(\left\lfloor \frac{lb}{l} \right\rfloor - 1\right) l - (l-1)b \right).$$

For $k \le \lfloor \frac{b}{l} \rfloor$ it is quite easy to determine the functions $\Psi_b^{\sigma_l,+}(x)$, one obtains for $k < \lfloor \frac{b}{l} \rfloor$

$$\Psi_b^{\sigma_l,+}(x) = \begin{cases} k - ((k-1)l+1)x & \text{if} \quad \frac{k}{b} \le x < \frac{k}{b-l} \\ (b-lk-1)x & \text{if} \quad \frac{k}{b-l} \le x < \frac{k+1}{b} \end{cases}$$

and for $k = \lfloor \frac{b}{l} \rfloor$

$$\Psi_b^{\sigma_l,+}(x) = \begin{cases} k - ((k-1)l+1)x & \text{if} \quad \frac{k}{b} \le x < \frac{1}{l} \\ 1-x & \text{if} \quad \frac{1}{l} \le x < \frac{k+1}{b} \end{cases}.$$

For $k > \lfloor \frac{b}{l} \rfloor$ the situation gets more complicated. If we take for example $k = \lfloor \frac{jb}{l} \rfloor + i$, $i \neq 0$ we have

$$\sigma_l(k) = \left(\left\lfloor \frac{jb}{l} \right\rfloor + i\right) l - jb.$$

We define numbers n_r, $0 \le r \le j-1$ as

$$n_r = \left(\left\lfloor \frac{rb}{l} \right\rfloor + 1 \right) l - rb$$

and with the help of those

$$a_r = \left| \{ h \quad \text{with} \quad 0 \le h \le j \quad \text{and} \quad n_h \le n_r \} \right| .$$

We need these figures in order to determine the functions $\varphi_{b,h}^{\sigma_l}$. To find the maximum of the h different functions $\varphi_{b,h}^{\sigma_l}$, we only have to consider those, where $A(h/b, k+1, Z_b^{\sigma_l})$ and $A([h/b, 1[, k+1, Z_b^{\sigma_l})$ change. So we define functions $g_{r,t}(x)$ and $\tilde{g}_{r,t}(x)$, which represent those functions $\varphi_{b,h}^{\sigma_l}$ where these changes occur. Let us first recall the definition of the functions:

$$\varphi_{b,h}^{\sigma}(x) = \begin{cases} A(h/b, k+1, Z_b^{\sigma}) - hx & \text{if} \quad 0 \le h \le \sigma(k), \\ (b-h)x - A([h/b, 1[, k+1, Z_b^{\sigma}) & \text{if} \quad \sigma(k) < h < b \end{cases} .$$

As it is more convenient in this context we augmented the index k by one compared with the original definition. We obtain for $t < i$

$$g_{r,t}(x) = (t-1)(j+1) + a_r + 1 - \left[\left(\left\lfloor \frac{rb}{l} \right\rfloor + t \right) l - rb + 1 \right] x$$

and for $t \ge i$

$$\tilde{g}_{r,t}(x) = \left[(r+1)b - \left(\left\lfloor \frac{rb}{l} \right\rfloor + t \right) l - 1 \right] x - \left\lfloor \frac{jb}{l} \right\rfloor + tj - (j - a_r + 1).$$

Taking the maxima over the t this gives us

$$\max_{1 \le t < i} g_{r,t}(x) = g_{r,i-1}(x)$$

and

$$\max_{i \le t < \lfloor \frac{(j+1)b}{l} \rfloor - \lfloor \frac{jb}{l} \rfloor} \tilde{g}_{r,t}(x) = \tilde{g}_{r,i}(x).$$

This still leaves us $2j$ functions to consider. Taking those maxima cannot be done in a general but we have to impose restrictions on l. We will demonstrate this for the case $j = 1$. There are four different functions we have to take into account:

$$g_1(x) = g_{0,i-1}(x) = 2i - 1 - [(i-1)l + 1]x$$

$$g_2(x) = g_{1,i-1}(x) = 2(i-1) - \left[\left(\left\lfloor \frac{b}{l} \right\rfloor + (i-1) \right) l - b + 1 \right] x$$

$$g_3(x) = g_{0,i}(x) = (b - il - 1)x - \left(\left\lfloor \frac{b}{l} \right\rfloor - i \right)$$

$$g_4(x) = g_{1,i}(x) = \left(2b - \left(\left\lfloor \frac{b}{l} \right\rfloor + i \right) l - 1 \right) x - \left(\left\lfloor \frac{b}{l} \right\rfloor - i + 1 \right)$$

We will denote the x coordinate of the point of intersection of two functions $g_i(x)$ and $g_j(x)$ by s_{ij}. In order to obtain the maximum we need to consider four s_{ij}

$$s_{12} = s_{34} = \frac{1}{b - \left\lfloor \frac{b}{l} \right\rfloor l}$$
$$s_{13} = s_{24} = \frac{k-1}{b-l}.$$

As $\frac{k-1}{b-l}$ lies in the interval $[k/b, (k+1)/b[$ we have to differentiate between two cases where either $\frac{1}{b-\lfloor \frac{b}{l} \rfloor l} > \frac{k-1}{b-l}$ or $\frac{1}{b-\lfloor \frac{b}{l} \rfloor l} < \frac{k-1}{b-l}$. So for $\lfloor \frac{b}{l} \rfloor < k < \lfloor \frac{2b}{l} \rfloor$ we obtain

$$\Psi_b^{\sigma_l,+}(x) = \begin{cases} 2i - 1 - [(i-1)l + 1]x & \text{if } \frac{k}{b} \leq x < \frac{k-1}{b-l} \\ (b - il - 1)x - \left(\lfloor \frac{b}{l} \rfloor - i\right) & \text{if } \frac{k-1}{b-l} \leq x < \frac{k+1}{b} \end{cases}$$

if $\frac{1}{b-\lfloor \frac{b}{l} \rfloor l} > \frac{k-1}{b-l}$ and

$$\Psi_b^{\sigma_l,+}(x) = \begin{cases} 2(i-1) - \left[\left(\lfloor \frac{b}{l} \rfloor + (i-1)\right) l - b + 1\right] x & \text{if } \frac{k}{b} \leq x < \frac{k-1}{b-l} \\ \left(2b - \left(\lfloor \frac{b}{l} \rfloor + i\right) l - 1\right) x - \left(\lfloor \frac{b}{l} \rfloor - i + 1\right) & \text{if } \frac{k-1}{b-l} \leq x < \frac{k+1}{b} \end{cases}$$

if $\frac{1}{b-\lfloor \frac{b}{l} \rfloor l} < \frac{k-1}{b-l}$.

For arbitrary j one has to take into account more functions g_i and gets more different cases. We will not analyze this problem further. Even if one would be able to describe the function $\Psi_b^{\sigma_l,+}$ in a useful fashion it still remains to determine the function $\Psi_b^{\sigma_l,-}$, of course there exist certain relations between these two functions but there is some work left.

5.2 Conclusions and Open Problems

It was the objective of this work to get a better understanding of the distribution properties of the generalized van der Corput sequences. We were able to find a probability model, which enables us to prove a central limit theorem for the discrepancy of these sequences under the condition we can show the variance of the discrepancy tends to infinity with at least order N^α for some $\alpha > 2/3$. Although we were able to obtain an explicit formula for the variance verifying this property proved to be more involved than we expected. We now want to discuss major problems and possible solutions.

Although we were able to show the variance can be written as the sum of variances and some additional terms we called error terms, it is not possible to derive any general results from this formula. This is due to the fact that the error terms sum up to order n and therefore are of the same order as the sum of the variances. From Theorems van:theorem1 and van:theorem7, we know it is possible to write $R_k = \frac{c_k}{b^{2k}}$ where c_k is a ultimately periodic sequence but the previous period makes it hard or even impossible to get some general results on the magnitude or sign of the error terms. Therefore it is necessary to study the functions $\Psi_b^{\sigma}(x)$ to get a better understanding of these terms.

As we have seen in Section 5.1 it is sometimes quite hard to find an explicit form of the functions $\Psi_b^{\sigma}(x)$. This is mainly caused by the necessity of taking the maxima over b different functions for each of the b intervals. We are already facing problems in this very limit case (only b different permutations are taken into account), so it is not very surprising this strategy does not provide results in the general case (where $b!$ permutations have to be considered). Therefore it is crucial to develop different approaches.

In this work we considered special bases and used a computer program (see appendix) to obtain the functions $\Psi_b^{\sigma}(x)$, compare Section 3. Theoretically this is possible for all bases but the complexity of the calculations grows rapidly with the base. For large bases it would be necessary to find more efficient algorithms.

Another strategy used in Section 4 is to find special permutations where the function $\Psi_b^{\sigma}(x)$ can be found comparatively easy. This approach demands a profound knowledge of the relations between the permutations and the resulting functions in order to make good guesses.

We already mentioned for bases 2, 3 and 4 we only obtained a small number of functions $\Psi_b^{\sigma}(x)$ although theoretically there could be $b!$ of them. A similar phenomenon we observed for bases 5 and 7 though we did not study them completely. This leads us to the conjecture that the number of functions we have to consider when we calculate the expectation and the variance of the discrepancy is small compared to $b!$. Most of them (or even all) are symmetric with respect to $x = 1/2$ and have their local extrema at points of the form $x = \frac{i}{b-j}$, $i = 1, \ldots, b-j-1$, $j = 1, \ldots, b-2$. Of course they have to be positive, continuous and the absolute values of their coefficients are bounded by $b - 1$.

So another strategy could consist of finding all piecewise linear functions satisfying the above conditions. Then one has to construct permutations for each of them. If this approach would lead to a complete solution of the problem remains doubtful as the completeness of the above functions needs to be proved. Still this could lead to some more results.

Appendix A

Maple Code

In Chapter 3 we used Maple to obtain the functions $\Psi_b^{\sigma,+}$ and $\Psi_b^{\sigma,-}$. This is achieved by the following algorithm. It returns a matrix A, where the first column contains the piecewise defined function and the second column the intervals. The input consists of the vector $Z = Z_b^\sigma$ defined in 2.1.2 and the basis b. For example we would obtain the following result for $b = 7$ and $Z = [0, 2/7, 4/7, 6/7, 1/7, 3/7, 5/7]$:

$$\begin{matrix} [6x & 0 & 0 & 0 & 0 & 0 & 0] \\ [1-x & 4x & 0 & 0 & 0 & 0 & 0] \\ [2-3x & 2x & 0 & 0 & 0 & 0 & 0] \\ [3-5x & 1-x & 0 & 0 & 0 & 0 & 0] \\ [1-x & -2+4x & -2+4x & 0 & 0 & 0 & 0] \\ [3-3x & -1+2x & -1+2x & 0 & 0 & 0 & 0] \\ [5-5x & 0 & 0 & 0 & 0 & 0 & 0] \end{matrix} \qquad \begin{matrix} [0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0] \\ [\frac{1}{7} & \frac{1}{5} & \frac{2}{7} & 0 & 0 & 0 & 0 & 0] \\ [\frac{2}{7} & \frac{2}{5} & \frac{3}{7} & 0 & 0 & 0 & 0 & 0] \\ [\frac{3}{7} & \frac{1}{2} & \frac{4}{7} & 0 & 0 & 0 & 0 & 0] \\ [\frac{4}{7} & \frac{3}{5} & \frac{2}{3} & \frac{5}{7} & 0 & 0 & 0 & 0] \\ [\frac{5}{7} & \frac{4}{5} & \frac{5}{6} & \frac{6}{7} & 0 & 0 & 0 & 0] \\ [\frac{6}{7} & 1 & 0 & 0 & 0 & 0 & 0 & 0] \end{matrix}$$

It is also possible to modify the algorithm so one only has to enter a number instead of the vector Z. The vector is then obtained by the command "permute$\left(\left[seq\left(\frac{i}{b}, i = 0..(b-1)\right)\right]\right)$" and consists of all possible permutations. The number gives the position of the permutation in Z. The following algorithm provides the results for $\Psi_b^{\sigma,+}$. It can easily be changed to return $\Psi_b^{\sigma,-}$.

```
Van_der_Corput1:=module()
export ps;
ps:=proc(Z, b)
local a, c, d, e, f, g, h, i, j, l, n, N, o, p, P, q, Q, r, s, S, t, u, v, w, x, φ, k, A;
uses LinearAlgebra;
A:=Matrix(b, 2);
```

```
for k from 1 to b do
    φ:=Matrix(b, b);
    w :=Matrix(b + 1, 1);
    o :=Matrix(b, b);
    s := 1;
    l := 0;
    for h from 0 to (b − 1) do
        if (h/b) <= Z[k] then
            for j from 1 to k do
                if Z[j] < h/b then
                    l := l + 1;
                end if;
            end do;
            f := x → l − h ∗ x;
            φ[h + 1, 1] := f(x);
            j := 0;
            l := 0;
        else
            for j from 1 to k do
                if Z[j] >= h/b then
                    l := l + 1;
                end if;
            end do;
            f := x → (b − h) ∗ x − l;
            φ[h + 1, 1] := f(x);
            j := 0;
            l := 0;
        end if;
    end do;
    w[1, 1] := ((k − 1))/b;
    w[2, 1] := k/b;
    for r from 1 to (b − 1) do
        for t from 1 to s do
            c := w[t, 1];
            d := w[t + 1, 1];
            if (φ[r, t] − φ[r + 1, t]) <> 0 then
                g := x → φ[r, t] − φ[r + 1, t];
                v :=roots(g(x), x);
                P :=verify(v,[[c..d,0..2]],'[[interval,interval]]') assuming x > c,
                    x < d;
                if P=true then
```

```
    for n from t+1 to s+1 do
        e := w[n+1,1];
        w[n+1,1] := w[t+1,1];
        w[t+1,1] := e;
    end do;
    w[t+1,1] := v[1][1];
    s := s+1;
    for S from 0 to s-t-1 do
        for N from r to b do
            φ[N,s-S] := φ[N,s-S-1];
        end do;
    end do;
    p:=verify(φ[r,t],φ[r+1,t],boolean(greater_than)) assuming
        x > c, x < v[1][1];
    if p= true then
        o[r+1,t] := p;
        φ[r+1,t] := φ[r,t];
    elif p=false then
        o[r+1,t] := p;
        φ[r+1,t] := φ[r+1,t];
    else
        o[r+1,t] := p;
    end if;
    q:=verify(φ[r,t],φ[r+1,t+1],boolean(greater_than))
        assuming x > v[1][1], x < d;
    if q=true then
        o[r+1,t+1] := q;
        φ[r+1,t+1] := φ[r,t];
    elif q=false then
        o[r+1,t+1] := q;
        φ[r+1,t+1] := φ[r+1,t+1];
    else
        o[r+1,t+1] := q;
    end if;
else
    Q:=verify(φ[r,t],φ[r+1,t],'greater_than') assuming
        x > c, x < d;
    if Q=true then
        o[r+1,t] := Q;
        φ[r+1,t] := φ[r,t];
    elif Q =false then
```

```
                    o[r + 1, t] := Q;
                    φ[r + 1, t] := φ[r + 1, t];
                else
                    o[r + 1, t] := Q;
                end if;
            end if;
        end if;
    end do;
  end do;
  A[k, 1] := φ[b, 1..b];
  A[k, 2] := w^%T;
  unassign('a','c','d','e','f','g', 'h', 'i', 'j');
  unassign('l', 'n', 'N', 'o', 'p','P', 'q','Q', 'r', 's');
  unassign('S', 't', 'u', 'v', 'w','x','phi');
end do;
A;
end proc;
end module;
```

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