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Vienna University of Technology

TECHNISCHE UNIVERSITÄT WIEN
Karlsplatz 13, 1040 Wien

DIPLOMARBEIT

Coalgebras, Hopf Algebras and Combinatorics

Betreuer:

Univ.-Prof. Dipl.-Ing. Dr. Michael Drmota
E104 Institut für Diskrete Mathematik und Geometrie
E-Mail-Adresse: michael.drmota@tuwien.ac.at

Autor:

Lukas Daniel Klausner BSc
066 400 Mathematik
Anschrift: Steinbachstraße 34-36
3001 Mauerbach
E-Mail-Adresse: lukas.d.klausner@tuwien.ac.at

In loving memory of my mother

“A mathematician is a machine for converting coffee into theorems.”

—Alfréd Rényi

“A comathematician, by categorical duality, is a machine for converting cotheorems into ffee.”

—anonymous

Preface

Hopf algebras are a relatively new concept in algebra, first encountered by their namesake Heinz Hopf in 1941 in the field of algebraic topology. In the 1960s, study of Hopf algebras increased, perhaps spurred by the findings of Moss Eisenberg Sweedler, and by the late 1980s and early 1990s, Hopf algebras gained further interest from mathematicians and even scientists in other fields, as connections with quantum mechanics became clearer. Hopf algebras are a particularly interesting algebraic concept in that they turn up in almost any field of study – number theory, algebraic geometry and algebraic topology, Lie theory and representation theory, Galois theory and the theory of separable field extensions, operator theory, distribution theory, quantum mechanics and last but not least combinatorics.

The aim of my diploma thesis is to give an introduction into the notation and paradigm of coalgebras and Hopf algebras and to present applications in and connections with combinatorics.

In Chapter 1, we briefly revisit well-known definitions and results from basic algebra and tensor products to lay the groundwork for the following chapters. Apart from a basic understanding of algebraic thinking and some basic notions from linear algebra, no knowledge on the part of the reader is assumed. The presentation of the content is along the lines of [Abe80], with some elements inspired by [DNR01].

In Chapter 2, we define coalgebras by dualizing the definition of algebras and introduce the sigma notation (or Sweedler notation, after the aforementioned algebraist), which simplifies the written treatment of coalgebras significantly. We then consider dual constructions of algebras and coalgebras and finally discuss a special subset of elements in a coalgebra, the grouplike elements. In Chapter 3, we present definitions and basic properties of bialgebras and Hopf algebras, as well as some examples. We follow the presentation of [DNR01].

Chapter 4 presents an important natural occurrence of Hopf algebras in the theory of combinatorial classes. The concepts of composition and decomposition of objects are linked to the multiplication and comultiplication maps. The results are due to [Bla10]. In Chapter 5, we consider applications of coalgebras and Hopf algebras in treating enumeration problems, as well as in formulating relations between polynomial sequences on the one hand and linear functionals and operators on the other hand, giving an introduction to the field of umbral calculus. The exposition is based on [JR82].

In addition to the material presented in this thesis, the following applications merit mention:

- An important event in the history of the study of Hopf algebras was the discovery of relations to quantum mechanics, namely the concept of quantum groups; for a detailed treatment of this topic, see [Kas95].
- Finally, also of interest is the concept of Hopf algebras enriched by so-called characters (multiplicative linear functionals) to gain extensive results about quasi-symmetric functions, as presented by [ABS06].

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Chapter 1

Preliminaries

1.1 Rings and Fields

Definition 1.1: A structure $\langle R, +, \cdot, 0, 1 \rangle$ is called a **unital ring** if it satisfies the following properties:

- $\langle R, +, 0 \rangle$ is an Abelian group.
- $\langle R, \cdot, 1 \rangle$ is a semigroup with unit element (or identity) 1.
- The distributive laws

$$\begin{aligned}r \cdot (s + t) &= r \cdot s + r \cdot t \\(r + s) \cdot t &= r \cdot t + s \cdot t\end{aligned}$$

hold for all $r, s, t \in R$.

A ring with identity whose multiplication is commutative is called a **commutative unital ring**. ▽

Remark 1.2: As we do not require nonunital rings in this thesis, unital rings are simply called **rings**; analogously, commutative unital rings are called **commutative rings**.

Definition 1.3: Assume $\langle R, +_R, \cdot_R, 0_R, 1_R \rangle, \langle S, +_S, \cdot_S, 0_S, 1_S \rangle$ are rings. If a map $\rho: R \rightarrow S$ satisfies

$$\begin{aligned}\rho(r_1 +_R r_2) &= \rho(r_1) +_S \rho(r_2) \\ \rho(r_1 \cdot_R r_2) &= \rho(r_1) \cdot_S \rho(r_2) \\ \rho(1_R) &= 1_S\end{aligned}$$

for $r_1, r_2 \in R$, then ρ is called a **ring morphism** from R to S . (These conditions already imply $\rho(0_R) = 0_S$ and $\rho(-r) = -\rho(r)$.) We call the category of rings **Ring** and the category

of commutative rings **CommRing**. The set of all ring morphisms from R to S is written $\mathbf{Ring}(R, S)$ (respectively $\mathbf{CommRing}(R, S)$ if R, S are commutative rings). ∇

Remark 1.4: We will not always distinguish operations or unit elements by structure from now on, omitting the index whenever it is unnecessary. Furthermore, the product $r \cdot s$ of $r, s \in R$ will be written rs for simplicity's sake.

Definition 1.5: The **characteristic** $\text{char}(R)$ of a ring R is the smallest positive integer $n \in \mathbb{N}^\times$ such that

$$\underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = 0$$

if such an n exists; if there is no such positive integer, we set $\text{char}(R) := 0$. ∇

Definition 1.6: A structure $\langle \mathbb{K}, +, \cdot, 0, 1 \rangle$ is called a **field** if it satisfies the following properties:

- $\langle \mathbb{K}, +, \cdot, 0, 1 \rangle$ is a commutative ring.
- For all $k \in \mathbb{K} \setminus \{0\}$, there is a multiplicative inverse k^{-1} .

Equivalently, $\langle \mathbb{K}, +, \cdot, 0, 1 \rangle$ is a field if it satisfies:

- $\langle \mathbb{K}, +, 0 \rangle$ is an Abelian group.
- $\langle \mathbb{K} \setminus \{0\}, \cdot, 1 \rangle$ is an Abelian group.
- The distributive law

$$k \cdot (\ell + m) = k \cdot \ell + k \cdot m$$

holds for all $k, \ell, m \in \mathbb{K}$.

∇

Definition 1.7: Assume $\langle \mathbb{K}, +_{\mathbb{K}}, \cdot_{\mathbb{K}}, 0_{\mathbb{K}}, 1_{\mathbb{K}} \rangle, \langle \mathbb{L}, +_{\mathbb{L}}, \cdot_{\mathbb{L}}, 0_{\mathbb{L}}, 1_{\mathbb{L}} \rangle$ are fields. If a map $\kappa: \mathbb{K} \rightarrow \mathbb{L}$ is a ring morphism, then κ is also a **field morphism** from \mathbb{K} to \mathbb{L} . (The conditions required for a ring morphism already imply $\kappa(k^{-1}) = (\kappa(k))^{-1}$.) We call the category of fields **Field**. The set of all field morphisms from \mathbb{K} to \mathbb{L} is written $\mathbf{Field}(\mathbb{K}, \mathbb{L})$. ∇

Remark 1.8: In the following, \mathbb{K} will always denote a field.

1.2 Modules

Definition 1.9: Let $\langle R, +, \cdot, 0, 1 \rangle$ be a ring and $\langle M, + \rangle$ an Abelian group; suppose we are given maps

$$\phi: R \times M \longrightarrow M \qquad \psi: M \times R \longrightarrow M.$$

For $r \in R, m \in M$ we write

$$\phi(r, m) =: rm \qquad \psi(m, r) =: mr.$$

We call M a **left R -module** (respectively **right R -module**) if the following properties hold for all $r, s \in R$ and $m, n \in M$:

$$\begin{array}{ll} r(m+n) = rm + rn & (m+n)r = mr + nr \\ (r+s)m = rm + sm & m(r+s) = mr + ms \\ (rs)m = r(sm) & m(rs) = (mr)s \\ 1m = m & m1 = m \end{array}$$

We then call ϕ (respectively ψ) the **structure map** of the left (respectively right) R -module. Furthermore, if R, S are rings and M is both a left R -module and a right S -module which also satisfies

$$(rm)s = r(ms)$$

for all $r \in R, s \in S$ and $m \in M$ we call M a **two-sided (R, S) -module**; a two-sided (R, R) -module is simply called a **two-sided R -module**. If R is commutative, left and right R -modules can be identified and are simply called **R -modules**. (For example, every Abelian group is a \mathbb{Z} -module.) ∇

Definition 1.10: If we consider a ring R and the multiplication map of the ring

$$\mu: R \times R \longrightarrow R: (r, s) \longmapsto rs$$

as the structure map, R becomes a left R -module and a right R -module, thus a two-sided R -module. In the case of a field \mathbb{K} , a \mathbb{K} -module is called a **\mathbb{K} -vector space** or **\mathbb{K} -linear space**. ∇

Definition 1.11: Let M, N be left R -modules. A map $f: M \longrightarrow N$ which satisfies

$$f(rm_1 + sm_2) = rf(m_1) + sf(m_2)$$

for all $r, s \in R$ and $m_1, m_2 \in M$ is called a **left R -module morphism** from M to N . A \mathbb{K} -module morphism is also called a **\mathbb{K} -linear map**. The category of left R -modules is called ${}_R\mathbf{Mod}$; the set of all left R -module morphisms from M to N is called ${}_R\mathbf{Mod}(M, N)$.

Similarly, for right R -modules M, N in the category \mathbf{Mod}_R we can define **right R -module morphisms** from M to N , the set of which we call $\mathbf{Mod}_R(M, N)$.

The set ${}_R\mathbf{Mod}(M, M)$ of **endomorphisms** (morphisms from M into itself) is also written as ${}_R\mathbf{End}(M)$ ($\mathbf{Mod}_R(M, M)$ as $\mathbf{End}_R(M)$, respectively). If $f \in {}_R\mathbf{Mod}(M, N)$ or $f \in \mathbf{Mod}_R(M, N)$ is bijective, f is called an **isomorphism**. (For example, the identity map $\mathbf{1}_M$ – or simply $\mathbf{1}$ – from M to M is an isomorphism.) ∇

Definition 1.12: Let $f, g \in {}_R\mathbf{Mod}(M, N)$. Defining

$$\begin{aligned}(f + g)(m) &:= f(m) + g(m) \\ (-f)(m) &:= -f(m)\end{aligned}$$

for all $m \in M$, both $f + g$ and $-f$ are left R -module morphisms from M to N . With these operations, ${}_R\mathbf{Mod}(M, N)$ is an Abelian group. If N is even a two-sided module, then by defining

$$(fr)(m) := f(m)r$$

for all $r \in R$ and $m \in M$ we have $fr \in {}_R\mathbf{Mod}(M, N)$ and thus ${}_R\mathbf{Mod}(M, N)$ is a right R -module. If we have $N = R$, then it is a two-sided module by Definition 1.10, and in this case the right R -module ${}_R\mathbf{Mod}(M, R)$ is called the **dual right R -module** of the left R -module M , which is written as M^* . If R is commutative, then ${}_R\mathbf{Mod}(M, N)$ can also be regarded as a left R -module. ∇

Definition 1.13: A subgroup N of a left R -module M is called a **left R -submodule** of M if

$$n \in N, r \in R \Rightarrow rn \in N;$$

the factor group M/N then also inherits a left R -module structure and is called the **factor left R -module** (similar nomenclature is used for right R -modules). If we regard a ring R as a left R -module (respectively right R -module respectively two-sided R -module), then an R -submodule is simply a **left ideal** (respectively **right ideal** respectively **two-sided ideal**).

If the only left R -submodules of a left R -module M are $\{o\}$ – where o denotes the neutral element in M – and M , we call M a **simple left R -module** (or **irreducible left R -module**). If $f: M \rightarrow N$ is a left R -module morphism, then the sets

$$\begin{aligned}\ker f &= \{m \in M \mid f(m) = o\} \\ \operatorname{im} f &= \{f(m) \in N \mid m \in M\}\end{aligned}$$

are left R -submodules of M and N , respectively, and are called **kernel** and **image** of f , respectively.

The analogue for the R -submodule in a \mathbb{K} -vector space is the **\mathbb{K} -subspace**.

The smallest left R -submodule which contains a subset $S \subseteq M$ of M is called the left R -submodule generated by S and written $\langle S \rangle$. ∇

1.3 Tensor Products

Remark 1.14: Given a set X , the **free \mathbb{Z} -module** generated by the set X can be understood as the set

$$\left\{ \sum_{i=1}^n z_i x_i \mid n \in \mathbb{N}, z_i \in \mathbb{Z} \forall i, x_i \in X \forall i \right\}$$

with the natural addition $+$.

Definition 1.15: Let R be a ring, M a right R -module and N a left R -module. Let $F(M \times N)$ be the free \mathbb{Z} -module generated by the set $M \times N$ and $G(M \times N)$ the \mathbb{Z} -submodule of $F(M \times N)$ generated by the set of all elements of the types

$$\begin{aligned} (m_1 + m_2, n) - (m_1, n) - (m_2, n) \\ (m, n_1 + n_2) - (m, n_1) - (m, n_2) \\ (mr, n) - (m, rn) \end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$. In this case, we call the factor group $F(M \times N)/G(M \times N)$ the **tensor product** of M and N over R , written $M \otimes_R N$ or simply $M \otimes N$ if R is clear from the context. The residual class $(m, n) + G(M \times N)$ containing the element (m, n) is denoted by $m \otimes n$. ∇

Remark 1.16: By definition, we have:

$$\begin{aligned} (m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n \\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2 \\ mr \otimes n &= m \otimes rn \end{aligned}$$

For every element of $M \otimes N$ there is a $k \in \mathbb{N}^\times$ such that the element can be written as

$$\sum_{i=1}^k m_i \otimes n_i$$

with $m_i \in M$ and $n_i \in N$ for all i . If R is commutative, we can define

$$r(m \otimes n) := mr \otimes n = m \otimes rn,$$

by which $M \otimes N$ is an R -module.

Remark 1.17: An alternative definition of the tensor product can be given using multilinear maps on vector spaces and the universal property of such maps; we will not discuss this further.

Definition 1.18: Assume R is a commutative ring and M, N, O are R -modules. We call a map $f: M \times N \longrightarrow O$ a **bilinear map** if it fulfils

$$\begin{aligned} f(rm_1 + sm_2, n) &= rf(m_1, n) + sf(m_2, n) \\ f(m, rn_1 + sn_2) &= rf(m, n_1) + sf(m, n_2) \end{aligned}$$

for all $r, s \in R$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$. The set of all bilinear maps from $M \times N$ to O is denoted by $\mathbf{BilMap}_R(M \times N, O)$.

By the definition of tensor products, the map

$$\varphi: M \times N \longrightarrow M \otimes N: (m, n) \longmapsto m \otimes n$$

is bilinear; it is called the **canonical bilinear map**. ▽

Remark 1.19: We can establish a bijective relationship between ${}_R\mathbf{Mod}(M \otimes N, O)$ and $\mathbf{BilMap}_R(M \times N, O)$ by using the following maps:

$$\begin{aligned} \Phi: {}_R\mathbf{Mod}(M \otimes N, O) &\longrightarrow \mathbf{BilMap}_R(M \times N, O): g \longmapsto g \circ \varphi \\ \Phi^{-1}: \mathbf{BilMap}_R(M \times N, O) &\longrightarrow {}_R\mathbf{Mod}(M \otimes N, O): f \longmapsto f \circ \varphi', \end{aligned}$$

where φ' maps $m \otimes n$ to (m, n) , which means that Φ^{-1} maps $f \in \mathbf{BilMap}_R(M \times N, O)$ to $g \in {}_R\mathbf{Mod}(M \otimes N, O)$ via $g(m \otimes n) = f(m, n)$.

Definition 1.20: Assume that we have right R -modules M, M' , left R -modules N, N' and R -module morphisms $f: M \longrightarrow M'$, $g: N \longrightarrow N'$. We define a map

$$\phi: M \times N \longrightarrow M' \otimes N': (m, n) \longmapsto f(m) \otimes g(n)$$

and see that the following properties hold for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$:

$$\begin{aligned} \phi(m_1 + m_2, n) &= \phi(m_1, n) + \phi(m_2, n) \\ \phi(m, n_1 + n_2) &= \phi(m, n_1) + \phi(m, n_2) \\ \phi(mr, n) &= \phi(m, rn) \end{aligned}$$

Consider the R -module morphism

$$\rho: M^* \otimes N^* \longrightarrow (M \otimes N)^*$$

defined by

$$\rho(f \otimes g): M \otimes N \longrightarrow M' \otimes N': m \otimes n \longmapsto f(m) \otimes g(n).$$

$\rho(f \otimes g)$ is well-defined and determined uniquely by the previous note and the properties of ϕ . We suppress ρ in the notation, write $\rho(f \otimes g)$ as $f \otimes g$ and call it the **tensor product** of f and g . We thus have

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

If R is commutative, $f \otimes g$ is an R -module morphism. ▽

Definition 1.21: We define the **twist map** τ as

$$\tau: M \otimes N \longrightarrow N \otimes M: m \otimes n \longmapsto n \otimes m.$$

▽

1.4 Algebras

Definition 1.22: Assume R is a commutative ring. If we have a ring A and a ring morphism $\eta_A: R \rightarrow A$, we can consider A a left A -module using the ring multiplication map. We can also view it as a left R -module by defining

$$ra := \eta_A(r)a$$

as the action of $r \in R$ on $a \in A$. We call A an R -**algebra** if

$$(ra)b = a(rb)$$

holds for all $r \in R$ and $a, b \in A$. This is equivalent to $\eta_A(R) \subseteq Z(A)$, where

$$Z(A) := \{a \in A \mid ab = ba \forall b \in A\}$$

denotes the **centre** of A . ▽

Remark 1.23: The map

$$f: A \times A \rightarrow A: (a, b) \mapsto ab$$

is bilinear if we consider A an R -module. We thus obtain an R -module morphism

$$\mu_A: A \otimes A \rightarrow A$$

and we can now see that there is an equivalent definition for an R -algebra A :

Definition 1.24: Let A be an R -module and

$$\begin{aligned} \eta_A: R &\rightarrow A \\ \mu_A: A \otimes A &\rightarrow A \end{aligned}$$

be R -module morphisms. We call $\langle A, \mu_A, \eta_A \rangle$ an R -algebra if diagrams 1.1 and 1.2 both commute.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathbf{1}_A \otimes \mu_A} & A \otimes A \\ \downarrow \mu_A \otimes \mathbf{1}_A & & \downarrow \mu_A \\ A \otimes A & \xrightarrow{\mu_A} & A \end{array}$$

Figure 1.1: Associative law

$$\begin{array}{ccccc} R \otimes A & \xrightarrow{\eta_A \otimes \mathbf{1}_A} & A \otimes A & \xleftarrow{\mathbf{1}_A \otimes \eta_A} & A \otimes R \\ & \searrow \sim & \downarrow \mu_A & \swarrow \sim & \\ & & A & & \end{array}$$

Figure 1.2: Unitary property

This means that if we identify $A \otimes R$ and $R \otimes A$ with A (this identification is denoted by \sim in diagram 1.2), we call A an R -algebra if the following equations hold:

$$\begin{aligned}\mu_A \circ (\mathbf{1}_A \otimes \mu_A) &= \mu_A \circ (\mu_A \otimes \mathbf{1}_A) \\ \mu_A \circ (\mathbf{1}_A \otimes \eta_A) &= \mu_A \circ (\eta_A \otimes \mathbf{1}_A) = \mathbf{1}_A\end{aligned}$$

We call μ_A the **multiplication map** and η_A the **unit map** of A , collectively known as the **structure maps** of A ; whenever we do not explicitly write $\langle A, \mu_A, \eta_A \rangle$ for A and its structure maps, the structure maps are assumed to be μ_A, η_A .

We will sometimes write the product of two elements $a, b \in A$ as ab and we will use the unit map η_A and the unit element e_A interchangeably (as η_A maps $r \in R$ to $re_A \in A$), depending on what is understandable more easily or makes notation simpler in the context. Furthermore, o_A will denote the neutral element with regard to $+$ in A . ∇

Remark 1.25: A narrower definition would define only a \mathbb{K} -algebra A , starting with a \mathbb{K} -vector space A instead. This definition is also consistent with the classical definition of a **unital algebra** over a field \mathbb{K} . In Chapter 2 the analogous definitions for coalgebras will only be made for a field \mathbb{K} .

Definition 1.26: Let $\langle A, \mu_A, \eta_A \rangle, \langle B, \mu_B, \eta_B \rangle$ be R -algebras. If a map $m: A \rightarrow B$ is a ring morphism and an R -module morphism, we call m an **R -algebra morphism**. An R -module morphism $m: A \rightarrow B$ is an R -algebra morphism if and only if diagrams 1.3 and 1.4 both commute.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ \downarrow m \otimes m & & \downarrow m \\ B \otimes B & \xrightarrow{\mu_B} & B \end{array}$$

Figure 1.3: Compatibility with μ

$$\begin{array}{ccc} & R & \\ \eta_A \swarrow & & \searrow \eta_B \\ A & \xrightarrow{m} & B \end{array}$$

Figure 1.4: Compatibility with η

Equivalently, this means if and only if the following two equations hold:

$$\begin{aligned}\mu_B \circ (m \otimes m) &= m \circ \mu_A \\ \eta_B &= m \circ \eta_A\end{aligned}$$

The category of R -algebras is called \mathbf{Alg}_R and the set of R -algebra morphisms from A to B is called $\mathbf{Alg}_R(A, B)$. ∇

Definition 1.27: Let $\langle A, \mu_A, \eta_A \rangle$ be an R -algebra. We call A **commutative** if and only if diagram 1.5 commutes.

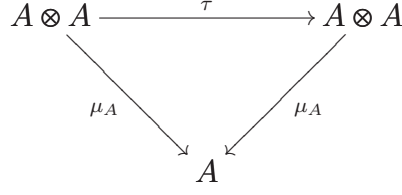


Figure 1.5: Commutativity

Equivalently, this means if and only if

$$\mu_A \circ \tau = \mu_A.$$

▽

Remark 1.28: If we have R -algebras A, B , the R -module $A \otimes B$ is also an R -algebra if we define its structure maps as follows:

$$\begin{aligned}
\mu_{A \otimes B} &:= (\mu_A \otimes \mu_B) \circ (\mathbf{1}_A \otimes \tau \otimes \mathbf{1}_B) \\
\eta_{A \otimes B} &:= \eta_A \otimes \eta_B
\end{aligned}$$

The multiplication map in $A \otimes B$ is then given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and that the unit element in $A \otimes B$ is given by $e_A \otimes e_B$. We call $A \otimes B$ the **tensor product** of A and B .

Definition 1.29: Let $\langle A, \mu_A, \eta_A \rangle$ be an R -algebra. If there are disjoint R -submodules $A_n, n \in \mathbb{N}$, such that

$$\begin{aligned}
A &= \bigoplus_{n \in \mathbb{N}} A_n \\
\mu_A(A_i \otimes A_j) &\subseteq A_{i+j} \quad \forall i, j \in \mathbb{N} \\
\eta_A(R) &\subseteq A_0,
\end{aligned}$$

we call A a **graded R -algebra**.

▽

Definition 1.30: Let $\langle A, \mu_A, \eta_A \rangle$ be an R -algebra. Then $\langle A, \mu_A \circ \tau, \eta_A \rangle$ is also an R -algebra, the **opposite algebra** A_{opp} of A .

▽

Lemma 1.31: Let $\langle A, \mu_A, \eta_A \rangle$ be an R -algebra and $S \supseteq R$ a commutative ring. If we consider $A_S := S \otimes_R A$ and define

$$\begin{aligned}
\mu'_A: (S \otimes_R A) \otimes_S (S \otimes_R A) &\longrightarrow S \otimes_R A: (s_1 \otimes_R a_1) \otimes_S (s_2 \otimes_R a_2) \longmapsto (s_1 s_2) \otimes_R \mu_A(a_1 \otimes a_2) \\
\eta'_A: S &\longrightarrow S \otimes_R A: s \longmapsto s \otimes_R \eta_A(1),
\end{aligned}$$

then $\langle A_S, \mu'_A, \eta'_A \rangle$ is an S -algebra.

PROOF. μ'_A and η'_A evidently fulfil the necessary conditions, as they are simply extensions of the structure maps μ_A and η_A to the ring extension S of R . ■

Chapter 2

Coalgebras

In the following, we only consider algebraic structures over a field \mathbb{K} , as is the custom in this field of study.

2.1 Definitions

The advantage of the definition of the \mathbb{K} -algebra as given above over the classical definition is that it is easily dualized; the following definition of \mathbb{K} -coalgebras is an exact dualization of the definition of \mathbb{K} -algebras.

Definition 2.1: Let C be a \mathbb{K} -vector space and $\Delta_C \in \mathbf{Mod}_{\mathbb{K}}(C, C \otimes C)$ and $\varepsilon_C \in \mathbf{Mod}_{\mathbb{K}}(C, \mathbb{K})$ \mathbb{K} -linear maps. We call $\langle C, \Delta_C, \varepsilon_C \rangle$ a \mathbb{K} -**coalgebra** if diagrams 2.1 and 2.2 both commute.

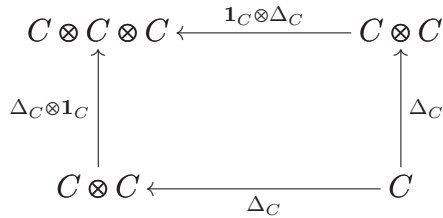


Figure 2.1: Coassociative law

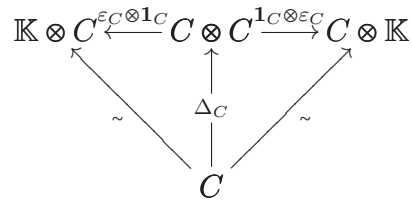


Figure 2.2: Counitary property

Again, if we identify $C \otimes \mathbb{K}$ and $\mathbb{K} \otimes C$ with C , we equivalently call C a \mathbb{K} -coalgebra if the following equations hold:

$$\begin{aligned} (\Delta_C \otimes \mathbf{1}_C) \circ \Delta_C &= (\mathbf{1}_C \otimes \Delta_C) \circ \Delta_C \\ (\varepsilon_C \otimes \mathbf{1}_C) \circ \Delta_C &= (\mathbf{1}_C \otimes \varepsilon_C) \circ \Delta_C = \mathbf{1}_C \end{aligned}$$

In this case, we call Δ_C the **comultiplication map** and ε_C the **counit map** of C , again collectively known as the **structure maps** of C . As with algebras, whenever we don't explicitly write $\langle C, \Delta_C, \varepsilon_C \rangle$ for C and its structure maps, the structure maps are assumed to be Δ_C, ε_C . ▽

Definition 2.2: Let $\langle C, \Delta_C, \varepsilon_C \rangle, \langle D, \Delta_D, \varepsilon_D \rangle$ be \mathbb{K} -coalgebras. A \mathbb{K} -linear map $m: C \rightarrow D$ is a \mathbb{K} -coalgebra **morphism** if and only if diagrams 2.3 and 2.4 both commute.

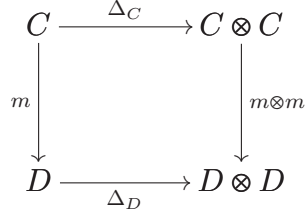


Figure 2.3: Compatibility with Δ

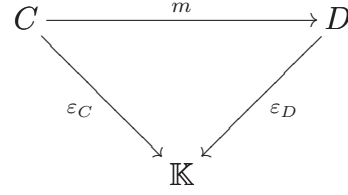


Figure 2.4: Compatibility with ε

Equivalently, this means if and only if the following two equations hold:

$$\begin{aligned}\Delta_D \circ m &= (m \otimes m) \circ \Delta_C \\ \varepsilon_D \circ m &= \varepsilon_C\end{aligned}$$

The category of \mathbb{K} -coalgebras is called $\mathbf{CoAlg}_{\mathbb{K}}$ and the set of \mathbb{K} -coalgebra morphisms from C to D is called $\mathbf{CoAlg}_{\mathbb{K}}(C, D)$. ∇

Definition 2.3: Let $\langle C, \Delta_C, \varepsilon_C \rangle$ be a \mathbb{K} -coalgebra. We call C **cocommutative** if and only if diagram 2.5 commutes.

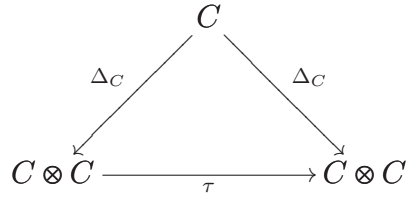


Figure 2.5: Cocommutativity

Equivalently, this means if and only if

$$\tau \circ \Delta_C = \Delta_C.$$

∇

Definition 2.4: An element $g \neq o$ of a \mathbb{K} -coalgebra C is called **grouplike** if it fulfils

$$\Delta(g) = g \otimes g.$$

An element p of a \mathbb{K} -coalgebra C is called **primitive** if it fulfils

$$\Delta(p) = p \otimes e + e \otimes p.$$

∇

Example 2.5: We give some examples of coalgebras:

- a) Let S be a nonempty set. Let $\mathbb{K}S$ be the \mathbb{K} -vector space with basis S . We can then define structure maps Δ and ε by

$$\begin{aligned}\Delta: \mathbb{K}S &\longrightarrow \mathbb{K}S \otimes \mathbb{K}S: s \longmapsto s \otimes s \\ \varepsilon: \mathbb{K}S &\longrightarrow \mathbb{K}: s \longmapsto 1\end{aligned}$$

for all $s \in S$. With this definition, any \mathbb{K} -vector space can be considered a \mathbb{K} -coalgebra.

- b) If we consider Example 2.5 a) with the set $S = \{s\}$ containing only one element, we get an interpretation of \mathbb{K} itself as a \mathbb{K} -coalgebra, with structure maps

$$\begin{aligned}\Delta: \mathbb{K} &\longrightarrow \mathbb{K} \otimes \mathbb{K}: k \longmapsto k \otimes k \\ \varepsilon: \mathbb{K} &\longrightarrow \mathbb{K}: k \longmapsto k.\end{aligned}$$

- c) Let C be a \mathbb{K} -vector space with a countable basis $\{c_n \mid n \in \mathbb{N}\}$. C is a \mathbb{K} -coalgebra with structure maps

$$\begin{aligned}\Delta(c_n) &:= \sum_{i=0}^n c_i \otimes c_{n-i} \\ \varepsilon(c_n) &:= \delta_{0n}\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

is the **Kronecker symbol**. This coalgebra is known as the **divided power coalgebra**.

- d) Consider a \mathbb{K} -vector space of dimension n^2 for an integer $n \geq 1$, which we will denote by $M^c(n, \mathbb{K})$. Let $\{e_{ij} \mid i, j = 1, \dots, n\}$ be a basis of $M^c(n, \mathbb{K})$. We then define structure maps Δ and ε by

$$\begin{aligned}\Delta(e_{ij}) &:= \sum_{1 \leq k \leq n} e_{ik} \otimes e_{kj} \\ \varepsilon(e_{ij}) &:= \delta_{ij}\end{aligned}$$

for all $i, j \in \{1, \dots, n\}$. The resulting coalgebra is called the **matrix coalgebra**.

- e) Assume we have a \mathbb{K} -vector space C with basis $\{g_i \mid i \in \mathbb{N}^\times\} \cup \{d_j \mid j \in \mathbb{N}^\times\}$. We define structure maps Δ and ε by

$$\begin{aligned}\Delta(g_i) &:= g_i \otimes g_i \\ \Delta(d_j) &:= g_j \otimes d_j + d_j \otimes g_{j+1} \\ \varepsilon(g_i) &:= 1 \\ \varepsilon(d_j) &:= 0\end{aligned}$$

for all $i, j \in \mathbb{N}^\times$. (Note that $g_i, i \in \mathbb{N}^\times$, are grouplike elements.) With these definitions, C is a \mathbb{K} -coalgebra.

f) Let C be a \mathbb{K} -vector space with basis $\{s, c\}$. We define structure maps Δ and ε by

$$\begin{aligned}\Delta(s) &:= s \otimes c + c \otimes s \\ \Delta(c) &:= c \otimes c - s \otimes s \\ \varepsilon(s) &:= 0 \\ \varepsilon(c) &:= 1.\end{aligned}$$

This coalgebra is called the **trigonometric coalgebra**, a term which comes from representation theory: If we consider the functions

$$\begin{aligned}\sin: \mathbb{R} &\longrightarrow \mathbb{R} \\ \cos: \mathbb{R} &\longrightarrow \mathbb{R},\end{aligned}$$

we see that

$$\begin{aligned}\sin(x + y) &= \sin(x) \cos(y) + \sin(y) \cos(x) \\ \cos(x + y) &= \cos(x) \cos(y) - \sin(x) \sin(y).\end{aligned}$$

The trigonometric coalgebra represents the subspace generated by \sin and \cos in the space of real functions. (For a detailed discussion, see [DNR01] pp. 40–42.)

//

Definition 2.6: For a coalgebra $\langle C, \Delta, \varepsilon \rangle$, we recursively define a sequence of maps $(\Delta_n)_{n \geq 1}$ as follows:

$$\begin{aligned}\Delta_1: C &\longrightarrow C \otimes C & \Delta_1 &:= \Delta \\ \Delta_2: C &\longrightarrow C \otimes C \otimes C & \Delta_2 &:= (\Delta \otimes \mathbf{1}) \circ \Delta_1 \\ \Delta_n: C &\longrightarrow \underbrace{C \otimes \cdots \otimes C}_{n+1 \text{ times}} & \Delta_n &:= (\Delta \otimes \mathbf{1}^{n-1}) \circ \Delta_{n-1}\end{aligned}$$

▽

It is well-known that for algebras, generalized associativity follows from associativity; dually to that, we have generalized coassociativity following from coassociativity for coalgebras.

Lemma 2.7: *Let $\langle C, \Delta, \varepsilon \rangle$ be a coalgebra. For any $n \geq 2$ and any $k \in \{0, \dots, n-1\}$ the following equality holds:*

$$\Delta_n = (\mathbf{1}^k \otimes \Delta \otimes \mathbf{1}^{n-1-k}) \circ \Delta_{n-1}$$

PROOF. We prove this lemma by induction on n (for any k). For the base case $n = 2$, we have to prove

$$(\Delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \Delta) \circ \Delta,$$

which is just the coassociativity of the comultiplication in diagram 2.1.

We now assume that the equality holds for n and any $k \in \{0, \dots, n-1\}$. First, let $k = 0$. By definition,

$$\Delta_{n+1} = (\Delta \otimes \mathbf{1}^n) \circ \Delta_n = (\mathbf{1}^k \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ \Delta_n,$$

and the equality thus holds for $n+1$. Now, let $k \in \{1, \dots, n\}$. By the induction hypothesis (IH), we have:

$$\begin{aligned} (\mathbf{1}^k \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ \Delta_n &\stackrel{\text{IH}}{=} (\mathbf{1}^k \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ \Delta_{n-1} \\ &= (\mathbf{1}^{k-1} \otimes \mathbf{1} \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ \Delta_{n-1} \\ &= (\mathbf{1}^{k-1} \otimes ((\mathbf{1} \otimes \Delta) \circ \Delta) \otimes \mathbf{1}^{n-k}) \circ \Delta_{n-1} \\ &\stackrel{\text{IH}}{=} (\mathbf{1}^{k-1} \otimes ((\Delta \otimes \mathbf{1}) \circ \Delta) \otimes \mathbf{1}^{n-k}) \circ \Delta_{n-1} \\ &= (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1} \otimes \mathbf{1}^{n-k}) \circ (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ \Delta_{n-1} \\ &= (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1}^{n+1-k}) \circ (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1}^{n-k}) \circ \Delta_{n-1} \\ &\stackrel{\text{IH}}{=} (\mathbf{1}^{k-1} \otimes \Delta \otimes \mathbf{1}^{n+1-k}) \circ \Delta_n \end{aligned}$$

By induction on k by -1 , we can reduce the above relation to $k = 0$, for which we have already proved the equality; thus the equality holds for all $k \in \{0, \dots, n\}$. \blacksquare

2.2 Sigma Notation

If we consider algebras, we see that multiplication reduces the number of elements involved in a computation; if we multiply two elements, we are left with a single one. In contrast, with coalgebras comultiplication has the opposite effect: comultiplying one element results in a finite family of pairs of elements. Computations are therefore “more complicated” in coalgebras than in algebras.

To reduce complexity of notation, the introduction of the following notation, usually referred to as the **sigma notation** (also known as the **Sweedler** or **Heyneman–Sweedler notation**), turns out to be quite useful.

Definition 2.8: Let $\langle C, \Delta, \varepsilon \rangle$ be a coalgebra. For $c \in C$, we write

$$\Delta(c) = \sum c_1 \otimes c_2.$$

Following the usual conventions for summation, this would read

$$\Delta(c) = \sum_{i=1}^n c_{i,1} \otimes c_{i,2};$$

the sigma notation thus suppresses the index i in the sum. Extending this notation, we also write

$$\Delta_k(c) = \sum c_1 \otimes \cdots \otimes c_{k+1}$$

for any $k \geq 1$ instead of

$$\Delta_k(c) = \sum_{i=1}^n c_{i,1} \otimes \cdots \otimes c_{i,k+1}.$$

The sigma notation thus helps to shorten lengthy calculations. ∇

Remark 2.9: Some authors suppress even the summation symbol \sum in the sigma notation; in my opinion, however, this is not conducive to better understanding on the part of the reader.

Example 2.10: For $n = 2$, the sigma notation gives us the following equality:

$$\Delta_2(c) = \sum \Delta(c_1) \otimes c_2 = \sum c_1 \otimes \Delta(c_2) = \sum c_1 \otimes c_2 \otimes c_3,$$

or, to emphasize the sigma notation even more,

$$\Delta_2(c) = \sum c_{1_1} \otimes c_{1_2} \otimes c_2 = \sum c_1 \otimes c_{2_1} \otimes c_{2_2} = \sum c_1 \otimes c_2 \otimes c_3.$$

Coassociativity is expressed in this way as

$$\sum c_{1_1} \otimes c_{1_2} \otimes c_2 = \sum c_1 \otimes c_{2_1} \otimes c_{2_2} = \sum c_1 \otimes c_2 \otimes c_3 \quad \forall c \in C,$$

as this is precisely the equality

$$(\Delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \Delta) \circ \Delta$$

from diagram 2.1. The counitary property can be expressed in formulae as

$$\mathbf{1} = \phi_{\text{left}} \circ (\varepsilon \otimes \mathbf{1}) \circ \Delta = \phi_{\text{right}} \circ (\mathbf{1} \otimes \varepsilon) \circ \Delta,$$

where

$$\begin{aligned} \phi_{\text{left}}: C \otimes \mathbb{K} &\xrightarrow{\sim} C \\ \phi_{\text{right}}: \mathbb{K} \otimes C &\xrightarrow{\sim} C \end{aligned}$$

are the canonical isomorphisms as in diagram 2.2. This becomes simpler by far using the sigma notation, with

$$\sum \varepsilon(c_1) c_2 = c = \sum c_1 \varepsilon(c_2) \quad \forall c \in C$$

expressing the same equalities. //

Remark 2.11: Using the sigma notation, we can concisely write diagrams 2.3 and 2.5 as

$$\Delta_D(m(c)) = \sum m(c)_1 \otimes m(c)_2 = \sum m(c_1) \otimes m(c_2)$$

and

$$\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1,$$

respectively.

To use the sigma notation to simplify calculations, the following lemma is required.

Lemma 2.12: *Let $\langle C, \Delta, \varepsilon \rangle$ be a coalgebra.*

1. *For any $i \geq 2$,*

$$\Delta_i = (\Delta_{i-1} \otimes \mathbf{1}) \circ \Delta.$$

2. *For any $n \geq 2$, $i \in \{1, \dots, n-1\}$ and $m \in \{0, \dots, n-i\}$, we have*

$$\Delta_n = (\mathbf{1}^m \otimes \Delta_i \otimes \mathbf{1}^{n-i-m}) \circ \Delta_{n-i}.$$

PROOF.

1. We prove by induction on i . For $i = 2$, this is simply the definition of Δ_2 . We now assume that the equality is true for i (IH). Then we have

$$\begin{aligned} \Delta_{i+1} &= (\Delta \otimes \mathbf{1}^i) \circ \Delta_i \\ &\stackrel{\text{IH}}{=} (\Delta \otimes \mathbf{1}^i) \circ (\Delta_{i-1} \otimes \mathbf{1}) \circ \Delta \\ &= (((\Delta \otimes \mathbf{1}^{i-1}) \circ \Delta_{i-1}) \otimes \mathbf{1}) \circ \Delta = (\Delta_i \otimes \mathbf{1}) \circ \Delta. \end{aligned}$$

2. We prove this for fixed $n \geq 2$ by induction on $i \in \{1, \dots, n-1\}$ for all $m \in \{0, \dots, n-i\}$. For $i = 1$ this is the generalized coassociativity (GC) as proved in Lemma 2.7. Assume that the equality holds for $i - 1$, with $i \geq 2$ (IH). For any $m \in \{0, \dots, n-i\} \subset \{0, \dots, n-(i-1)\}$ we have

$$\begin{aligned} \Delta_n &\stackrel{\text{IH}}{=} (\mathbf{1}^m \otimes \Delta_{i-1} \otimes \mathbf{1}^{n-(i-1)-m}) \circ \Delta_{n-(i-1)} \\ &\stackrel{\text{GC}}{=} (\mathbf{1}^m \otimes \Delta_{i-1} \otimes \mathbf{1}^{n-(i-1)-m}) \circ (\mathbf{1}^m \otimes \Delta \otimes \mathbf{1}^{n-i-m}) \circ \Delta_{n-i} \\ &= (\mathbf{1}^m \otimes ((\Delta_{i-1} \otimes \mathbf{1}) \circ \Delta) \otimes \mathbf{1}^{n-i-m}) \circ \Delta_{n-i} \\ &\stackrel{1}{=} (\mathbf{1}^m \otimes \Delta_i \otimes \mathbf{1}^{n-i-m}) \circ \Delta_{n-i}. \end{aligned}$$

■

We can now use the sigma notation to provide us with the following computation rule, which is often used for computations in coalgebras.

Proposition 2.13: *Let $\langle C, \Delta, \varepsilon \rangle$ be a coalgebra and $i \geq 1$. Let*

$$\begin{aligned} f: \underbrace{C \otimes \dots \otimes C}_{i+1 \text{ times}} &\longrightarrow C \\ \bar{f}: C &\longrightarrow C \end{aligned}$$

be linear maps such that $f \circ \Delta_i = \bar{f}$. If $n \geq i$, V is a \mathbb{K} -vector space and

$$g: \underbrace{C \otimes \dots \otimes C}_{n+1 \text{ times}} \longrightarrow V$$

is a \mathbb{K} -linear map, then we have

$$\begin{aligned} & \sum g(c_1 \otimes \cdots \otimes c_{j-1} \otimes f(c_j \otimes \cdots \otimes c_{j+i}) \otimes c_{j+i+1} \otimes \cdots \otimes c_{n+i+1}) \\ &= \sum g(c_1 \otimes \cdots \otimes c_{j-1} \otimes \bar{f}(c_j) \otimes c_{j+1} \otimes \cdots \otimes c_{n+1}) \end{aligned}$$

for any $1 \leq j \leq n+1$ and $c \in C$.

PROOF. This follows from the following equalities:

$$\begin{aligned} & \sum g(c_1 \otimes \cdots \otimes c_{j-1} \otimes f(c_j \otimes \cdots \otimes c_{j+i}) \otimes c_{j+i+1} \otimes \cdots \otimes c_{n+i+1}) \\ &= g \circ (\mathbf{1}^{j-1} \otimes f \otimes \mathbf{1}^{n-j+1}) \circ \Delta_{n+i}(c) \\ &= g \circ (\mathbf{1}^{j-1} \otimes f \otimes \mathbf{1}^{n-j+1}) \circ (\mathbf{1}^{j-1} \otimes \Delta_i \otimes \mathbf{1}^{n-j+1}) \circ \Delta_n(c) \\ &= g \circ (\mathbf{1}^{j-1} \otimes (f \circ \Delta_i) \otimes \mathbf{1}^{n-j+1}) \circ \Delta_n(c) = g \circ (\mathbf{1}^{j-1} \otimes \bar{f} \otimes \mathbf{1}^{n-j+1}) \circ \Delta_n(c) \\ &= \sum g(c_1 \otimes \cdots \otimes c_{j-1} \otimes \bar{f}(c_j) \otimes c_{j+1} \otimes \cdots \otimes c_{n+1}) \end{aligned}$$

■

Theorem 2.14: *If we have \mathbb{K} -coalgebras C, D , the \mathbb{K} -module $C \otimes D$ is also a \mathbb{K} -coalgebra (the **tensor product** of C and D) if we define its structure maps as follows:*

$$\begin{aligned} \Delta_{C \otimes D} &:= (\mathbf{1}_C \otimes \tau \otimes \mathbf{1}_D) \circ (\Delta_C \otimes \Delta_D) \\ \varepsilon_{C \otimes D} &:= \varepsilon_C \otimes \varepsilon_D \end{aligned}$$

Furthermore, we define the projection maps

$$\begin{aligned} \pi_C: C \otimes D &\longrightarrow C: c \otimes d \longmapsto \varepsilon_D(d)c \\ \pi_D: C \otimes D &\longrightarrow D: c \otimes d \longmapsto \varepsilon_C(c)d. \end{aligned}$$

Then π_C and π_D are \mathbb{K} -coalgebra morphisms.

PROOF. Using the sigma notation, we have

$$\begin{aligned} \Delta_{C \otimes D}(c \otimes d) &= \sum c_1 \otimes d_1 \otimes c_2 \otimes d_2 \\ \varepsilon_{C \otimes D}(c \otimes d) &= \varepsilon_C(c)\varepsilon_D(d) \end{aligned}$$

for any $c \in C$ and $d \in D$. Using this, we now have

$$\begin{aligned} ((\Delta_{C \otimes D} \otimes \mathbf{1}_{C \otimes D}) \circ \Delta_{C \otimes D})(c \otimes d) &= (\Delta_{C \otimes D} \otimes \mathbf{1}_{C \otimes D}) \left(\sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \right) \\ &= \sum ((c_1)_1 \otimes (d_1)_1) \otimes ((c_1)_2 \otimes (d_1)_2) \otimes (c_2 \otimes d_2) \\ &= \sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \otimes (c_3 \otimes d_3) \end{aligned}$$

as well as

$$\begin{aligned} ((\mathbf{1}_{C \otimes D} \otimes \Delta_{C \otimes D}) \circ \Delta_{C \otimes D})(c \otimes d) &= (\mathbf{1}_{C \otimes D} \otimes \Delta_{C \otimes D}) \left(\sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \right) \\ &= \sum (c_1 \otimes d_1) \otimes ((c_2)_1 \otimes (d_2)_1) \otimes ((c_2)_2 \otimes (d_2)_2) \\ &= \sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \otimes (c_3 \otimes d_3), \end{aligned}$$

which shows that $\Delta_{C \otimes D}$ is coassociative.

Moreover, we have

$$\begin{aligned}
((\mathbf{1}_{C \otimes D} \otimes \varepsilon_{C \otimes D}) \circ \Delta_{C \otimes D})(c \otimes d) &= (\mathbf{1}_{C \otimes D} \otimes \varepsilon_{C \otimes D}) \left(\sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \right) \\
&= \sum (c_1 \otimes d_1) \varepsilon_{C \otimes D}(c_2 \otimes d_2) = \sum (c_1 \otimes d_1) \varepsilon_C(c_2) \varepsilon_D(d_2) \\
&= \left(\sum c_1 \varepsilon_C(c_2) \right) \otimes \left(\sum d_1 \varepsilon_D(d_2) \right) = c \otimes d
\end{aligned}$$

and analogously

$$((\varepsilon_{C \otimes D} \otimes \mathbf{1}_{C \otimes D}) \circ \Delta_{C \otimes D})(c \otimes d) = c \otimes d,$$

whereby $\varepsilon_{C \otimes D}$ is counitary, and thus $C \otimes D$ is a \mathbb{K} -coalgebra.

Finally, consider π_C . We have

$$\begin{aligned}
((\pi_C \otimes \pi_C) \circ \Delta_{C \otimes D})(c \otimes d) &= (\pi_C \otimes \pi_C) \left(\sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \right) \\
&= \sum \varepsilon_D(d_1) c_1 \otimes \varepsilon_D(d_2) c_2 = \sum \varepsilon_D(\varepsilon_D(d_1) d_2) c_1 \otimes c_2 \\
&= \varepsilon_D(d) \sum c_1 \otimes c_2 = \varepsilon_D(d) \Delta_C(c) = \Delta_C(\pi_C(c \otimes d))
\end{aligned}$$

and

$$(\varepsilon_C \circ \pi_C)(c \otimes d) = \varepsilon_C(\varepsilon_D(d) c) = \varepsilon_C(c) \varepsilon_D(d) = \varepsilon_{C \otimes D}(c \otimes d),$$

which makes π_C a \mathbb{K} -coalgebra morphism; the proof for π_D is analogous. \blacksquare

Definition 2.15: Let $\langle C, \Delta_C, \varepsilon_C \rangle$ be a \mathbb{K} -coalgebra. If there are disjoint \mathbb{K} -submodules $C_n, n \in \mathbb{N}$, such that

$$\begin{aligned}
C &= \bigoplus_{n \in \mathbb{N}} C_n \\
\Delta_C(C_k) &\subseteq \bigoplus_{i+j=k} C_i \otimes C_j \quad \forall k \in \mathbb{N} \\
\varepsilon_C(C_i) &= \{0\} \quad \forall i \neq 0,
\end{aligned}$$

we call C a **graded \mathbb{K} -coalgebra**. ∇

Lemma 2.16: Let $\langle C, \Delta_C, \varepsilon_C \rangle$ be a \mathbb{K} -coalgebra and $\mathbb{L} \supseteq \mathbb{K}$ a field. If we consider $C_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{K}} C$ and define

$$\begin{aligned}
\Delta'_C: \mathbb{L} \otimes_{\mathbb{K}} C &\longrightarrow (\mathbb{L} \otimes_{\mathbb{K}} C) \otimes_{\mathbb{L}} (\mathbb{L} \otimes_{\mathbb{K}} C): \ell \otimes_{\mathbb{K}} c \longmapsto \sum (\ell \otimes_{\mathbb{K}} c_1) \otimes_{\mathbb{L}} (1 \otimes_{\mathbb{K}} c_2) \\
\varepsilon'_C: \mathbb{L} \otimes_{\mathbb{K}} C &\longrightarrow \mathbb{L}: \ell \otimes_{\mathbb{K}} c \longmapsto \ell \varepsilon_C(c),
\end{aligned}$$

then $\langle C_{\mathbb{L}}, \Delta'_C, \varepsilon'_C \rangle$ is an \mathbb{L} -coalgebra.

PROOF. Analogously to the proof of Lemma 1.31. \blacksquare

2.3 Dual Constructions

As the structure of algebras and coalgebras is dual in a natural way, the questions arises whether this can be used to construct an algebra from a coalgebra, and vice versa. We require the following two concepts from linear algebra:

Definition 2.17: Let V be a \mathbb{K} -vector space. Then

$$V^* := \mathbf{Mod}_{\mathbb{K}}(V, \mathbb{K})$$

is also a \mathbb{K} -vector space, the **dual vector space** of all \mathbb{K} -linear maps from V to \mathbb{K} . ∇

Definition 2.18: Let V, W be \mathbb{K} -vector spaces and $v: V \rightarrow W$ a \mathbb{K} -linear map. Then the map

$$v^*: W^* \rightarrow V^*: f \mapsto v^*(f) := f \circ v$$

is also a \mathbb{K} -linear map, the **dual linear map**. ∇

Consider the function ρ in Definition 1.20; we use (without proof) that ρ is injective, and an isomorphism if N is finite-dimensional.

We first consider the construction of an algebra from a coalgebra.

Theorem 2.19: Let $\langle C, \Delta, \varepsilon \rangle$ be a \mathbb{K} -coalgebra. We define maps

$$\begin{aligned} \mu: C^* \otimes C^* &\rightarrow C^*: f \otimes g \mapsto \mu(f \otimes g) := \Delta^* \circ \rho(f \otimes g) \\ \eta: \mathbb{K} &\rightarrow C^*: k \mapsto \eta(k) := \varepsilon^* \circ \iota(k), \end{aligned}$$

where $\iota: \mathbb{K} \rightarrow \mathbb{K}^*$ is the canonical isomorphism which maps k to the map

$$k^* := \iota(k): \mathbb{K} \rightarrow \mathbb{K}: \ell \mapsto k^*(\ell) := k\ell.$$

Then $\langle C^*, \mu, \eta \rangle$ is a \mathbb{K} -algebra, the **dual algebra** of C .

PROOF. Let $f, g, h \in C^*$. We write $f * g$ for $\mu(f \otimes g)$. By the definition of μ , we have

$$(f * g)(c) = \Delta^* \circ \rho(f \otimes g)(c) = \rho(f \otimes g)(\Delta(c)) = \sum f(c_1)g(c_2)$$

for $c \in C$. Having shown this, we have

$$\begin{aligned} ((f * g) * h)(c) &= \sum (f * g)(c_1)h(c_2) = \sum f(c_1)g(c_2)h(c_3) \\ &= \sum f(c_1)(g * h)(c_2) = (f * (g * h))(c) \end{aligned}$$

for $c \in C$, and thus we have associativity.

To show the unitary property, it is sufficient to show that $\eta(1)$ is the unit element for the multiplication map μ via

$$\eta(1) * f = f = f * \eta(1)$$

for $f \in C^*$. As we have

$$\eta(k)(c) = k\varepsilon(c)$$

for $k \in K$ and $c \in C$, and as

$$\sum \varepsilon(c_1)c_2 = c = \sum c_1\varepsilon(c_2)$$

from the counitary property, we can combine these to show

$$\begin{aligned} (\eta(1) * f)(c) &= \sum \eta(1)(c_1)f(c_2) = \sum \varepsilon(c_1)f(c_2) \\ &= \sum f(\varepsilon(c_1)c_2) = f\left(\sum \varepsilon(c_1)c_2\right) = f(c) \end{aligned}$$

for $c \in C$. Analogously, we have $f * \eta(1) = f$, and thus the desired equality. \blacksquare

Example 2.20: We give the dual algebras of two previously defined coalgebras:

- a) Let S be a nonempty set and $\mathbb{K}S$ the \mathbb{K} -coalgebra defined in Example 2.5 a). As the dual algebra, we get $(\mathbb{K}S)^* = \mathbf{Mod}_{\mathbb{K}}(\mathbb{K}S, \mathbb{K})$ with structure maps defined by

$$\begin{aligned} (f * g)(s) &= f(s)g(s) \\ \eta(k) &= k\varepsilon \end{aligned}$$

for all $f, g \in (\mathbb{K}S)^*$, $s \in S$ and $k \in \mathbb{K}$.

- b) Let C be the divided power coalgebra defined in Example 2.5 c). Then we have the dual algebra C^* with structure maps

$$\begin{aligned} (f * g)(c_n) &= \sum_{i=0}^n f(c_i)g(c_{n-i}) \\ \eta(k)(c_n) &= k\delta_{0n} \end{aligned}$$

for all $f, g \in C^*$, $n \in \mathbb{N}$ and $k \in \mathbb{K}$. C^* is isomorphic to the algebra of formal power series $\mathbb{K}[[x]]$ with the canonical isomorphism

$$\iota: C^* \longrightarrow \mathbb{K}[[x]]: f \longmapsto \sum_{n \geq 0} f(c_n)x^n.$$

//

On the other hand, it is not always possible to define a natural coalgebra structure on the dual space A^* of an algebra A , as there is not necessarily a canonical morphism

$$\bar{\rho}: (A \otimes A)^* \longrightarrow A^* \otimes A^*$$

which we could use. If, however, A is finite-dimensional, then

$$\rho: A^* \otimes A^* \longrightarrow (A \otimes A)^*$$

is an isomorphism and we can use ρ^{-1} for $\bar{\rho}$.

Theorem 2.21: *Let $\langle A, \mu, \eta \rangle$ be a finite-dimensional \mathbb{K} -algebra. We define maps*

$$\begin{aligned} \Delta: A^* &\longrightarrow A^* \otimes A^*: f \longmapsto \Delta(f) := \rho^{-1} \circ \mu^*(f) \\ \varepsilon: A^* &\longrightarrow \mathbb{K}: f \longmapsto \varepsilon(f) := \iota^{-1} \circ \eta^*(f), \end{aligned}$$

where ι^{-1} is the canonical isomorphism

$$\iota^{-1}: \mathbb{K}^* \longrightarrow \mathbb{K}: f \longmapsto \iota^{-1}(f) := f(1).$$

Then $\langle A^*, \Delta, \varepsilon \rangle$ is a \mathbb{K} -coalgebra, the **dual coalgebra** of A .

Remark 2.22: Before we prove the theorem, we make two brief remarks.

- If for $f \in A^*$

$$\Delta(f) = \sum_{i \in I} g_i \otimes h_i$$

with $g_i, h_i \in A^*$, $i \in I$ (I finite), then we have

$$f(ab) = \sum_{i \in I} g_i(a)h_i(b)$$

for all $a, b \in A$. If, on the other hand, we have $g'_j, h'_j \in A^*$, $j \in J$ (J finite), such that

$$f(ab) = \sum_{j \in J} g'_j(a)h'_j(b)$$

for all $a, b \in B$, then we must have

$$\sum_{i \in I} g_i \otimes h_i = \sum_{j \in J} g'_j \otimes h'_j$$

due to the injectivity of ρ . We can therefore define

$$\Delta(f) = \sum_{i \in I} g_i \otimes h_i$$

for any $g_i, h_i \in A^*$, $i \in I$, such that

$$f(ab) = \sum_{i \in I} g_i(a)h_i(b)$$

for all $a, b \in A$.

- In the following proof, we require the map

$$\rho^{(3)}: (A^* \otimes A^* \otimes A^*) \longrightarrow (A \otimes A \otimes A)^*$$

defined by

$$\rho^{(3)}(f \otimes g \otimes h)(a \otimes b \otimes c) := f(a)g(b)h(c)$$

for any $f, g, h \in A^*$ and all $a, b, c \in A$. This map is also injective, which immediately follows from the injectivity of ρ .

PROOF. Using the remark, we can now proceed to prove the theorem. Let $f \in A^*$ and $\Delta(f) = \sum_{i \in I} g_i \otimes h_i$ per the remark above. Let furthermore

$$\begin{aligned} \Delta(g_i) &= \sum_{j_1 \in J_1} g'_{i,j_1} \otimes g''_{i,j_1} \\ \Delta(h_i) &= \sum_{j_2 \in J_2} h'_{i,j_2} \otimes h''_{i,j_2}. \end{aligned}$$

From this we have

$$\begin{aligned} (\Delta \otimes \mathbf{1}) \circ \Delta(f) &= \sum_{i \in I, j_1 \in J_1} g'_{i,j_1} \otimes g''_{i,j_1} \otimes h_i \\ (\mathbf{1} \otimes \Delta) \circ \Delta(f) &= \sum_{i \in I, j_2 \in J_2} g_i \otimes h'_{i,j_2} \otimes h''_{i,j_2}. \end{aligned}$$

Let $a, b, c \in A$. On the one hand, we now have

$$\begin{aligned} \rho^{(3)} \left(\sum_{i \in I, j_1 \in J_1} g'_{i,j_1} \otimes g''_{i,j_1} \otimes h_i \right) (a \otimes b \otimes c) &= \sum_{i \in I, j_1 \in J_1} g'_{i,j_1}(a) g''_{i,j_1}(b) h_i(c) \\ &= \sum_{i \in I} g_i(a \otimes b) h_i(c) = f(abc), \end{aligned}$$

while on the other hand

$$\begin{aligned} \rho^{(3)} \left(\sum_{i \in I, j_2 \in J_2} g_i \otimes h'_{i,j_2} \otimes h''_{i,j_2} \right) (a \otimes b \otimes c) &= \sum_{i \in I, j_2 \in J_2} g_i(a) h'_{i,j_2}(b) h''_{i,j_2}(c) \\ &= \sum_{i \in I} g_i(a) h_i(bc) = f(abc). \end{aligned}$$

Due to the injectivity of $\rho^{(3)}$, we can conclude

$$\sum_{i \in I, j_1 \in J_1} g'_{i,j_1} \otimes g''_{i,j_1} \otimes h_i = \sum_{i \in I, j_2 \in J_2} g_i \otimes h'_{i,j_2} \otimes h''_{i,j_2},$$

and have thus shown the coassociativity of Δ .

Again using the above equality for f , we have

$$\begin{aligned} (\varepsilon \otimes \mathbf{1}) \circ \Delta(f)(a) &= (\varepsilon \otimes \mathbf{1}) \left(\sum_{i \in I} g_i \otimes h_i \right) (a) = \left(\sum_{i \in I} \varepsilon(g_i) h_i \right) (a) \\ &= \sum_{i \in I} g_i(1) h_i(a) = f(a) \end{aligned}$$

and analogously $(\mathbf{1} \otimes \varepsilon) \circ \Delta(f)(a) = f(a)$, both for all $a \in A$, from which we have

$$\sum_{i \in I} \varepsilon(g_i) h_i = f = \sum_{i \in I} g_i \varepsilon(h_i),$$

which is the counitary property. ■

Definition 2.23: Let $\langle A, \mu, \eta \rangle$ be a finite-dimensional \mathbb{K} -algebra and $(b_i)_{i \in I}$ a basis of A . Define b_i^* by

$$b_i^*(b_j) = \delta_{ij};$$

then $(b_i^*)_{i \in I}$ is a basis of the dual \mathbb{K} -coalgebra $\langle A^*, \Delta, \varepsilon \rangle$, the **dual basis**. ▽

Remark 2.24: Using the dual basis, we can easily express the comultiplication map in the dual \mathbb{K} -coalgebra. In addition to the above, $(b_i^* \otimes b_j^*)_{i,j \in I}$ is a basis of $A^* \otimes A^*$. Considering an element $f \in A^*$, there exist unique scalars $(a_{ij})_{i,j \in I}$ such that

$$\Delta(f) = \sum_{i,j \in I} a_{ij} b_i^* \otimes b_j^*.$$

If we then take fixed $s, t \in I$ and consider the definition of the comultiplication map in the dual \mathbb{K} -coalgebra, we have

$$f(b_s b_t) = \sum_{i,j \in I} a_{ij} b_i^*(b_s) b_j^*(b_t) = a_{st},$$

and thus

$$\Delta(f) = \sum_{i,j \in I} f(b_i b_j) b_i^* \otimes b_j^*.$$

Example 2.25: Let $A = M_n(\mathbb{K})$ be the \mathbb{K} -algebra of $(n \times n)$ - \mathbb{K} -matrices (with matrix multiplication, and the identity matrix I as the unit element). We construct the dual \mathbb{K} -coalgebra.

Let $(E_{ij})_{i,j=1}^n$ be the canonical basis of $M_n(\mathbb{K})$ (where E_{ij} is the matrix having 1 at the intersection of the i -th row and the j -th column, and 0 everywhere else) and $(E_{ij}^*)_{i,j=1}^n$ the dual basis of A^* . We have

$$E_{ij} E_{kl} = \delta_{jk} E_{il}$$

for any $i, j, k, l \in \{1, \dots, n\}$. Using this equality and the previous remark, we have

$$\begin{aligned} \Delta(E_{ij}^*) &= \sum_{r,s,t,u=1}^n E_{ij}^*(E_{rs} E_{tu}) E_{rs}^* \otimes E_{tu}^* \\ &= \sum_{r,s,t,u=1}^n E_{ij}^*(\delta_{st} E_{ru}) E_{rs}^* \otimes E_{tu}^* \\ &= \sum_{r,s,u=1}^n E_{ij}^*(E_{ru}) E_{rs}^* \otimes E_{su}^* \\ &= \sum_{r,s,u=1}^n \delta_{ir} \delta_{ju} E_{rs}^* \otimes E_{su}^* = \sum_{s=1}^n E_{is}^* \otimes E_{sj}^*. \end{aligned}$$

Furthermore, we have

$$\varepsilon(E_{ij}^*) = \iota^{-1} \circ \eta^* \circ E_{ij}^* = E_{ij}^*(I) = E_{ij}^* \left(\sum_{k=1}^n E_{kk} \right) = \delta_{ij}.$$

We have thus shown that the dual \mathbb{K} -coalgebra of the \mathbb{K} -algebra of $(n \times n)$ - \mathbb{K} -matrices $M_n(\mathbb{K})$ is isomorphic to the matrix coalgebra $M^c(n, \mathbb{K})$ defined in Example 2.5 d). //

The construction of the dual \mathbb{K} -algebra or dual \mathbb{K} -coalgebra preserves morphisms as formulated in the following theorem.

Theorem 2.26: *Let C, D be \mathbb{K} -coalgebras and let A, B be finite-dimensional \mathbb{K} -algebras. If $f: C \rightarrow D$ is a \mathbb{K} -coalgebra morphism, then $f^*: D^* \rightarrow C^*$ is a \mathbb{K} -algebra morphism. If $g: A \rightarrow B$ is a \mathbb{K} -algebra morphism, then $g^*: B^* \rightarrow A^*$ is a \mathbb{K} -coalgebra morphism.*

PROOF. Consider f^* . Let $d^*, d'^* \in D^*$ and $c \in C$. As f is a \mathbb{K} -coalgebra morphism, we have

$$\begin{aligned} (f^*(d^* * d'^*))(c) &= (d^* * d'^*)(f(c)) = \sum d^*(f(c_1))d'^*(f(c_2)) \\ &= \sum (f^*(d^*))(c_1)(f^*(d'^*))(c_2) = (f^*(d^*) * f^*(d'^*))(c), \end{aligned}$$

whence we have $f^*(d^* * d'^*) = f^*(d^*) * f^*(d'^*)$. We also have

$$f^*(\varepsilon_D) = \varepsilon_D \circ f = \varepsilon_C,$$

and thus f^* is a \mathbb{K} -coalgebra morphism.

Now consider g^* . We first have to show that diagram 2.6 commutes.

$$\begin{array}{ccc} B^* & \xrightarrow{\Delta_{B^*}} & B^* \otimes B^* \\ g^* \downarrow & & \downarrow g^* \otimes g^* \\ A^* & \xrightarrow{\Delta_{A^*}} & A^* \otimes A^* \end{array}$$

Figure 2.6: Compatibility with Δ

To this end, let $b^* \in B^*$ as well as

$$\begin{aligned} (\Delta_{A^*} \circ g^*)(b^*) &= \Delta_{A^*}(b^* \circ g) = \sum_{i \in I} g_i \otimes h_i \\ \Delta_{B^*}(b^*) &= \sum_{j \in J} p_j \otimes q_j. \end{aligned}$$

Let ρ again be the injective map $\rho: A^* \otimes A^* \rightarrow (A \otimes A)^*$ and let $a_1, a_2 \in A$. Then we have

$$\rho((\Delta_{A^*} \circ g^*)(b^*))(a_1 \otimes a_2) = \sum_{i \in I} g_i(a_1)h_i(a_2) = (b^* \circ g)(a_1 a_2)$$

as well as

$$\begin{aligned}
\rho((g^* \otimes g^*) \circ \Delta_{B^*} \circ b^*)(a_1 \otimes a_2) &= \rho\left(\sum_{j \in J} (p_j \circ g) \otimes (q_j \circ g)\right)(a_1 \otimes a_2) \\
&= \sum_{j \in J} (p_j \circ g)(a_1)(q_j \circ g)(a_2) = \sum_{j \in J} p_j(g(a_1))q_j(g(a_2)) \\
&= b^*(g(a_1)g(a_2)) = b^*(g(a_1a_2)) = (b^* \circ g)(a_1a_2).
\end{aligned}$$

Therefore, diagram 2.6 commutes.

Additionally, we have

$$(\varepsilon_{A^*} \circ g^*)(b^*) = \varepsilon_{A^*}(b^* \circ g) = (b^* \circ g)(1) = b^*(g(1)) = b^*(1) = \varepsilon_{B^*}(b^*)$$

and hence g^* is a \mathbb{K} -algebra morphism. ■

Definition 2.27: Let V be a finite-dimensional \mathbb{K} -vector space. Then the map

$$\theta_V: V \longrightarrow V^{**}$$

defined by

$$\theta_V(v)(v^*) := v^*(v)$$

for all $v \in V, v^* \in V^*$ is an isomorphism of \mathbb{K} -vector spaces. ▽

Theorem 2.28: Let $\langle A, \mu, \eta \rangle$ be a finite-dimensional \mathbb{K} -coalgebra and $\langle C, \Delta, \varepsilon \rangle$ a finite-dimensional \mathbb{K} -coalgebra. Then we have:

- $\theta_A: A \longrightarrow A^{**}$ is a \mathbb{K} -algebra isomorphism.
- $\theta_C: C \longrightarrow C^{**}$ is a \mathbb{K} -coalgebra isomorphism.

PROOF. We first consider θ_A . It suffices to show that θ_A is a \mathbb{K} -algebra morphism. Let $a, b \in A$ and $a^* \in A^*$; let Δ_A be the comultiplication in A^* and let

$$\Delta_A(a^*) = \sum_{i \in I} f_i \otimes g_i.$$

With $*$ denoting the multiplication in A^{**} , we have:

$$\begin{aligned}
(\theta_A(a) * \theta_A(b))(a^*) &= \sum_{i \in I} \theta_A(a)(f_i) \theta_A(b)(g_i) = \sum_{i \in I} f_i(a) g_i(b) \\
&= a^*(ab) = \theta_A(ab)(a^*)
\end{aligned}$$

Thereby θ_A is multiplicative.

Furthermore, if e is the unit element in A , then we have

$$\theta_A(e)(a^*) = a^*(e) = \varepsilon_{A^*}(a^*),$$

and therefore

$$\theta_A(e) = \varepsilon_{A^*}.$$

θ_A thus also preserves the unit map; altogether, this means that θ_A is a \mathbb{K} -algebra morphism and thus a \mathbb{K} -algebra isomorphism.

Let us now consider θ_C . It is again sufficient to show that θ_C is a \mathbb{K} -coalgebra morphism. We denote the comultiplication in C^{**} by Δ'' and now have to show that diagram 2.7 commutes.

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \theta_C \downarrow & & \downarrow \theta_C \otimes \theta_C \\ C^{**} & \xrightarrow{\Delta''} & C^{**} \otimes C^{**} \end{array}$$

Figure 2.7: Compatibility with Δ

To show this, let

$$\rho: C^{**} \otimes C^{**} \longrightarrow (C^* \otimes C^*)^*$$

be the canonical isomorphism; let furthermore $c \in C$ and $c^*, d^* \in C^*$, and finally

$$\Delta''(\theta_C(c)) = \sum_{i \in I} f_i \otimes g_i.$$

If $*$ denotes the multiplication in C^* , then we have:

$$\begin{aligned} \rho((\Delta'' \circ \theta_C)(c))(c^* \otimes d^*) &= \sum_{i \in I} \rho(f_i \otimes g_i)(c^* \otimes d^*) = \sum_{i \in I} f_i(c^*)g_i(d^*) \\ &= \theta_C(c^* * d^*) = (c^* * d^*)(c) \end{aligned}$$

Going the other way in the diagram yields:

$$\begin{aligned} \rho((\theta_C \otimes \theta_C) \circ \Delta)(c)(c^* \otimes d^*) &= \sum \rho(\theta_C(c_1) \otimes \theta_C(c_2))(c^* \otimes d^*) \\ &= \sum c^*(c_1)d^*(c_2) = (c^* * d^*)(c) \end{aligned}$$

This proves the commutativity of diagram 2.7.

Additionally, we have

$$(\varepsilon_{C^{**}} \circ \theta_C)(c) = \varepsilon_{C^{**}}(\theta_C(c)) = \theta_C(c)(\varepsilon_C) = \varepsilon_C(c),$$

and thus

$$\varepsilon_{C^{**}} \circ \theta_C = \varepsilon_C.$$

Hence θ_C also preserves the counit map, is thus a \mathbb{K} -coalgebra morphism and therefore a \mathbb{K} -coalgebra isomorphism. ■

Example 2.29: The previous example showed $M_n(\mathbb{K})^* \cong M^c(n, \mathbb{K})$. Using the proposition we have just proved, we now also know that $M^c(n, \mathbb{K})^* \cong M_n(\mathbb{K})^{**} \cong M_n(\mathbb{K})$. //

Remark 2.30: We have only defined the dual coalgebra for finite-dimensional algebras; there is, in fact, an extension of this notion, the **finite dual coalgebra** A° of an algebra A . (For details, see [DNR01] pp. 33–39.)

2.4 Grouplike Elements

Definition 2.31: Recalling the definition of grouplike elements ($g \in C \setminus \{o\}$ such that $\Delta(g) = g \otimes g$), we denote the set of grouplike elements of a \mathbb{K} -coalgebra by $G(C)$. ∇

Lemma 2.32: For all $g \in G(C)$, we have $\varepsilon(g) = 1$.

PROOF. Per the counitary property, we have

$$(\varepsilon \otimes \mathbf{1}) \circ \Delta = \mathbf{1};$$

applying this to $g \in G(C)$ yields

$$((\varepsilon \otimes \mathbf{1}) \circ \Delta)(g) = (\varepsilon \otimes \mathbf{1})(g \otimes g) = \varepsilon(g) \otimes g \stackrel{!}{=} g,$$

and therefore $\varepsilon(g) = 1$. ■

Proposition 2.33: Let $\langle C, \Delta, \varepsilon \rangle$ be a \mathbb{K} -coalgebra. $G(C)$ then consists of linearly independent elements.

PROOF. We prove this indirectly. Assume that $G(C)$ is a linearly dependent family, and let n be the smallest integer such that there are distinct $g_1, \dots, g_n, g \in G(C)$ and $\alpha_i \in \mathbb{K}, i \in \{1, \dots, n\}$, such that

$$g = \sum_{i=1}^n \alpha_i g_i.$$

If $n = 1$, then we have

$$g = \alpha_1 g_1$$

and applying ε yields $\alpha_1 = 1$, therefore $g = g_1$, which is a contradiction.

Therefore, we have $n \geq 2$; all α_i are also distinct from 0 (otherwise n would not be smallest), and g_1, \dots, g_n are linearly independent (otherwise, again, n would not be smallest). By applying Δ to the equality $g = \sum_{i=1}^n \alpha_i g_i$, we have

$$g \otimes g = \sum_{i=1}^n \alpha_i g_i \otimes g_i.$$

Replacing g in this equality yields

$$\sum_{i,j=1}^n \alpha_i \alpha_j g_i \otimes g_j = \sum_{i=1}^n \alpha_i g_i \otimes g_i.$$

It follows that $\alpha_i \alpha_j = 0$ for $i \neq j$, which is also a contradiction. \blacksquare

For finite-dimensional \mathbb{K} -algebras, the dual \mathbb{K} -coalgebra's grouplike elements are especially interesting.

Proposition 2.34: *Let $\langle A, \mu, \eta \rangle$ be a finite-dimensional \mathbb{K} -algebra and A^* its dual \mathbb{K} -coalgebra. Then $G(A^*) = \mathbf{Alg}_{\mathbb{K}}(A, \mathbb{K})$.*

PROOF. Let $f \in A^*$. We have $f \in G(A^*)$ if and only if $\Delta(f) = f \otimes f$; per the definition of the dual \mathbb{K} -coalgebra, this means

$$f(ab) = f(a)f(b)$$

for all $a, b \in A$. In addition, we have

$$f(e) = \varepsilon(f) = 1,$$

which means $f \in G(A^*)$ if and only if f is a \mathbb{K} -algebra morphism. \blacksquare

Finally, we consider the grouplike elements in two examples we previously discussed.

Example 2.35: Let S be a nonempty set, and consider the \mathbb{K} -coalgebra $\mathbb{K}S$ from Example 2.5 a). We now show that $G(\mathbb{K}S) = S$. Let

$$g = \sum_{i \in I} \alpha_i s_i \in G(\mathbb{K}S);$$

Then we have

$$\Delta(g) = \sum_{i \in I} \alpha_i s_i \otimes s_i \stackrel{!}{=} \sum_{i,j \in I} \alpha_i \alpha_j s_i \otimes s_j = g \otimes g.$$

Let us assume that there are $k, \ell \in I$ with $k \neq \ell$ and $\alpha_k \neq 0 \neq \alpha_\ell$. For $r \in I$, we define functions

$$d_r: S \longrightarrow \mathbb{K}: s \longmapsto \delta_{s, s_r}.$$

Applying $d_k \otimes d_\ell$ to the above equality yields

$$\begin{aligned} (d_k \otimes d_\ell)(\Delta(g)) &= \sum_{i \in I} \alpha_i d_k(s_i) d_\ell(s_i) = 0 \\ (d_k \otimes d_\ell)(g \otimes g) &= \sum_{i,j \in I} \alpha_i \alpha_j d_k(s_i) d_\ell(s_j) = \alpha_k \alpha_\ell, \end{aligned}$$

and therefore $0 = \alpha_k \alpha_\ell$, which is a contradiction. We thus have $g = \alpha s$ for some $\alpha \in \mathbb{K}$ and $s \in S$; by $\varepsilon(g) = 1$ we have $\alpha = 1$, and therefore $g \in S$. Taken together, this means $G(\mathbb{K}S) \subseteq S$; $S \subseteq G(\mathbb{K}S)$ follows from the definition of Δ and ε . \parallel

Example 2.36: Consider the matrix coalgebra $M^c(n, \mathbb{K})$ for $n \geq 2$. As $M^c(n, \mathbb{K})$ is the dual \mathbb{K} -coalgebra of the matrix algebra $M_n(\mathbb{K})$, we have

$$G(M^c(n, \mathbb{K})) = \mathbf{Alg}_{\mathbb{K}}(M_n(\mathbb{K}), \mathbb{K}).$$

However, $\mathbf{Alg}_{\mathbb{K}}(M_n(\mathbb{K}), \mathbb{K})$ is in fact empty: Consider a \mathbb{K} -algebra map $f: M_n(\mathbb{K}) \rightarrow \mathbb{K}$. Using the fact that $M_n(\mathbb{K})$ is simple (it only has the trivial two-sided ideals $\{o\}$ and $M_n(\mathbb{K})$) and that $\ker f$ is such a two-sided ideal, we have only two cases:

- $\ker f = \{o\}$ means that f is injective, which is impossible due to the dimensions involved.
- $\ker f = M_n(\mathbb{K})$ means that $f(M_n(\mathbb{K})) = \{0\}$, which is impossible because $f(I) = 1$.

Therefore, $G(M^c(n, \mathbb{K})) = \emptyset$.

We can also show this directly: Let $X \in G(M^c(n, \mathbb{K}))$ with

$$X = \sum_{i,j=1}^n \alpha_{ij} E_{ij}^*.$$

Then we have

$$\Delta(X) = \sum_{i,j,k=1}^n \alpha_{ij} E_{ik}^* \otimes E_{kj}^* = \sum_{i,j,k,l=1}^n \alpha_{ij} \alpha_{kl} E_{ij}^* \otimes E_{kl}^* = X \otimes X.$$

Let $r, s \in \{1, \dots, n\}$ with $r \neq s$, and let $f, f^R \in M^c(n, \mathbb{K})^*$ be the maps which fulfil

$$\begin{aligned} f(E_{ij}^*) &= \delta_{ir} \delta_{js} \\ f^R(E_{ij}^*) &= \delta_{is} \delta_{jr}. \end{aligned}$$

If we now apply $f \otimes f$ to the equality above, we have

$$\begin{aligned} (f \otimes f)(\Delta(X)) &= \sum_{i,j,k=1}^n \alpha_{ij} f(E_{ik}^*) f(E_{kj}^*) = 0 \\ (f \otimes f)(X \otimes X) &= \sum_{i,j,k,l=1}^n \alpha_{ij} \alpha_{kl} f(E_{ij}^*) f(E_{kl}^*) = \alpha_{rs}^2. \end{aligned}$$

On the other hand, applying $f \otimes f^R$ to the same equality yields

$$\begin{aligned} (f \otimes f^R)(\Delta(X)) &= \sum_{i,j,k=1}^n \alpha_{ij} f(E_{ik}^*) f^R(E_{kj}^*) = \alpha_{rr} \\ (f \otimes f^R)(X \otimes X) &= \sum_{i,j,k,l=1}^n \alpha_{ij} \alpha_{kl} f(E_{ij}^*) f^R(E_{kl}^*) = \alpha_{rs} \alpha_{sr}. \end{aligned}$$

Taken together, this means that for arbitrary r, s with $r \neq s$ we have both $\alpha_{rs} = 0$ and $\alpha_{rr} = 0$; this means $X = o$, which is a contradiction (as o is by definition not grouplike). //

Definition 2.37: Let $\langle C, \Delta_C, \varepsilon_C \rangle$ be a \mathbb{K} -coalgebra. Then $\langle C, \tau \circ \Delta_C, \varepsilon_C \rangle$ is also a \mathbb{K} -coalgebra, the **coopposite coalgebra** C_{coop} of C . ∇

Lemma 2.38: Let $\langle C, \Delta_C, \varepsilon_C \rangle$ be a \mathbb{K} -coalgebra. Then the \mathbb{K} -algebras $(C_{\text{coop}})^*$ and $(C^*)_{\text{opp}}$ are equal.

PROOF. Let μ_1 and μ_2 be the multiplication maps in $(C_{\text{coop}})^*$ and $(C^*)_{\text{opp}}$, respectively. Let also $c^*, d^* \in C^*$ and $c \in C$. Then we have

$$\begin{aligned}\mu_1(c^* \otimes d^*)(c) &= (c^* \otimes d^*)((\tau \circ \Delta_C)(c)) = \sum c^*(c_2)d^*(c_1) \\ \mu_2(c^* \otimes d^*)(c) &= (d^* \otimes c^*)(\Delta_C(c)) = \sum d^*(c_1)c^*(c_2),\end{aligned}$$

whence the two \mathbb{K} -algebras are equal (due to the commutativity in \mathbb{K} for the expressions on the right-hand sides). \blacksquare

Chapter 3

Bialgebras and Hopf Algebras

3.1 Bialgebras

Suppose we are given a \mathbb{K} -vector space B that is both a \mathbb{K} -algebra $\langle B, \mu, \eta \rangle$ and a \mathbb{K} -coalgebra $\langle B, \Delta, \varepsilon \rangle$. We now consider the necessary conditions for these two structures on B to be compatible. For the following, we recall that $B \otimes B$ is both a \mathbb{K} -algebra and a \mathbb{K} -coalgebra and that \mathbb{K} allows a \mathbb{K} -coalgebra structure as described in Example 2.5 b). (\mathbb{K} obviously also allows a trivial \mathbb{K} -algebra structure.)

Lemma 3.1: *The following two conditions are equivalent:*

- μ and η are \mathbb{K} -coalgebra morphisms.
- Δ and ε are \mathbb{K} -algebra morphisms.

PROOF. Consider diagrams 3.1–3.4.

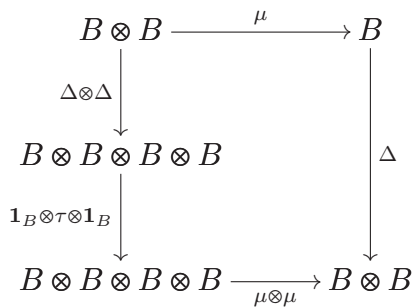


Figure 3.1: Compatibility of μ and Δ

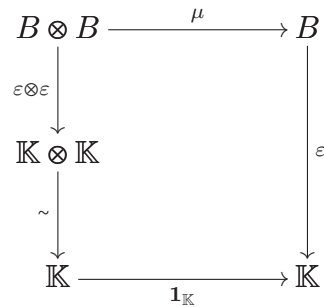


Figure 3.2: Compatibility of μ and ε

If we recall the required conditions for \mathbb{K} -algebra morphisms (see diagrams 1.3 and 1.4) and \mathbb{K} -coalgebra morphisms (see diagrams 2.3 and 2.4), we make the following observations:

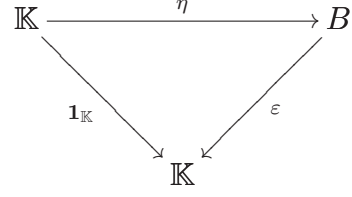
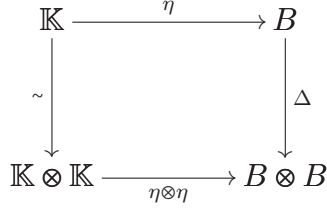


Figure 3.3: Compatibility of η and Δ Figure 3.4: Compatibility of η and ε

- μ is a \mathbb{K} -coalgebra morphism if and only if diagrams 3.1 and 3.2 both commute.
- η is a \mathbb{K} -coalgebra morphism if and only if diagrams 3.3 and 3.4 both commute.
- Δ is a \mathbb{K} -algebra morphism if and only if diagrams 3.1 and 3.3 both commute.
- ε is a \mathbb{K} -algebra morphism if and only if diagrams 3.2 and 3.4 both commute.

Taken together, these observations prove the equivalence. ■

Remark 3.2: Using the sigma notation and denoting the unit element of B and \mathbb{K} by e_B and $\mathbf{1}_{\mathbb{K}}$, the necessary conditions on Δ and ε so that they are \mathbb{K} -algebra morphisms are

$$\begin{aligned}
\Delta(gh) &= \sum g_1 h_1 \otimes g_2 h_2 \\
\varepsilon(gh) &= \varepsilon(g)\varepsilon(h) \\
\Delta(e_B) &= e_B \otimes e_B \\
\varepsilon(e_B) &= \mathbf{1}_{\mathbb{K}}
\end{aligned}$$

for all $g, h \in B$.

Definition 3.3: Let B be a \mathbb{K} -vector space that is both a \mathbb{K} -algebra $\langle B, \mu, \eta \rangle$ and a \mathbb{K} -coalgebra $\langle B, \Delta, \varepsilon \rangle$. We call $\langle B, \mu, \eta, \Delta, \varepsilon \rangle$ a \mathbb{K} -**bialgebra** if μ and η are \mathbb{K} -coalgebra morphisms (equivalently, if Δ and ε are \mathbb{K} -algebra morphisms). ▽

Definition 3.4: Let $\langle B_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1 \rangle$ and $\langle B_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2 \rangle$ be two \mathbb{K} -bialgebras. A \mathbb{K} -linear map $m: B_1 \rightarrow B_2$ is a \mathbb{K} -**bialgebra morphism** if it is a \mathbb{K} -algebra morphism on the \mathbb{K} -algebras $\langle B_1, \mu_1, \eta_1 \rangle$ and $\langle B_2, \mu_2, \eta_2 \rangle$ and a \mathbb{K} -coalgebra morphism on the \mathbb{K} -coalgebras $\langle B_1, \Delta_1, \varepsilon_1 \rangle$ and $\langle B_2, \Delta_2, \varepsilon_2 \rangle$.

The category of \mathbb{K} -bialgebras is called $\mathbf{BiAlg}_{\mathbb{K}}$ and the set of \mathbb{K} -bialgebra morphisms from B_1 to B_2 is called $\mathbf{BiAlg}_{\mathbb{K}}(B_1, B_2)$. ▽

Remark 3.5: If we have \mathbb{K} -bialgebras B_1, B_2 , the \mathbb{K} -module $B_1 \otimes B_2$ is also a \mathbb{K} -bialgebra if we use the tensor product of \mathbb{K} -algebras and the tensor product of \mathbb{K} -coalgebras to define its structure maps. We also call it **tensor product**.

Definition 3.6: Let $\langle B, \mu, \eta, \Delta, \varepsilon \rangle$ be a \mathbb{K} -bialgebra. If there are disjoint \mathbb{K} -submodules $B_n, n \in \mathbb{N}$, such that B is a graded \mathbb{K} -algebra with respect to $B_n, n \in \mathbb{N}$, as well as a graded \mathbb{K} -coalgebra with respect to $B_n, n \in \mathbb{N}$, we call B a **graded \mathbb{K} -bialgebra**. ∇

Example 3.7: We briefly mention two simple examples of bialgebras, recalling the coalgebras presented in 2.5:

- a) The field \mathbb{K} with its trivial \mathbb{K} -algebra structure and the canonical \mathbb{K} -coalgebra structure defined in Example 2.5 b) is a \mathbb{K} -bialgebra.
- b) If G is a monoid, the semigroup algebra $\mathbb{K}G$ with the \mathbb{K} -coalgebra structure as defined in Example 2.5 a) is a \mathbb{K} -bialgebra.

//

Example 3.8: Not every \mathbb{K} -algebra is suited to be part of a \mathbb{K} -bialgebra structure. Consider the matrix algebra $M_n(\mathbb{K})$ for $n \geq 2$, and suppose there is a \mathbb{K} -bialgebra structure on $M_n(\mathbb{K})$ with the matrix algebra as its underlying \mathbb{K} -algebra structure. Then the counit map $\varepsilon: M_n(\mathbb{K}) \rightarrow \mathbb{K}$ is a \mathbb{K} -algebra morphism. Recalling that $M_n(\mathbb{K})$ is simple and that $\ker \varepsilon$ is a two-sided ideal, we again have either $\ker \varepsilon = \{o\}$ or $\ker \varepsilon = M_n(\mathbb{K})$. As $\varepsilon(I) = 1$, we have $\ker \varepsilon = \{o\}$; this would mean that ε is injective, which then is a contradiction to $\dim M_n(\mathbb{K}) > \dim \mathbb{K}$. //

Remark 3.9: Let $\langle B, \mu, \eta, \Delta, \varepsilon \rangle$ be a \mathbb{K} -bialgebra. Recalling the definitions of the opposite algebra and coopposite coalgebra, we can define B_{opp} , B_{coop} and $B_{\text{opp,coop}}$ as follows:

- B_{opp} has the \mathbb{K} -algebra structure opposite to the one on B and the same \mathbb{K} -coalgebra structure as B .
- B_{coop} has the same \mathbb{K} -algebra structure as B and the \mathbb{K} -coalgebra structure coopposite to the one on B .
- $B_{\text{opp,coop}}$ has the \mathbb{K} -algebra structure opposite to the one on H and the \mathbb{K} -coalgebra structure coopposite to the one on B .

All three of these are \mathbb{K} -bialgebras.

Theorem 3.10: *Let $\langle B, \mu, \eta, \Delta, \varepsilon \rangle$ be a finite-dimensional \mathbb{K} -bialgebra. If we take the \mathbb{K} -algebra structure on B^* which is dual to the \mathbb{K} -coalgebra structure on B , and the \mathbb{K} -coalgebra structure on B^* which is dual to the \mathbb{K} -algebra structure on B , then B^* is a \mathbb{K} -bialgebra, the **dual bialgebra** of B .*

PROOF. Let δ and ε be the comultiplication and counit maps on B^* . We have to show that both of these are \mathbb{K} -algebra morphisms.

We recall that for $b^* \in B^*$ with

$$b^*(gh) = \sum b_1^*(g)b_2^*(h)$$

for all $g, h \in B$, we have

$$\begin{aligned} E(b^*) &= b^*(1_B) \\ \delta(b^*) &= \sum b_1^* \otimes b_2^*. \end{aligned}$$

Let $b^*, c^* \in B^*$ such that

$$\begin{aligned} \delta(b^*) &= \sum b_1^* b_2^* \\ \delta(c^*) &= \sum c_1^* c_2^* \end{aligned}$$

and $g, h \in B$, then

$$\begin{aligned} (b^*c^*)(gh) &= \sum b^*(g_1h_1)c^*(g_2h_2) = \sum b_1^*(g_1)b_2^*(h_1)c_1^*(g_2)c_2^*(h_2) \\ &= \sum (b_1^*c_1^*)(g)(b_2^*c_2^*)(h) \end{aligned}$$

and hence

$$\delta(b^*c^*) = \sum b_1^*c_1^* \otimes b_2^*c_2^* = \delta(b^*)\delta(c^*).$$

Additionally, we have $\varepsilon(gh) = \varepsilon(g)\varepsilon(h)$ for all $g, h \in B$, and therefore $\delta(\varepsilon) = \varepsilon \otimes \varepsilon$, whereby δ is a \mathbb{K} -algebra morphism.

Now consider E . First, we have

$$E(b^*c^*) = (b^*c^*)(e_B) = b^*(e_B)c^*(e_B) = E(b^*)E(c^*)$$

and additionally

$$E(\varepsilon) = \varepsilon(e_B) = 1_{\mathbb{K}},$$

whereby E is also a \mathbb{K} -algebra morphism. ■

3.2 Hopf Algebras and the Antipode

We now develop the concept of Hopf algebras. Given a \mathbb{K} -algebra $\langle A, \mu, \eta \rangle$ (with unit element e_A) and a \mathbb{K} -coalgebra $\langle C, \Delta, \varepsilon \rangle$, we define a \mathbb{K} -algebra structure on the set of homomorphisms $\mathbf{Mod}_{\mathbb{K}}(C, A)$ with the multiplication map denoted by $*$ and defined as

$$f * g := \mu \circ (f \otimes g) \circ \Delta$$

or element-wise in the sigma notation by

$$(f * g)(c) := \sum f(c_1)g(c_2)$$

for all $f, g \in \mathbf{Mod}_{\mathbb{K}}(C, A)$ and $c \in C$. The multiplication is associative, as for all $f, g, h \in \mathbf{Mod}_{\mathbb{K}}(C, A)$ and $c \in C$ we have

$$\begin{aligned} ((f * g) * h)(c) &= \sum (f * g)(c_1)h(c_2) = \sum f(c_1)g(c_2)h(c_3) \\ &= \sum f(c_1)(g * h)(c_2) = (f * (g * h))(c) \end{aligned}$$

by the coassociativity of Δ . The unit element of the algebra $\mathbf{Mod}_{\mathbb{K}}(C, A)$ is $\eta \circ \varepsilon \in \mathbf{Mod}_{\mathbb{K}}(C, A)$, as for all $f \in \mathbf{Mod}_{\mathbb{K}}(C, A)$ we have both

$$(f * (\eta \circ \varepsilon))(c) = \sum f(c_1)(\eta \circ \varepsilon)(c_2) = \sum f(c_1)\varepsilon(c_2) \cdot e_A = f(c)$$

and

$$((\eta \circ \varepsilon) * f)(c) = \sum (\eta \circ \varepsilon)(c_1)f(c_2) = \sum \varepsilon(c_1) \cdot e_A f(c_2) = f(c),$$

and thus both $f * (\eta \circ \varepsilon) = f$ and $(\eta \circ \varepsilon) * f = f$. The multiplication map $*$ is called the **convolution product**.

Having defined the convolution product in the general case of $\mathbf{Mod}_{\mathbb{K}}(C, A)$, let us now consider a special case of this construction. Let H be a \mathbb{K} -bialgebra, and denote the underlying \mathbb{K} -algebra and \mathbb{K} -coalgebra by H_{Alg} and H_{CoAlg} , respectively. We can then define an algebra structure on $\mathbf{Mod}_{\mathbb{K}}(H_{\text{CoAlg}}, H_{\text{Alg}})$ as above. As the identity map $\mathbf{1}_H: H \rightarrow H$ is an element of $\mathbf{Mod}_{\mathbb{K}}(H_{\text{CoAlg}}, H_{\text{Alg}})$, we can now define the following concept:

Definition 3.11: Let $\langle H, \mu, \eta, \Delta, \varepsilon \rangle$ be a \mathbb{K} -bialgebra. A \mathbb{K} -linear map $S: H \rightarrow H$ is called an **antipode** of the \mathbb{K} -bialgebra H if it is the inverse of the identity map $\mathbf{1}_H: H \rightarrow H$ in $\mathbf{Mod}_{\mathbb{K}}(H_{\text{CoAlg}}, H_{\text{Alg}})$ with respect to the convolution product $*$.

Put another way, S is an antipode if it fulfils the equation

$$S * \mathbf{1}_H = \eta \circ \varepsilon = \mathbf{1}_H * S$$

or equivalently

$$\mu \circ (S \otimes \mathbf{1}_H) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathbf{1}_H \otimes S) \circ \Delta$$

or equivalently if diagram 3.5 commutes.

$$\begin{array}{ccccc} & & H \otimes H & \xrightarrow{S \otimes \mathbf{1}} & H \otimes H & & \\ & \nearrow \Delta & & & & \searrow \mu & \\ H & \xrightarrow{\varepsilon} & \mathbb{K} & \xrightarrow{\eta} & H & & \\ & \searrow \Delta & & & & \nearrow \mu & \\ & & H \otimes H & \xrightarrow{\mathbf{1} \otimes S} & H \otimes H & & \end{array}$$

Figure 3.5: Antipode

A \mathbb{K} -bialgebra H which has an antipode is called a **\mathbb{K} -Hopf algebra**. ∇

Remark 3.12: The antipode in a \mathbb{K} -Hopf algebra H is by necessity unique, as it is the inverse of the element $\mathbf{1}_H$ in the algebra $\mathbf{Mod}_{\mathbb{K}}(H_{\text{CoAlg}}, H_{\text{Alg}})$. The fact that $S: H \rightarrow H$ is the antipode can also be formulated using the sigma notation as

$$\sum S(h_1)h_2 = \varepsilon(h)1 = \sum h_1S(h_2)$$

for all $h \in H$.

Definition 3.13: Let $\langle H_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1 \rangle$ and $\langle H_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2 \rangle$ be two \mathbb{K} -Hopf algebras. A \mathbb{K} -linear map $m: H_1 \rightarrow H_2$ is a **\mathbb{K} -Hopf algebra morphism** if it is a \mathbb{K} -bialgebra morphism on the \mathbb{K} -bialgebras $\langle H_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1 \rangle$ and $\langle H_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2 \rangle$.

The category of \mathbb{K} -Hopf algebras is called **$\mathbf{HopfAlg}_{\mathbb{K}}$** and the set of \mathbb{K} -Hopf algebra morphisms from H_1 to H_2 is called **$\mathbf{HopfAlg}_{\mathbb{K}}(H_1, H_2)$** . ∇

Note that we did not require that \mathbb{K} -Hopf algebra morphisms preserve the antipode; the following theorem shows that this is always the case.

Theorem 3.14: *Let $\langle H_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1 \rangle$ and $\langle H_2, \mu_2, \eta_2, \Delta_2, \varepsilon_2, S_2 \rangle$ be two \mathbb{K} -Hopf algebras and $f: H_1 \rightarrow H_2$ a \mathbb{K} -bialgebra morphism. Then*

$$S_2 \circ f = f \circ S_1,$$

that means that f preserves the antipode.

PROOF. Consider the \mathbb{K} -algebra $\mathbf{Mod}_{\mathbb{K}}(H_1, H_2)$ with the convolution product $*$, and consider its elements $f, S_2 \circ f, f \circ S_1$. We show that f is invertible and that both $S_2 \circ f$ and $f \circ S_1$ are inverses of f , and thus must be equal.

Let $h \in H_1$. On the one hand we have

$$\begin{aligned} ((S_2 \circ f) * f)(h) &= \sum (S_2 \circ f)(h_1)f(h_2) = \sum S_2(f(h_1))f(h_2) = (S_2 * \mathbf{1}_{H_2})(f(h)) \\ &= \sum S_2(f(h)_1)f(h)_2 = \varepsilon_2(f(h))e_{H_2} = \varepsilon_1(h)e_{H_2} = \eta_2 \circ \varepsilon_1(h), \end{aligned}$$

and thus $S_2 \circ f$ is a left inverse of f . On the other hand

$$\begin{aligned} (f * (f \circ S_1))(h) &= \sum f(h_1)(f \circ S_1)(h_2) = \sum f(h_1)f(S_1(h_2)) = f\left(\sum h_1S_1(h_2)\right) \\ &= f(\varepsilon_1(h)e_{H_1}) = \varepsilon_1(h)e_{H_2} = \eta_2 \circ \varepsilon_1(h), \end{aligned}$$

and therefore $f \circ S_1$ is a right inverse of f . Taken together, we have that f is invertible with respect to the convolution product $*$, and thus its left and right inverses must be equal. \blacksquare

Remark 3.15: If we have \mathbb{K} -Hopf algebras H_1, H_2 , then the \mathbb{K} -bialgebra $H_1 \otimes H_2$ is also a \mathbb{K} -Hopf algebra if we define its antipode as $S_1 \otimes S_2$; yet again, this is called the **tensor product**.

Definition 3.16: Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a \mathbb{K} -Hopf algebra. If the underlying \mathbb{K} -bialgebra $\langle H, \mu, \eta, \Delta, \varepsilon \rangle$ is graded, we call H a **graded \mathbb{K} -Hopf algebra**. ∇

Theorem 3.17: Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a \mathbb{K} -Hopf algebra. We show a few properties of the antipode S .

1. $S \circ \mu = \mu \circ (S \otimes S) \circ \tau$, which means that for all $g, h \in H$ we have

$$S(hg) = S(g)S(h).$$

2. $S \circ \eta = \eta$ or equivalently $S(e_H) = e_H$.

3. $\Delta \circ S = (S \otimes S) \circ \tau \circ \Delta$, which means that for all $h \in H$ we have

$$\Delta(S(h)) = \sum S(h_2) \otimes S(h_1).$$

4. $\varepsilon \circ S = \varepsilon$.

The first two properties mean that S is an antihomomorphism of \mathbb{K} -algebras; the second two properties mean that S is also an antihomomorphism of \mathbb{K} -coalgebras. (The term “antihomomorphism” refers to the fact that the order of the elements on the right-hand sides is reversed.)

PROOF. In the following calculations, $\stackrel{S}{\equiv}$ denotes a step which uses the defining equality of the antipode S , namely

$$\sum S(h_1)h_2 = \varepsilon(h)e_H = \sum h_1S(h_2),$$

and $\stackrel{\varepsilon}{\equiv}$ denotes a step which employs the counit property.

1. Regard $H \otimes H$ as a \mathbb{K} -coalgebra (with the \mathbb{K} -coalgebra structure of the tensor product of \mathbb{K} -coalgebras) and H as a \mathbb{K} -algebra. It is then sensible to consider the \mathbb{K} -algebra $\mathbf{Mod}_{\mathbb{K}}(H \otimes H, H)$ with the convolution product $*$. The unit element in this algebra is then $\eta_H \circ \varepsilon_{H \otimes H}$.

We define maps $F, G: H \otimes H \rightarrow H$ by

$$\begin{aligned} F(h \otimes g) &:= S(g)S(h) \\ G(h \otimes g) &:= S(hg) \end{aligned}$$

for $g, h \in H$. Similar to the last proof, we show that the multiplication map μ is invertible and that both F and G are inverses of μ , and therefore equal.

Let $g, h \in H$. We have

$$\begin{aligned} (\mu * F)(h \otimes g) &= \sum \mu((h \otimes g)_1)F((h \otimes g)_2) = \sum \mu(h_1 \otimes g_1)F(h_2 \otimes g_2) \\ &= \sum h_1g_1S(g_2)S(h_2) \stackrel{S}{\equiv} \sum h_1\varepsilon_H(g)e_HS(h_2) \stackrel{S}{\equiv} \varepsilon_H(h)\varepsilon_H(g)e_H \\ &= \varepsilon_{H \otimes H}(h \otimes g)e_H = \eta \circ \varepsilon_{H \otimes H}(h \otimes g) \end{aligned}$$

as well as

$$\begin{aligned}
(G * \mu)(h \otimes g) &= \sum G((h \otimes g)_1) \mu((h \otimes g)_2) = \sum G(h_1 \otimes g_1) \mu(h_2 \otimes g_2) \\
&= \sum S(h_1 g_1) h_2 g_2 = \sum S((hg)_1) (hg)_2 \stackrel{S}{=} \varepsilon_H(hg) e_H = \varepsilon_{H \otimes H}(h \otimes g) e_H \\
&= \eta \circ \varepsilon_{H \otimes H}(h \otimes g).
\end{aligned}$$

Therefore, G is a left inverse of μ , F is a right inverse of μ , hence μ is invertible with respect to the convolution product $*$ and its inverses F and G must be equal. Thus we have

$$S(hg) = S(g)S(h)$$

for all $g, h \in H$.

2. Applying the defining property of the antipode to the element $e_H \in H$ yields

$$S(e_H)e_H = \varepsilon_H(e_H)e_H = e_H$$

and applying ε_H to this equality gives us

$$\varepsilon_H(S(e_H)e_H) = \varepsilon_H(S(e_H))\varepsilon_H(e_H) = \varepsilon_H(S(e_H)) = \varepsilon_H(e_H) = 1,$$

and thus $S(e_H) = e_H$ or $S \circ \eta = \eta$.

3. Regard H as a \mathbb{K} -coalgebra and $H \otimes H$ as a \mathbb{K} -algebra (with the \mathbb{K} -algebra structure of the tensor product of \mathbb{K} -algebras). We can then consider the \mathbb{K} -algebra $\mathbf{Mod}_{\mathbb{K}}(H, H \otimes H)$ with the convolution product $*$; the unit element is $\eta_{H \otimes H} \circ \varepsilon_H$.

We again define maps $F, G: H \rightarrow H \otimes H$ by

$$\begin{aligned}
F(h) &:= \Delta(S(h)) \\
G(h) &:= \sum S(h_2) \otimes S(h_1)
\end{aligned}$$

for $h \in H$. Analogously to the proof of 1., we show that the comultiplication map Δ is invertible and that both F and G are inverses of Δ , and hence equal.

Let $h \in H$. We have

$$\begin{aligned}
(\Delta * F)(h) &= \sum \Delta(h_1) F(h_2) = \sum \Delta(h_1) \Delta(S(h_2)) = \Delta\left(\sum h_1 S(h_2)\right) \\
&= \Delta(\varepsilon_H(h) e_H) = \varepsilon_H(h) e_H \otimes e_H = \eta_{H \otimes H} \circ \varepsilon_H(h)
\end{aligned}$$

and also

$$\begin{aligned}
(G * \Delta)(h) &= \sum G(h_1) \Delta(h_2) = \sum (S((h_1)_2) \otimes S((h_1)_1)) ((h_2)_1 \otimes (h_2)_2) \\
&= \sum (S(h_2) \otimes S(h_1)) (h_3 \otimes h_4) = \sum S(h_2) h_3 \otimes S(h_1) h_4 \\
&= \sum S(h_2)_1 (h_2)_2 \otimes S(h_1) h_3 \stackrel{S}{=} \sum \varepsilon_H(h_2) e_H \otimes S(h_1) h_3 \\
&= \sum e_H \otimes S(h_1) \varepsilon_H(h_2) h_3 = \sum e_H \otimes S(h_1) \varepsilon_H((h_2)_1) (h_2)_2 \\
&\stackrel{S}{=} \sum e_H \otimes S(h_1) h_2 \stackrel{S}{=} e_H \otimes \varepsilon_H(h) e_H = \varepsilon_H(h) e_H \otimes e_H = \eta_{H \otimes H} \circ \varepsilon_H(h).
\end{aligned}$$

We have thus shown that G is a left inverse of Δ , F is a right inverse of Δ , whereby Δ is invertible with respect to the convolution product $*$ and its inverses F and G are equal. Therefore we have the equality

$$\Delta(S(h)) = \sum S(h_2) \otimes S(h_1)$$

for all $h \in H$.

4. We apply ε_H to the defining property of the antipode, giving us

$$\sum \varepsilon_H(h_1) \varepsilon_H(S(h_2)) = \varepsilon_H(h) \varepsilon_H(e_H) = \varepsilon_H(h).$$

As both ε_H and S are linear maps, we have

$$\begin{aligned} \sum \varepsilon_H(h_1) \varepsilon_H(S(h_2)) &= \varepsilon_H \left(\sum \varepsilon_H(h_1) S(h_2) \right) \\ &= \varepsilon_H \left(S \left(\sum \varepsilon_H(h_1) h_2 \right) \right) \stackrel{\varepsilon}{=} \varepsilon_H(S(h)), \end{aligned}$$

and thus $\varepsilon_H(S(h)) = \varepsilon_H(h)$. ■

Theorem 3.18: *Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a \mathbb{K} -Hopf algebra. Then the following statements are equivalent:*

- (i) $\sum S(h_2) h_1 = \varepsilon(h) e_H$ for all $h \in H$.
- (ii) $\sum h_2 S(h_1) = \varepsilon(h) e_H$ for all $h \in H$.
- (iii) $S^2 := S \circ S = \mathbf{1}_H$.

PROOF. (i) \Rightarrow (iii): $\mathbf{1}_H$ is the inverse of S with respect to the convolution product $*$. We show that S^2 is also a right inverse of S , hence it must be equal to $\mathbf{1}_H$. Let $h \in H$, then (using that S is a \mathbb{K} -algebra antihomomorphism)

$$\begin{aligned} (S * S^2)(h) &= \sum S(h_1) S^2(h_2) = \sum S(S(h_2) h_1) \\ &= S \left(\sum S(h_2) h_1 \right) = S(\varepsilon(h) e_H) = \varepsilon(h) e_H = \eta \circ \varepsilon(h), \end{aligned}$$

which shows that S^2 is a right inverse of S and thus equal to $\mathbf{1}_H$.

(ii) \Rightarrow (iii): Similarly, we show that S^2 is a left inverse of S . Let again $h \in H$, then

$$\begin{aligned} (S^2 * S)(h) &= \sum S^2(h_1) S(h_2) = \sum S(h_2 S(h_1)) \\ &= S \left(\sum h_2 S(h_1) \right) = S(\varepsilon(h) e_H) = \varepsilon(h) e_H = \eta \circ \varepsilon(h). \end{aligned}$$

(iii) \Rightarrow (i): Applying S to the defining equality of the antipode yields

$$\sum S(h_2) h_1 = \sum S(h_2) S^2(h_1) = S \left(\sum S(h_1) h_2 \right) = S(\varepsilon(h) e_H) = \varepsilon(h) e_H,$$

which is what we wanted to show.

(iii) \Rightarrow (ii): Similarly, applying S to the converse equality yields

$$\sum h_2 S(h_1) = \sum S^2(h_2) S(h_1) = S\left(\sum h_1 S(h_2)\right) = S(\varepsilon(h)e_H) = \varepsilon(h)e_H,$$

which completes the proof. \blacksquare

Corollary 3.19: *Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a \mathbb{K} -Hopf algebra. If H is commutative or cocommutative, then $S^2 = \mathbf{1}_H$.*

PROOF. Let H be commutative, then from

$$\sum S(h_1)h_2 = \varepsilon(h)e_H$$

we immediately have

$$\sum h_2 S(h_1) = \varepsilon(h)e_H$$

by commutativity, which is condition (ii) in the preceding theorem.

If H is instead cocommutative, that is,

$$\sum h_1 \otimes h_2 = \sum h_2 \otimes h_1,$$

then we have

$$\begin{aligned} \varepsilon(h)e_H &= \sum S(h_1)h_2 = (\mu \circ (S \otimes \mathbf{1}_H))\left(\sum h_1 \otimes h_2\right) \\ &= (\mu \circ (S \otimes \mathbf{1}_H))\left(\sum h_2 \otimes h_1\right) = \sum S(h_2)h_1, \end{aligned}$$

which is condition (i) in the preceding theorem. \blacksquare

Lemma 3.20: *Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a \mathbb{K} -Hopf algebra. Then the set $G(H)$ of grouplike elements of H is a group with the multiplication induced by μ .*

PROOF. By the compatibility of η and Δ , we have

$$\Delta(e_H) = e_H \otimes e_H,$$

and thus $e_H \in G(H)$; e_H is naturally the unit element in the group structure of $G(H)$.

If $g, h \in G(H)$, then

$$\begin{aligned} \Delta(gh) &= \Delta(\mu(g \otimes h)) = ((\mu \otimes \mu) \circ (\mathbf{1}_H \otimes \tau \otimes \mathbf{1}_H) \circ (\Delta \otimes \Delta))(g \otimes h) \\ &= ((\mu \otimes \mu) \circ (\mathbf{1}_H \otimes \tau \otimes \mathbf{1}_H))(\Delta(g) \otimes \Delta(h)) \\ &= ((\mu \otimes \mu) \circ (\mathbf{1}_H \otimes \tau \otimes \mathbf{1}_H))(g \otimes g \otimes h \otimes h) = (\mu \otimes \mu)(g \otimes h \otimes g \otimes h) = gh \otimes gh, \end{aligned}$$

and therefore $gh \in G(H)$.

Finally, from

$$\begin{aligned} S(g)g &= \sum S(g_1)g_2 = \varepsilon(g)e_H = e_H \\ gS(g) &= \sum g_1S(g_2) = \varepsilon(g)e_H = e_H \end{aligned}$$

for $g \in G(H)$, we have inverse elements in $G(H)$ by defining $g^{-1} := S(g)$. \blacksquare

Remark 3.21: Using the definitions of the opposite algebra and coopposite coalgebra, we have previously defined \mathbb{K} -bialgebras H_{opp} , H_{coop} and $H_{\text{opp,coop}}$. If S is bijective, then H_{opp} and H_{coop} are \mathbb{K} -Hopf algebras with antipode S^{-1} ; $H_{\text{opp,coop}}$ is always a \mathbb{K} -Hopf algebra with antipode S .

Theorem 3.22: *Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a finite-dimensional \mathbb{K} -Hopf algebra. Then the \mathbb{K} -bialgebra H^* is a \mathbb{K} -Hopf algebra with antipode S^* , the **dual Hopf algebra**.*

PROOF. Let δ and E denote the comultiplication and counit maps in H^* , let $h^* \in H^*$ with

$$\delta(h^*) = \sum h_1^* \otimes h_2^*$$

and let $h \in H$. Then we have

$$\begin{aligned} \sum (S^*(h_1^*)h_2^*)(h) &= \sum S^*(h_1^*)(h_1)h_2^*(h_2) = \sum h_1^*(S(h_1))h_2^*(h_2) \\ &= \sum h^*(S(h_1)h_2) = h^*(\varepsilon(h)e_H) = \varepsilon(h)h^*(e_H) = E(h^*)\varepsilon(h) \end{aligned}$$

and have shown that

$$\sum S^*(h_1^*)h_2^* = E(h^*)\varepsilon;$$

the converse is proved analogously, and therefore S^* is the antipode of H^* , making it a \mathbb{K} -Hopf algebra. ■

Lemma 3.23: *Let $\langle H, \mu, \eta, \Delta, \varepsilon, S \rangle$ be a \mathbb{K} -Hopf algebra and $\mathbb{L} \supseteq \mathbb{K}$ a field. If we consider $H_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{K}} H$, then we have an \mathbb{L} -bialgebra structure on $H_{\mathbb{L}}$ by Lemma 1.31 and Lemma 2.16.*

By defining

$$S': \mathbb{L} \otimes_{\mathbb{K}} H \longrightarrow \mathbb{L} \otimes_{\mathbb{K}} H: \ell \otimes_{\mathbb{K}} h \longmapsto \ell \otimes_{\mathbb{K}} S(h)$$

we have that $H_{\mathbb{L}}$ is an \mathbb{L} -Hopf algebra.

PROOF. This is obvious. ■

3.3 Examples

We finally give some examples of Hopf algebras.

Example 3.24: Let G be a group and $\mathbb{K}G$ the resulting group algebra. By Example 3.7 b), we have a \mathbb{K} -bialgebra structure on $\mathbb{K}G$. However, we now have the additional property that elements of G are invertible. Using this to define

$$S: \mathbb{K}G \longrightarrow \mathbb{K}G: g \longmapsto g^{-1}$$

and extending the map S by linearity, we have an antipode on $\mathbb{K}G$, as

$$\begin{aligned}(S * \mathbf{1})(g) &= \sum S(g_1)g_2 = S(g)g = g^{-1}g = e = \varepsilon(g)e \\ (\mathbf{1} * S)(g) &= \sum g_1S(g_2) = gS(g) = gg^{-1} = e = \varepsilon(g)e\end{aligned}$$

for all $g \in G$. This Hopf algebra structure is known as the **group Hopf algebra**. //

Example 3.25: We recall the divided power coalgebra from Example 2.5 c). Given such a divided power coalgebra $\langle H, \Delta, \varepsilon \rangle$ with countable \mathbb{K} -vector space basis $\{c_n \mid n \in \mathbb{N}\}$, we now define an accompanying \mathbb{K} -algebra structure on H to first yield a \mathbb{K} -bialgebra and finally a \mathbb{K} -Hopf algebra.

We first define

$$\mu(c_k, c_\ell) := \binom{k+\ell}{k} c_{k+\ell}$$

for any $k, \ell \in \mathbb{N}$ and extend this definition by linearity. It is clear that c_0 is a unit element with respect to this multiplication map, hence we have our unit element $e = c_0$. We now have to check that the multiplication map is indeed associative, that is,

$$\mu(\mu(c_k, c_\ell), c_m) = \mu(c_k, \mu(c_\ell, c_m))$$

for any $k, \ell, m \in \mathbb{N}$. This equality holds because

$$\begin{aligned}\mu(\mu(c_k, c_\ell), c_m) &= \mu\left(\binom{k+\ell}{k} c_{k+\ell}, c_m\right) = \binom{k+\ell}{k} \binom{k+\ell+m}{k+\ell} c_{k+\ell+m} \\ &= \frac{(k+\ell)! (k+\ell+m)!}{k! \ell! (k+\ell)! m!} c_{k+\ell+m} = \frac{(k+\ell+m)!}{k! \ell! m!} c_{k+\ell+m} \\ &= \frac{(k+\ell+m)! (\ell+m)!}{k! (\ell+m)! \ell! m!} c_{k+\ell+m} = \binom{k+\ell+m}{\ell+m} \binom{\ell+m}{\ell} c_{k+\ell+m} \\ &= \mu\left(c_k, \binom{\ell+m}{\ell} c_{\ell+m}\right) = \mu(c_k, \mu(c_\ell, c_m)).\end{aligned}$$

We now have to show that H is a \mathbb{K} -bialgebra with the above structure maps. First, it is evident that the counit ε is a \mathbb{K} -algebra morphism, as

$$\begin{aligned}\varepsilon(\mu(c_k, c_\ell)) &= \varepsilon\left(\binom{k+\ell}{k} c_{k+\ell}\right) = \delta_{0, k+\ell} = \delta_{0k} \delta_{0\ell} = \varepsilon(c_k) \varepsilon(c_\ell) \\ \varepsilon(e) &= \varepsilon(c_0) = \delta_{00} = 1_{\mathbb{K}}.\end{aligned}$$

It thus remains to show that Δ is a \mathbb{K} -algebra morphism. Writing $\bar{\mu}$ for $(\mu \otimes \mu) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1})$,

we have

$$\begin{aligned}
\bar{\mu}(\Delta(c_k), \Delta(c_\ell)) &= \bar{\mu}\left(\sum_{i=0}^k c_i \otimes c_{k-i}, \sum_{j=0}^{\ell} c_j \otimes c_{\ell-j}\right) \\
&= \sum_{i=0}^k \sum_{j=0}^{\ell} \binom{i+j}{i} \binom{k+\ell-i-j}{k-i} c_{i+j} \otimes c_{k+\ell-i-j} \\
&= \sum_{r=0}^{k+\ell} \sum_{s=0}^r \binom{r}{s} \binom{k+\ell-r}{k-s} c_r \otimes c_{k+\ell-r} = \sum_{r=0}^{k+\ell} \binom{k+\ell}{k} c_r \otimes c_{k+\ell-r} \\
&= \binom{k+\ell}{k} \sum_{r=0}^{k+\ell} c_r \otimes c_{k+\ell-r} = \binom{k+\ell}{k} \Delta(c_{k+\ell}) \\
&= \Delta\left(\binom{k+\ell}{k} c_{k+\ell}\right) = \Delta(\mu(c_k, c_\ell)).
\end{aligned}$$

Additionally, we have

$$\Delta(e) = \Delta(c_0) = c_0 \otimes c_0 = e \otimes e,$$

and thus H is a \mathbb{K} -bialgebra with the structure maps we have defined.

Finally, we want to define an antipode S to make H a \mathbb{K} -Hopf algebra. We remark that by the cocommutativity of the comultiplication map, it is sufficient to show

$$\sum \mu(S(h_1), h_2) = \varepsilon(h)e$$

for any h in the basis of H ; by linearity and cocommutativity, the defining property of the antipode is then fulfilled on all of H .

We define S on the basis $\{c_n \mid n \in \mathbb{N}\}$ of H , setting

$$S(c_n) := (-1)^n c_n.$$

First, we note that this definition fulfils the required equality for c_0 (with $S(c_0) = c_0 = e$), as

$$\sum \mu(S(e_1), e_2) = \mu(S(c_0), c_0) = \mu(c_0, c_0) = c_0 = e = \delta_{00}e = \varepsilon(c_0)e.$$

For $n \geq 1$, the required equality becomes

$$(S * \mathbf{1})(c_n) = \varepsilon(c_n)e = \delta_{0n}e = 0,$$

and we do indeed have

$$\begin{aligned}
(S * \mathbf{1})(c_n) &= \sum_{i=0}^n \mu(S(c_i), c_{n-i}) = \sum_{i=0}^n (-1)^i \mu(c_i, c_{n-i}) \\
&= \sum_{i=0}^n \binom{n}{i} (-1)^i c_n = \left(\sum_{i=0}^n \binom{n}{i} (-1)^i\right) c_n = (1 + (-1))^n c_n = 0.
\end{aligned}$$

We have thus shown that H has a \mathbb{K} -Hopf algebra structure (both commutative and cocommutative), which is called the **divided power Hopf algebra**. //

Example 3.26: Assume that $\text{char}(\mathbb{K}) \neq 2$. Let H be the \mathbb{K} -algebra defined by the generators c and x which satisfy the relations

$$\begin{aligned}c^2 &= e \\x^2 &= 0 \\xc &= -cx.\end{aligned}$$

H then is of dimension 4 with basis $\{e, c, x, cx\}$. We define a \mathbb{K} -coalgebra structure on H as induced by

$$\begin{aligned}\Delta(c) &:= c \otimes c \\ \Delta(x) &:= c \otimes x + x \otimes e \\ \varepsilon(c) &:= 1 \\ \varepsilon(x) &:= 0,\end{aligned}$$

and thus have a \mathbb{K} -bialgebra structure on H , which we can extend to a \mathbb{K} -Hopf algebra by defining the antipode as

$$\begin{aligned}S(c) &:= c^{-1} \\ S(x) &:= -cx.\end{aligned}$$

This example, originally due to Sweedler, was the first example of a Hopf algebra that is both noncommutative and noncocommutative, see [DNR01] p. 166. //

Chapter 4

Composition and Decomposition

4.1 Definitions

The concept of Hopf algebras occurs in a natural way in the field of combinatorial constructions. We establish the connection by linking the ideas of composition and decomposition of combinatorial objects to the multiplication and comultiplication maps of Hopf algebras. For a rigorous description, we require some additional definitions.

Definition 4.1: A **combinatorial class** $(\mathcal{C}, |\cdot|)$ is a denumerable collection of objects together with a **size function** $|\cdot|: \mathcal{C} \rightarrow \mathbb{N}$ which in some way counts a characteristic or attribute carried by the objects in the class. Via the size function, \mathcal{C} decomposes into disjoint subclasses $\mathcal{C}_n := \{\gamma \in \mathcal{C} \mid |\gamma| = n\}$ consisting only of objects of size n ; thus, we have

$$\mathcal{C} = \bigcup_{n \geq 0} \mathcal{C}_n.$$

Typically, one denotes the cardinality of \mathcal{C}_n by c_n ; for reasons of regularity, one usually demands that the size function be defined in such a way that $c_n \in \mathbb{N}$. ∇

Example 4.2: Classical examples of combinatorial classes in discrete mathematics are certain sets of graphs and trees, of which there are a wide variety.

A **graph** $G = \langle V, E \rangle$ consists of a set of vertices or nodes V and a set of edges E by which they are connected; multiple edges between the same vertices and loops (edges connecting a vertex with itself) are possible. A graph that does not have multiple edges nor loops is called **simple**. For our purposes, we only require **undirected graphs** (edges do not distinguish between the two vertices they connect).

Trees are undirected simple graphs that are **connected** (for each pair of vertices, there is a list of distinct edges which connects the two vertices, a so-called **path**) and **acyclic** (there is no path from any vertex to itself). A **forest** is a union of trees, that is, an undirected simple graph that is acyclic but not necessarily connected.

Usually, for any class of graphs the size $|G|$ of a graph G is the number of vertices $\#V$. //

Definition 4.3: Let A be a set and $m_A: A \rightarrow \mathbb{N}^\times$. We then call (A, m_A) a **multiset**; A is the set of distinct elements and m_A is the function counting the multiplicities of said

elements. Using this notation, basic concepts of set theory transfer easily from sets to multisets by also considering the multiplicity of elements:

- The **sum** of two multisets (A, m_A) and (B, m_B) is the multiset

$$A \uplus B = (A \cup B, m_{A \cup B}),$$

with the multiplicity function defined by

$$m_{A \cup B}(x) = \begin{cases} m_A(x) + m_B(x) & \text{if } x \in A \cap B, \\ m_A(x) & \text{if } x \in A \setminus B, \\ m_B(x) & \text{if } x \in B \setminus A. \end{cases}$$

- The **product** of two multisets (A, m_A) and (B, m_B) is the multiset

$$A \times B = (A \times B, m_{A \times B}),$$

with the multiplicity function defined by

$$m_{A \times B}(a, b) = m_A(a) \cdot m_B(b).$$

- **Inclusion** of multisets is defined as follows: $(A, m_A) \subseteq (B, m_B)$ if and only if $A \subseteq B$ and $m_A(x) \leq m_B(x)$ for all $x \in A$.

The multiplicity function m_A is usually dropped in the denotation of the multiset, leaving just A for (A, m_A) . ▽

Definition 4.4: The combinatorial construction we will be using in this chapter is the **multiset construction**. Given a combinatorial class $(\mathcal{C}, |\cdot|)$, a new class $\mathbf{MSet}(\mathcal{C})$ is defined, where the objects are multisets of objects in \mathcal{C} ; the size of an object $\Gamma \in \mathbf{MSet}(\mathcal{C})$ is canonically defined as

$$|(\Gamma, m_\Gamma)| := \sum_{\gamma \in \Gamma} m_\Gamma(\gamma) \cdot |\gamma|,$$

the sum of the sizes of all its elements, taking into account multiplicity. ▽

4.2 Composition

In the following, we consider a combinatorial class \mathcal{C} whose objects can compose and decompose into objects within the class.

One basic example for motivating the following theoretical notions is a jigsaw puzzle. Consider puzzle pieces with identical, square outlines, differing only in the layout of tabs and blanks.

The concept of composition in combinatorics is a rule specifying how two objects $\Gamma_1, \Gamma_2 \in \mathcal{C}$ can be combined to make another object in \mathcal{C} . In principle, this rule need not be unambiguous. (For example, given two puzzle pieces, they could be combined in a number of different ways by joining any tab of the first piece with any blank of the second piece.) Additionally, different possibilities of combining these objects could result in the same outcome. (Again, consider puzzle pieces.)

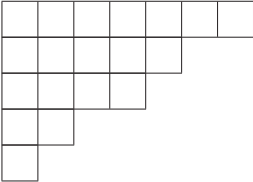
Another example are **partitions** of positive integers. A partition of a number n is a way of writing n as the sum of positive integers, without considering the order of the summands:

$$n = n_1 + n_2 + \dots + n_k$$

As the order of summands does not matter, they are typically ordered decreasingly. Partitions are visualized through **Ferrers diagrams**. The Ferrers diagram of a partition of n represents each summand in the partition as a row of boxes or bullets; the s -th row of a Ferrers diagram thus represents the s -th summand. Consider the partition

$$19 = 7 + 5 + 4 + 2 + 1.$$

The Ferrers diagram of this partition is:



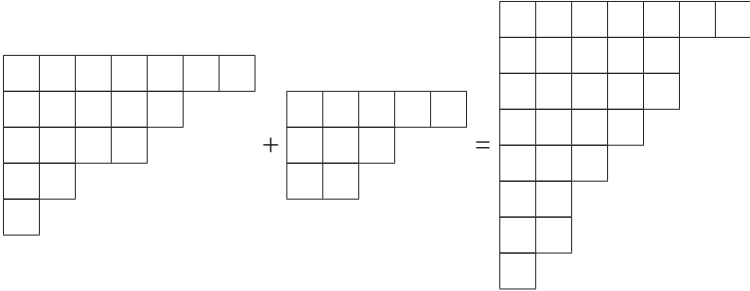
If we now consider partitions of two positive integers m, n , we could compose the partitions into a partition of $m + n$ by taking the summands (or rows) of the partition of m and the summands (or rows) of the partition of n and joining them in a single sum (or Ferrers diagram), rearranging the summands (or rows) so as to preserve the decreasing order. For example, if we have the partitions

$$\begin{aligned} 19 &= 7 + 5 + 4 + 2 + 1 \\ 10 &= 5 + 3 + 2, \end{aligned}$$

we could compose them into the partition

$$29 = 7 + 5 + 5 + 4 + 3 + 2 + 2 + 1.$$

Expressed in Ferrers diagrams, this could be written as



One could also define more complicated notions of composition on Ferrers diagrams by adding the boxes of the “second” Ferrers diagram in the composition to the “first” Ferrers diagram in different ways.

A rigorous description of this notion of composition therefore has to take account of all possible options and multiplicities of outcomes. The multiset construction provides exactly this functionality.

Definition 4.5: Given a combinatorial class \mathcal{C} , the **composition rule** is a map

$$*: \mathcal{C} \times \mathcal{C} \longrightarrow \mathbf{MSet}(\mathcal{C})$$

which assigns to each pair of objects $\gamma_1, \gamma_2 \in \mathcal{C}$ the multiset $\gamma_1 * \gamma_2$ consisting of all possible compositions of γ_1 with γ_2 . Multiple occurrences of the same object count the number of distinct ways in which an outcome occurs in the composition.

The map is easily extended to a map

$$*: \mathbf{MSet}(\mathcal{C}) \times \mathbf{MSet}(\mathcal{C}) \longrightarrow \mathbf{MSet}(\mathcal{C})$$

by taking each element of each multiset $\Gamma_1, \Gamma_2 \in \mathbf{MSet}(\mathcal{C})$, composing piecewise and then collecting the results:

$$\Gamma_1 * \Gamma_2 = \bigsqcup_{\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2} \gamma_1 * \gamma_2$$

▽

Definition 4.6: To make further use of the concept feasible, we have to consider a number of additional constraints.

(C1) **Finiteness:** The possible compositions of two objects are limited in number; this means that

$$\#(\gamma_1 * \gamma_2) < \infty$$

for all $\gamma_1, \gamma_2 \in \mathcal{C}$.

(C2) **Triple composition:** Assume we want to compose more than two objects, namely, we consider

$$\gamma_1 * \gamma_2 * \gamma_3$$

for $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{C}$. We can do this in two possible ways: Either we first compose the first two by

$$\gamma_1 * \gamma_2 = \Gamma'$$

and then compose the result of this composition with the last object by

$$\Gamma' * \gamma_3 = \Gamma,$$

or we first compose

$$\gamma_2 \star \gamma_3 = \Gamma''$$

and then compose the result with the first object by

$$\gamma_1 \star \Gamma'' = \Gamma.$$

It is sensible to demand that the two results are the same; this means that \star fulfils an associativity property

$$\gamma_1 \star (\gamma_2 \star \gamma_3) = (\gamma_1 \star \gamma_2) \star \gamma_3,$$

which can be generalized to n -fold compositions, $n \geq 3$, in a straightforward way.

(C3) **Neutral object:** This constraint assumes there exists a neutral object \emptyset which composes only trivially with all objects in the class; this means that

$$\emptyset \star \gamma = \gamma \star \emptyset = \gamma$$

for all $\gamma \in \mathcal{C}$. Obviously, \emptyset is unique if it exists:

$$\emptyset_1 = \emptyset_1 \star \emptyset_2 = \emptyset_2$$

(C4) **Symmetry:** In some cases, the composition rule does not distinguish between the two objects involved; this means that \star fulfils a commutativity property

$$\gamma_1 \star \gamma_2 = \gamma_2 \star \gamma_1$$

for all $\gamma_1, \gamma_2 \in \mathcal{C}$.

▽

4.3 Decomposition

Dually to the previous concept, consider decomposition in a combinatorial class; this means splitting an object of \mathcal{C} into a number of ordered pairs of objects in the same class. Again, puzzle pieces provide a useful and simple example for this: For any given constellation of puzzle pieces, any division of the pieces involved into disjoint constellations while still keeping track of spatial relations between the pieces would be part of such a decomposition.

Considering partitions and Ferrers diagrams, decomposition could be taken to mean choosing any subset of rows (or, more complicated, even any subset of boxes), building a new partition from the selected rows (or boxes) and rearranging the remainder so as to constitute a valid partition, as well.

Again, we have to consider all possible outcomes of such a decomposition. The multiset construction once again is the appropriate way of describing this.

Definition 4.7: Given a combinatorial class \mathcal{C} , the **decomposition rule** is a map

$$\langle \cdot \rangle: \mathcal{C} \longrightarrow \mathbf{MSet}(\mathcal{C} \times \mathcal{C})$$

which assigns to objects $\gamma \in \mathcal{C}$ the multiset $\langle \gamma \rangle$ consisting of all possible pairs of objects (γ_1, γ_2) which are splittings of γ . Multiple occurrences of the same object count the number of distinct ways in which a splitting occurs in the decomposition.

Again, the map is easily extended to a map

$$\langle \cdot \rangle: \mathbf{MSet}(\mathcal{C}) \longrightarrow \mathbf{MSet}(\mathcal{C} \times \mathcal{C})$$

by taking each element of the multiset $\Gamma \in \mathbf{MSet}(\mathcal{C})$, decomposing piecewise and then collecting the results:

$$\langle \Gamma \rangle = \bigsqcup_{\gamma \in \Gamma} \langle \gamma \rangle$$

We sometimes write

$$\gamma \longrightarrow (\gamma_1, \gamma_2)$$

to signify $(\gamma_1, \gamma_2) \in \langle \gamma \rangle$. ▽

Definition 4.8: As with the composition rule, we consider a number of general conditions we might additionally require the decomposition rule to fulfil. The first four are dual in some sense to the conditions we formulated for the composition rule; while this reflects the fact that composition and decomposition are related in a straightforward way, the two are not related so far (as we do not require that both composition and decomposition rules be defined) and the conditions should be considered independent.

(D1) **Finiteness:** The possible decompositions of an object are finite; this means that

$$\#\langle \gamma \rangle < \infty$$

for all $\gamma \in \mathcal{C}$.

(D2) **Triple decomposition:** The decomposition into pairs of objects can be extended in a natural way to the decomposition into triples, by simply applying the decomposition rule repeatedly. After splitting γ into (γ_1, γ_2) , we can apply the decomposition rule a second time, either to γ_1 or to γ_2 ; it is reasonable to demand that the result be the same either way. This translates to a coassociativity property for the decomposition rule, namely

$$\bigsqcup_{(\gamma_1, \gamma_2) \in \langle \gamma \rangle} \langle \gamma_1 \rangle \times \langle \gamma_2 \rangle = \bigsqcup_{(\gamma_1, \gamma_2) \in \langle \gamma \rangle} \langle \gamma_1 \rangle \times \langle \gamma_2 \rangle$$

for $\gamma \in \mathcal{C}$. This can be generalized to splitting γ into n pieces by iterated decomposition, with the coassociativity for $(n-1)$ -fold decomposition (that is, into n pieces) following from coassociativity for double decomposition (that is, into triples). We can therefore define an iterated decomposition rule $\langle \cdot \rangle^{(n)}$ by

$$\langle \gamma \rangle^{(n)} := \bigsqcup_{\gamma \longrightarrow (\gamma_1, \dots, \gamma_{n+1})} \{(\gamma_1, \dots, \gamma_{n+1})\}$$

for all $\gamma \in \mathcal{C}$.

(D3) **Void object:** This constraint assumes that there exists a void or empty object \emptyset such that any object $\gamma \neq \emptyset$ splits into pairs containing either \emptyset or γ in only two ways, namely

$$\gamma \longrightarrow (\emptyset, \gamma)$$

and

$$\gamma \longrightarrow (\gamma, \emptyset).$$

It also follows that \emptyset splits only into (\emptyset, \emptyset) . Again, \emptyset is unique if it exists.

(D4) **Symmetry:** Sometimes the order of the components an objects splits into is insignificant; this means that

$$(\gamma_1, \gamma_2) \in \langle \gamma \rangle \iff (\gamma_2, \gamma_1) \in \langle \gamma \rangle$$

for all $\gamma \in \mathcal{C}$, and that the multiplicities of (γ_1, γ_2) and (γ_2, γ_1) in the multiset $\langle \gamma \rangle$ are the same.

(D5) **Finiteness of iterated decomposition:** Using iterated decomposition $\langle \cdot \rangle^{(n)}$ as defined in condition (D3) above, we can decompose an object into any number of components. However, if we wish to consider only nontrivial decompositions (those which do not contain any void components \emptyset), it is often the case that the process terminates after a finite number of steps. This means that for each $\gamma \in \mathcal{C}$ there exists an $N_\gamma \in \mathbb{N}$ such that

$$\{(\gamma_1, \dots, \gamma_n) \mid \gamma \longrightarrow (\gamma_1, \dots, \gamma_n), \gamma_1 \neq \emptyset, \dots, \gamma_n \neq \emptyset\} = \emptyset$$

for all $n \geq N_\gamma$.

▽

4.4 Compatibility

In the following, we consider combinatorial classes which have both a composition and a decomposition rule. Two conditions of compatibility are then necessary to allow the two rules to operate consistently.

Definition 4.9:

(E1) **Composition–decomposition compatibility:** Assume we are given a pair of objects $(\gamma_1, \gamma_2) \in \mathcal{C} \times \mathcal{C}$. If we now want to have all pairs of objects which are equivalent to this pair with regard to our composition and decomposition rules, we could proceed with two different decomposition schemes which both have composition as an intermediate step: We could either first compose the two objects and then decompose the resulting object into all possible pairs, or we could first decompose both objects of the pair separately and then compose them component-wise. We would

expect to get the same result either way, so it is sensible to demand the following equality of multisets:

$$\langle \gamma_1 \star \gamma_2 \rangle = \bigsqcup_{\substack{(\gamma'_1, \gamma''_1) \in \langle \gamma_1 \rangle, \\ (\gamma'_2, \gamma''_2) \in \langle \gamma_2 \rangle}} \{(\gamma'_1 \star \gamma'_2) \times (\gamma''_1 \star \gamma''_2)\}$$

This property furthermore implies that the neutral object from (C3) and the void object from (D3) are the same, hence they are denoted by the same symbol \emptyset .

(E2) **Compatibility with size:** Considering the notion of size in combinatorial classes, it is again sensible to demand that the composition and decomposition rules preserve size. For composition, this means that the sizes of the objects being combined add up:

$$\gamma \in \gamma_1 \star \gamma_2 \implies |\gamma| = |\gamma_1| + |\gamma_2|$$

The parallel condition for decomposition is that when splitting an object, the size of the original object is also distributed between the parts:

$$(\gamma_1, \gamma_2) \in \langle \gamma \rangle \implies |\gamma_1| + |\gamma_2| = |\gamma|$$

If we consider the partition of \mathcal{C} into $\mathcal{C}_i, i \geq 0$, this means that the composition rule can be considered a map

$$\star: \mathcal{C}_i \times \mathcal{C}_j \longrightarrow \mathbf{MSet}(\mathcal{C}_{i+j})$$

and the decomposition rule can be considered a map

$$\langle \cdot \rangle: \mathcal{C}_k \longrightarrow \mathbf{MSet} \left(\bigsqcup_{i+j=k} \mathcal{C}_i \times \mathcal{C}_j \right)$$

for all $k \in \mathbb{N}$.

▽

In the following, we assume that there is a single object of size 0, which means $\mathcal{C}_0 = \{\emptyset\}$.

4.5 Natural Algebraic Structures

Using the definitions, constraints and conditions of the previous sections, we now outline a straightforward way to provide combinatorial objects with a natural algebraic structure by using the composition and decomposition rules. In this way, we are able to systematically construct algebra, coalgebra, bialgebra and Hopf algebra structures for combinatorial classes.

Definition 4.10: Given a combinatorial class \mathcal{C} , we define \mathbf{C} as the \mathbb{K} -vector space consisting of the (finite) linear combinations of objects in \mathcal{C} :

$$\mathbf{C} := \mathbb{K}\mathcal{C} = \left\{ \sum_i a_i \gamma_i \mid a_i \in \mathbb{K}, \gamma_i \in \mathcal{C} \right\}$$

Addition of elements and multiplication with scalars are defined in \mathbf{C} as usual by

$$\begin{aligned}\sum_i a_i \gamma_i + \sum_i b_i \gamma_i &= \sum_i (a_i + b_i) \gamma_i \\ k \cdot \sum_i a_i \gamma_i &= \sum_i (ka_i) \gamma_i.\end{aligned}$$

It is clear that the elements of \mathcal{C} are linearly independent and span all of \mathbf{C} , and that \mathcal{C} is thus a basis of \mathbf{C} ; as \mathcal{C} also has a combinatorial meaning, \mathcal{C} is called the **combinatorial basis** of \mathbf{C} . ∇

Suppose that \mathcal{C} has a composition rule \star as previously discussed. We can then define a bilinear map \diamond on basis elements $\gamma_1, \gamma_2 \in \mathcal{C}$ as the sum of all possible compositions of γ_1 with γ_2 :

$$\diamond: \mathcal{C} \times \mathcal{C} \longrightarrow \mathbf{C}: (\gamma_1, \gamma_2) \longmapsto \gamma_1 \diamond \gamma_2 := \sum_{\gamma \in \gamma_1 \star \gamma_2} \gamma$$

This map can straightforwardly be extended to a map on all of $\mathbf{C} \times \mathbf{C}$ by bilinearity. Despite the fact that all coefficients in the defining sum are 1, the terms in the right-hand side may in fact appear several times, as $\gamma_1 \star \gamma_2$ is a multiset. The multiplicities of elements in such multisets have the role of structure constants in the resulting algebra.

If we additionally define the map η by

$$\eta: \mathbb{K} \longrightarrow \mathbf{C}: k \longmapsto k\emptyset,$$

we have the following result:

Theorem 4.11: *The \mathbb{K} -vector space \mathbf{C} is a \mathbb{K} -algebra $\langle \mathbf{C}, \diamond, \eta \rangle$ if conditions (C1)–(C3) hold. It is commutative if condition (C4) holds.*

PROOF. Condition (C1) guarantees that the sums in the definition of \diamond are finite, hence the multiplication map \diamond is well-defined. Condition (C3) translates directly into the existence of the unit element \emptyset , and thus η is the desired unit map, as

$$\begin{aligned}k\gamma &= (k\emptyset) \diamond \gamma = \eta(k) \diamond \gamma \\ k\gamma &= \gamma \diamond (k\emptyset) = \gamma \diamond \eta(k).\end{aligned}$$

Using condition (C2), we can easily verify for basis elements $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{C}$ that

$$\gamma_1 \diamond (\gamma_2 \diamond \gamma_3) = \sum_{\gamma \in \gamma_1 \star \gamma_2 \star \gamma_3} \gamma = (\gamma_1 \diamond \gamma_2) \diamond \gamma_3,$$

and associativity for all of \mathbf{C} follows from the bilinearity of the multiplication map \diamond . Condition (C4) is obviously equivalent to the commutativity of \mathbf{C} . \blacksquare

Suppose now that \mathcal{C} has a decomposition rule $\langle \cdot \rangle$. We define a linear map Δ on basis elements $\gamma \in \mathcal{C}$ as the sum of all splittings of γ into pairs:

$$\Delta: \mathcal{C} \longrightarrow \mathbf{C} \otimes \mathbf{C}: \gamma \longmapsto \Delta(\gamma) := \sum_{(\gamma_1, \gamma_2) \in \langle \gamma \rangle} \gamma_1 \otimes \gamma_2$$

Again, this map is easily extended to a map on all of \mathbf{C} by bilinearity. Collection of repeated terms in the equations leads to coefficients which, again, are the multiplicities in the multiset $\langle \gamma \rangle$; these are sometimes called **section coefficients** (see also Definition 5.3).

By also defining the linear map ε on basis elements $\gamma \in \mathcal{C}$ by

$$\varepsilon: \mathcal{C} \longrightarrow \mathbb{K}: \gamma \longmapsto \begin{cases} 1 & \text{if } \gamma = \emptyset, \\ 0 & \text{if } \gamma \neq \emptyset \end{cases}$$

and extending it to a map on all of \mathbf{C} by virtue of linearity, we get the following theorem:

Theorem 4.12: *The \mathbb{K} -vector space \mathbf{C} is a \mathbb{K} -coalgebra $\langle \mathbf{C}, \Delta, \varepsilon \rangle$ if conditions (D1)–(D3) are satisfied. It is cocommutative if condition (D4) is satisfied.*

PROOF. That the sums in the definition of Δ are finite follows from condition (D1), meaning the comultiplication map Δ is well-defined. Using condition (D2), we have

$$(\Delta \otimes \mathbf{1}) \circ \Delta(\gamma) = \sum_{\gamma \rightarrow (\gamma_1, \gamma_2, \gamma_3)} \gamma_1 \otimes \gamma_2 \otimes \gamma_3 = (\mathbf{1} \otimes \Delta) \circ \Delta(\gamma)$$

for basis elements $\gamma \in \mathcal{C}$ via the equivalence of triple splittings, and the coassociativity of Δ on all of \mathbf{C} follows from the linearity of Δ . Regarding the counit ε , we show that

$$(\varepsilon \otimes \mathbf{1}) \circ \Delta(\gamma) = \gamma = (\mathbf{1} \otimes \varepsilon) \circ \Delta(\gamma)$$

for all $\gamma \in \mathcal{C}$, which extends to the same property on all of \mathbf{C} by linearity of ε . Using condition (D3), we have

$$\begin{aligned} (\varepsilon \otimes \mathbf{1}) \circ \Delta(\gamma) &= \sum_{(\gamma_1, \gamma_2) \in \langle \gamma \rangle} \varepsilon(\gamma_1) \otimes \gamma_2 = \varepsilon(\emptyset) \otimes \gamma + \sum_{\substack{(\gamma_1, \gamma_2) \in \langle \gamma \rangle, \\ \gamma_1 \neq \emptyset}} \varepsilon(\gamma_1) \otimes \gamma_2 \\ &= \mathbf{1} \otimes \gamma + \sum_{\substack{(\gamma_1, \gamma_2) \in \langle \gamma \rangle, \\ \gamma_1 \neq \emptyset}} 0 \otimes \gamma_2 = \gamma \end{aligned}$$

by definition of ε and by again identifying $\mathbb{K} \otimes \mathbf{C}$ with \mathbf{C} ; the converse property is verified analogously. Finally, the cocommutativity of the comultiplication map clearly follows from condition (D4). \blacksquare

We have seen how the existence of a composition rule and a decomposition rule make \mathbf{C} an algebra and coalgebra, respectively; the existence of both rules provides \mathbf{C} with a bialgebra structure.

Theorem 4.13: *Assuming the previous conditions, the \mathbb{K} -algebra and \mathbb{K} -coalgebra \mathbf{C} is a \mathbb{K} -bialgebra $\langle \mathbf{C}, \diamond, \eta, \Delta, \varepsilon \rangle$ if condition (E1) holds.*

PROOF. We have to show that both the comultiplication map Δ and the counit map ε preserve the multiplication map and the unit element in \mathbf{C} . For the multiplication map, it is sufficient to prove

$$\Delta(\gamma_1 \diamond \gamma_2) = \Delta(\gamma_1) \diamond \Delta(\gamma_2)$$

(with multiplication on the right-hand side occurring component-wise in the tensor product $\mathbf{C} \otimes \mathbf{C}$) and

$$\varepsilon(\gamma_1 \diamond \gamma_2) = \varepsilon(\gamma_1)\varepsilon(\gamma_2)$$

(with multiplication on the right-hand side in the field \mathbb{K}) for basis elements $\gamma_1, \gamma_2 \in \mathcal{C}$. For the unit, we have to show that

$$\begin{aligned}\Delta(\emptyset) &= \emptyset \otimes \emptyset \\ \varepsilon(\emptyset) &= 1,\end{aligned}$$

both of which follow easily from the definitions.

We first prove that comultiplication preserves multiplication by expanding both sides of the equation independently:

$$\begin{aligned}\Delta(\gamma_1 \diamond \gamma_2) &= \sum_{\gamma \in \gamma_1 \star \gamma_2} \Delta(\gamma) = \sum_{\substack{\gamma \in \gamma_1 \star \gamma_2, \\ (\gamma', \gamma'') \in \langle \gamma \rangle}} \gamma' \otimes \gamma'' = \sum_{(\gamma', \gamma'') \in \langle \gamma_1 \star \gamma_2 \rangle} \gamma' \otimes \gamma'' \\ \Delta(\gamma_1) \diamond \Delta(\gamma_2) &= \sum_{\substack{(\gamma'_1, \gamma''_1) \in \langle \gamma_1 \rangle, \\ (\gamma'_2, \gamma''_2) \in \langle \gamma_2 \rangle}} (\gamma'_1 \otimes \gamma''_1) \diamond (\gamma'_2 \otimes \gamma''_2) = \sum_{\substack{(\gamma'_1, \gamma''_1) \in \langle \gamma_1 \rangle, \\ (\gamma'_2, \gamma''_2) \in \langle \gamma_2 \rangle}} (\gamma'_1 \diamond \gamma'_2) \otimes (\gamma''_1 \diamond \gamma''_2) \\ &= \sum_{\substack{(\gamma'_1, \gamma''_1) \in \langle \gamma_1 \rangle, \\ (\gamma'_2, \gamma''_2) \in \langle \gamma_2 \rangle}} \sum_{\gamma'' \in \gamma'_1 \diamond \gamma'_2} \gamma' \otimes \gamma''\end{aligned}$$

The equality of the right-most terms in these two equations is exactly condition (E1).

To prove that the counit preserves multiplication, we first remark that for the right-hand side we have

$$\varepsilon(\gamma_1)\varepsilon(\gamma_2) = \begin{cases} 1 & \text{if } \gamma_1 = \gamma_2 = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Using conditions (C3), (D3) and (E1), we see that

$$\gamma_1 \diamond \gamma_2 = \emptyset$$

if and only if $\gamma_1 = \gamma_2 = \emptyset$, thus ending the proof. ■

We require one further definition to provide \mathbf{C} with a Hopf algebra structure.

Definition 4.14: Define a linear map

$$S: \mathbf{C} \longrightarrow \mathbf{C}$$

on basis elements $\gamma \in \mathcal{C}$ as the alternating sum of all products of nontrivial decompositions of γ (for $\gamma \neq \emptyset$), that is

$$S(\gamma) := \sum_{\substack{\gamma \longrightarrow (\gamma_1, \dots, \gamma_n), \\ \gamma_1 \neq \emptyset, \dots, \gamma_n \neq \emptyset}} (-1)^n \gamma_1 \diamond \dots \diamond \gamma_n,$$

and $S(\emptyset) := \emptyset$; the extension from \mathcal{C} to all of \mathbf{C} is again due to linearity. The map S will fulfil the role of the antipode on \mathbf{C} . ∇

Lemma 4.15: *If V is a vector space,*

$$L: V \longrightarrow V$$

is a linear map, $\Lambda := L - \mathbf{1}$ and

$$L' := \sum_{n=0}^{\infty} (-\Lambda)^n$$

is well-defined, then $L^{-1} = L'$.

PROOF. We have to show that

$$L \circ L' = \mathbf{1} = L' \circ L.$$

A straightforward calculation gives us

$$L \circ L' = (\mathbf{1} + \Lambda) \circ \sum_{n=0}^{\infty} (-\Lambda)^n = \sum_{n=0}^{\infty} (-\Lambda)^n + \sum_{n=0}^{\infty} (-\Lambda)^{n+1} = \mathbf{1}$$

and $L' \circ L$ is proven analogously. \blacksquare

Theorem 4.16: *Assuming the previous conditions, the \mathbb{K} -bialgebra \mathbf{C} is a \mathbb{K} -Hopf algebra $\langle \mathbf{C}, \diamond, \eta, \Delta, \varepsilon, S \rangle$ if condition (D5) holds.*

PROOF. In the following, we denote the multiplication map \diamond by μ for easier notation, with

$$\mu(\gamma_1 \otimes \gamma_2) = \gamma_1 \diamond \gamma_2$$

for all $\gamma_1, \gamma_2 \in \mathbf{C}$; $\pi = \pi_{\emptyset} := \emptyset \varepsilon$ will denote the projection onto the subspace spanned by \emptyset :

$$\pi(\gamma) = \begin{cases} \gamma & \text{if } \gamma = \emptyset, \\ 0 & \text{otherwise;} \end{cases}$$

and finally $\mathbf{End}(\mathbf{C}) := \mathbf{BiAlg}(\mathbf{C}, \mathbf{C})$.

To prove that the endomorphism S is the antipode of \mathbf{C} , we have to show that

$$\mu \circ (S \otimes \mathbf{1}) \circ \Delta = \pi = \mu \circ (\mathbf{1} \otimes S) \circ \Delta.$$

To show this, we define an auxiliary map

$$\Phi: \mathbf{End}(\mathbf{C}) \longrightarrow \mathbf{End}(\mathbf{C}): f \longmapsto \Phi(f) := \mu \circ (f \otimes \mathbf{1}) \circ \Delta$$

and note that the desired equality can now be formulated as

$$S = \Phi^{-1}(\pi),$$

assuming that Φ is invertible. We now prove that Φ is indeed invertible and calculate its inverse.

First, we rewrite Φ as

$$\Phi = \mathbf{1} + \Psi$$

by extracting the identity map $\mathbf{1}$. We now define $\bar{\pi}$ as the complementary projection $\bar{\pi} := \mathbf{1} - \pi$ onto the subspace spanned by all $\gamma \neq \emptyset$:

$$\bar{\pi}(\gamma) = \begin{cases} 0 & \text{if } \gamma = \emptyset, \\ \gamma & \text{otherwise.} \end{cases}$$

Ψ can then be written as

$$\Psi(f) = \mu \circ (f \otimes \bar{\pi}) \circ \Delta$$

for all $f \in \mathbf{End}(\mathbf{C})$. Using the previous lemma, we now have to show that

$$\Phi' = \sum_{n=0}^{\infty} (-\Psi)^n$$

is well-defined. To this end, we analyse one term of the sum and deduce that for the n -th iteration of Ψ we have

$$\Psi^n(f)(\gamma) = \sum_{\substack{\gamma \rightarrow (\gamma_1, \gamma_2, \dots, \gamma_{n+1}), \\ \gamma_2 \neq \emptyset, \dots, \gamma_{n+1} \neq \emptyset}} f(\gamma_1) \diamond \gamma_2 \diamond \dots \diamond \gamma_{n+1}.$$

In this equation, we have used that the comultiplication map Δ preserves the multiplication map \diamond and the definition of $\bar{\pi}$ (which results in the exclusion of all terms \emptyset but the first one). This exclusion of \emptyset together with condition (D5) thus shows us that for each γ we have

$$\Psi^n(f)(\gamma) = 0$$

for all $n \geq N_\gamma$, and hence the sum in the definition of Φ' is actually finite. As Φ' is therefore well-defined, we have $\Phi' = \Phi^{-1}$ and can explicitly calculate $\Phi^{-1}(\pi)$, which matches the original definition of S .

To summarize, we have thereby shown that the previously defined linear map S fulfils the first equality; the second equality follows analogously and S is therefore proven to be the antipode of \mathbf{C} , providing it with a Hopf algebra structure. \blacksquare

Theorem 4.17: *Assuming the conditions necessary for \mathbf{C} to be a \mathbb{K} -bialgebra, \mathbf{C} is a graded \mathbb{K} -Hopf algebra with grading given by size if condition (E2) is fulfilled, with*

$$\mathbf{C} = \bigoplus_{n \in \mathbb{N}} \mathbf{C}_n$$

where

$$\begin{aligned}\mathcal{C}_n &:= \{\gamma \in \mathcal{C} \mid |\gamma| = n\} \\ \mathbf{C}_n &:= \text{span}(\mathcal{C}_n)\end{aligned}$$

for all $n \in \mathbb{N}$, where $\text{span}(X)$ is the vector space spanned by X .

PROOF. We first remark that condition (E2) implies condition (D5), and thus the \mathbb{K} -bialgebra \mathbf{C} is indeed a \mathbb{K} -Hopf algebra. Furthermore, ε and η fulfil the necessary conditions by definition, and \diamond and Δ fulfil

$$\mathbf{C}_i \diamond \mathbf{C}_j \subseteq \mathbf{C}_{i+j}$$

for all $i, j \in \mathbb{N}$ and

$$\Delta(\mathbf{C}_k) \subseteq \bigoplus_{i+j=k} \mathbf{C}_i \otimes \mathbf{C}_j$$

for all $k \in \mathbb{N}$ by condition (E2), leading to the desired grading. \blacksquare

Lemma 4.18: *In a graded bialgebra, the antipode can be computed recursively and is in fact already uniquely determined by the bialgebra structure. Assume that we have for any $n \in \mathbb{N}^\times$ and for any $\gamma \in \mathbf{C}_n$*

$$\Delta(\gamma) = \gamma \otimes \emptyset + \sum_{i=1}^k \alpha_i \otimes \beta_i$$

with $\alpha_i \in \mathbf{C}_{\ell_i}$, $\beta_i \in \mathbf{C}_{n-\ell_i}$, $\ell_i \in \{1, \dots, n\}$ for all $i \in \{1, \dots, k\}$. Then we have

$$S(\gamma) = - \sum_{i=1}^k S(\alpha_i) \diamond \beta_i,$$

with $S(\emptyset) = \emptyset$ as the initial condition.

PROOF. The necessity of $S(\emptyset) = \emptyset$ is clear. Using the fact that $\varepsilon(\emptyset) = 1$ and $\varepsilon(\gamma) = 0$ for all $\gamma \neq \emptyset$ and the property of the antipode, we have the following:

$$\begin{aligned}\mu \circ (S \otimes \mathbf{1}) \circ \Delta(\gamma) &= \eta \circ \varepsilon(\gamma) \\ (\mu \circ (S \otimes \mathbf{1}))(\gamma \otimes \emptyset + \sum_{i=1}^k \alpha_i \otimes \beta_i) &= o \\ \mu(S(\gamma) \otimes \emptyset + \sum_{i=1}^k S(\alpha_i) \otimes \beta_i) &= o \\ S(\gamma) + \sum_{i=1}^k S(\alpha_i) \diamond \beta_i &= o \\ S(\gamma) &= - \sum_{i=1}^k S(\alpha_i) \diamond \beta_i\end{aligned}$$

Thus the antipode must fulfil this recursive definition. \blacksquare

4.6 Monoid as a Special Case

We briefly consider a special case by using a composition law of the form

$$\star: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

which means that two objects compose in exactly one way – the multiset $\gamma_1 \star \gamma_2$ is thus a singleton for all $\gamma_1, \gamma_2 \in \mathcal{C}$. Conditions (C2) and (C3) are then equivalent to the condition that $\langle \mathcal{C}, \star \rangle$ is a monoid.

It is furthermore convenient to consider subclasses $\mathfrak{C} \subseteq \mathcal{C}$ such that all elements of \mathcal{C} can be constructed from \mathfrak{C} by composing a finite number of elements; this means

$$\mathcal{C} = \{\gamma_1 \star \dots \star \gamma_n \mid n \in \mathbb{N}, \gamma_1, \dots, \gamma_n \in \mathfrak{C}\}.$$

(At the very least, the full class \mathcal{C} has this property.) We call \mathfrak{C} a **generating class** if it is the smallest subclass (with respect to inclusion) of \mathcal{C} which has this finite composition property. Such a generating class has the benefit of making the definition of an accompanying decomposition rule such that condition (E1) holds more straightforward: We can define the decomposition rule on \mathfrak{C} in an arbitrary way as a map

$$\langle \cdot \rangle: \mathfrak{C} \longrightarrow \mathbf{MSet}(\mathcal{C} \times \mathcal{C})$$

and then extend it to all of \mathcal{C} by using the equation in condition (E1) to define

$$\langle \gamma_1 \star \dots \star \gamma_n \rangle := \biguplus_{\substack{(\gamma'_1, \gamma''_1) \in \langle \gamma_1 \rangle, \\ \vdots \\ (\gamma'_n, \gamma''_n) \in \langle \gamma_n \rangle}} \{(\gamma'_1 \star \dots \star \gamma'_n) \times (\gamma''_1 \star \dots \star \gamma''_n)\}.$$

This way of introducing the decomposition rule is a simplification in that it reduces the number of objects one has to take into account (for example, if composition preserves size, checking that decomposition preserves size on \mathfrak{C} suffices to prove condition (E2) on all of \mathcal{C}) and in that condition (E1) is fulfilled by construction.

Finally, we can define the decomposition rule in a canonical way in this case by specifying that generating elements $\gamma \in \mathfrak{C}$ decompose primitively:

$$\langle \gamma \rangle := \{(\emptyset, \gamma), (\gamma, \emptyset)\}$$

(This means that all generating elements $\gamma \in \mathfrak{C}$ are primitive elements.) This decomposition rule almost trivially fulfils conditions (D1)–(D5) and thus allows a Hopf algebra structure.

4.7 Examples

We now discuss a few examples which show how the general theory of composition and decomposition can be used in practical applications.

4.7.1 Words

Consider a finite set

$$\mathcal{A} = \{\ell_1, \dots, \ell_n\}$$

of letters, which we call an alphabet. The combinatorial class based on this is the class \mathcal{W} of (finite) words over the alphabet \mathcal{A} , namely

$$\mathcal{W} := \mathcal{A}^* = \{\emptyset, \ell_1, \dots, \ell_n, \ell_1\ell_2, \ell_1\ell_3, \dots, \ell_1\ell_n, \ell_2\ell_1, \dots, \ell_{i_1} \dots \ell_{i_k}, \dots\},$$

where \emptyset is the empty word. The size function on \mathcal{W} is defined as the length of a word (the number of letters):

$$\begin{aligned} |\ell_{i_1} \dots \ell_{i_k}| &:= k \\ |\emptyset| &:= 0 \end{aligned}$$

In the following, we adopt the convention that the sequence of letters indexed by the empty set is identical to the empty word \emptyset , such that the term $\ell_{i_1} \dots \ell_{i_k}$ can also represent \emptyset (considering $k = 0$).

There are now a number of different ways in which algebraic structures can be built on \mathcal{W} , three of which we discuss here.

Example 4.19: The most straightforward composition rule on words is **concatenation**, with the natural definition

$$\ell_{i_1} \dots \ell_{i_m} \star \ell_{j_1} \dots \ell_{j_n} := \ell_{i_1} \dots \ell_{i_m} \ell_{j_1} \dots \ell_{j_n}.$$

With this definition, $\langle \mathcal{W}, \star \rangle$ becomes a monoid with \mathcal{A} as a generating class. As per the special case, we define decomposition on letters (which are generating elements) primitively by

$$\langle \ell_i \rangle := \{(\emptyset, \ell_i), (\ell_i, \emptyset)\}$$

and again extend this definition to all of \mathcal{W} . Decompositions of a word are thus defined by the choice of a subword which constitutes one part of the splitting, with the remaining subword forming the other part of the splitting. This comultiplication is sometimes called **excision** comultiplication. (“Subword” is to be understood in the meaning of subsequence here.) Expressing the decomposition rule on words in equation form leads to

$$\langle \ell_{i_1} \dots \ell_{i_k} \rangle = \bigoplus_{\substack{j_1 < \dots < j_m, \\ j_{m+1} < \dots < j_k}} \{(\ell_{i_{j_1}} \dots \ell_{i_{j_m}}, \ell_{i_{j_{m+1}}} \dots \ell_{i_{j_k}})\},$$

where $\{j_1, \dots, j_k\}$ is a permutation of $\{1, \dots, k\}$.

The composition and decomposition rules which are thereby defined are compatible with size; condition (E2) obviously holds on letters, and on words by extension. If we construct

the Hopf algebra structure as previously described, we have the following maps as a result:

$$\begin{aligned}
\ell_{i_1} \dots \ell_{i_m} \diamond \ell_{j_1} \dots \ell_{j_n} &= \ell_{i_1} \dots \ell_{i_m} \ell_{j_1} \dots \ell_{j_n} \\
\eta(k) &= k\emptyset \\
\Delta(\ell_{i_1} \dots \ell_{i_k}) &= \sum_{\substack{j_1 < \dots < j_m, \\ j_{m+1} < \dots < j_k, \\ \{j_1, \dots, j_k\} = \{1, \dots, k\}}} \ell_{i_{j_1}} \dots \ell_{i_{j_m}} \otimes \ell_{i_{j_{m+1}}} \dots \ell_{i_{j_k}} \\
\varepsilon(\emptyset) &= 1 \\
\varepsilon(\ell_{i_1} \dots \ell_{i_k}) &= 0 \quad \forall k \geq 1 \\
S(\ell_{i_1} \dots \ell_{i_k}) &= (-1)^k \ell_{i_k} \dots \ell_{i_1}
\end{aligned}$$

The bialgebra structure is obvious. For the antipode, consider the following:

$$\begin{aligned}
\langle \ell_{i_1} \ell_{i_2} \rangle &= \{(\emptyset, \ell_{i_1} \ell_{i_2}), (\ell_{i_1}, \ell_{i_2}), (\ell_{i_2}, \ell_{i_1}), (\ell_{i_1} \ell_{i_2}, \emptyset)\} \\
S(\ell_{i_1} \ell_{i_2}) &= -\ell_{i_1} \ell_{i_2} + (\ell_{i_1} \ell_{i_2} + \ell_{i_2} \ell_{i_1}) = \ell_{i_2} \ell_{i_1} \\
\langle \ell_{i_1} \ell_{i_2} \ell_{i_3} \rangle &= \{(\emptyset, \ell_{i_1} \ell_{i_2} \ell_{i_3}), (\ell_{i_1}, \ell_{i_2} \ell_{i_3}), (\ell_{i_2}, \ell_{i_1} \ell_{i_3}), (\ell_{i_3}, \ell_{i_1} \ell_{i_2}), \\
&\quad (\ell_{i_2} \ell_{i_3}, \ell_{i_1}), (\ell_{i_1} \ell_{i_3}, \ell_{i_2}), (\ell_{i_1} \ell_{i_2}, \ell_{i_3}), (\ell_{i_1} \ell_{i_2} \ell_{i_3}, \emptyset)\} \\
S(\ell_{i_1} \ell_{i_2} \ell_{i_3}) &= -\ell_{i_1} \ell_{i_2} \ell_{i_3} + (\ell_{i_1} \ell_{i_2} \ell_{i_3} + \ell_{i_2} \ell_{i_1} \ell_{i_3} + \ell_{i_3} \ell_{i_1} \ell_{i_2} + \ell_{i_2} \ell_{i_3} \ell_{i_1} + \ell_{i_1} \ell_{i_3} \ell_{i_2} + \ell_{i_1} \ell_{i_2} \ell_{i_3}) \\
&\quad - (\ell_{i_1} \ell_{i_2} \ell_{i_3} + \ell_{i_1} \ell_{i_3} \ell_{i_2} + \ell_{i_2} \ell_{i_1} \ell_{i_3} + \ell_{i_2} \ell_{i_3} \ell_{i_1} + \ell_{i_3} \ell_{i_1} \ell_{i_2} + \ell_{i_3} \ell_{i_2} \ell_{i_1}) = -\ell_{i_3} \ell_{i_2} \ell_{i_1}
\end{aligned}$$

An inductive reasoning gives the above definition of the antipode.

These maps make \mathcal{W} into a graded and cocommutative Hopf algebra, the **free algebra**.

If \mathcal{A} consists of more than one letter, then \mathcal{W} is noncommutative.

As a special case, consider an alphabet which is a singleton, $\mathcal{A} = \{x\}$; the words then take the form

$$\mathcal{P} = \{\emptyset, x, x^2, x^3, \dots\}$$

and the algebraic structure is the algebra of polynomials in one variable

$$\mathbb{K}[x] = \left\{ \sum_{i=0}^n a_i x^i \mid n \in \mathbb{N}, a_i \in \mathbb{K} \right\}.$$

In this case, the maps take the following form:

$$\begin{aligned}
x^i \diamond x^j &= x^{i+j} \\
\eta(k) &= k\emptyset \\
\Delta(x^n) &= \sum_{i=0}^n \binom{n}{i} x^i \otimes x^{n-i} \\
\varepsilon(x^n) &= \delta_{0n} \\
S(x^n) &= (-1)^n x^n
\end{aligned}$$

//

Example 4.20: As a variation of the previous example, let the alphabet $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ carry a linear order $\ell_1 < \dots < \ell_n$. We now consider only words whose letters are ordered, defining a combinatorial class

$$\mathcal{S} := \{\ell_{i_1} \dots \ell_{i_k} \mid 1 \leq i_1 \leq \dots \leq i_k \leq n\}.$$

Concatenation alone is not a viable composition rule for \mathcal{S} , as it does not preserve order; instead, we have to reorder the letters, yielding a composition rule

$$\ell_{i_1} \dots \ell_{i_m} \star \ell_{i_{m+1}} \dots \ell_{i_{m+k}} := \ell_{i_{\sigma(1)}} \dots \ell_{i_{\sigma(m+k)}}$$

with σ being an arbitrary permutation of $\{1, \dots, m+k\}$ such that

$$i_{\sigma(1)} \leq \dots \leq i_{\sigma(m+k)}.$$

Again, $\langle \mathcal{S}, \star \rangle$ is a monoid with \mathcal{A} as a generating class. Once again defining the decomposition rule on the letters as

$$\langle \ell_i \rangle := \{(\emptyset, \ell_i), (\ell_i, \emptyset)\}$$

by extension leads to the decomposition rule

$$\langle \ell_{i_1} \dots \ell_{i_k} \rangle = \biguplus_{\substack{j_1 < \dots < j_m, \\ j_{m+1} < \dots < j_k}} \{(\ell_{i_{j_1}} \dots \ell_{i_{j_m}}, \ell_{i_{j_{m+1}}} \dots \ell_{i_{j_k}})\},$$

where $\{j_1, \dots, j_k\}$ again is a permutation of $\{1, \dots, k\}$. This decomposition rule is exactly the same as in the previous example. Note, however, that in this example the rule guarantees that the resulting words are ordered.

Analogously deriving the Hopf algebra structure on \mathcal{S} gives us the following maps:

$$\begin{aligned} \ell_1^{i_1} \dots \ell_n^{i_n} \diamond \ell_1^{j_1} \dots \ell_n^{j_n} &= \ell_1^{i_1+j_1} \dots \ell_n^{i_n+j_n} \\ \eta(k) &= k\emptyset \\ \Delta(\ell_{i_1} \dots \ell_{i_k}) &= \sum_{\substack{j_1 < \dots < j_m, \\ j_{m+1} < \dots < j_k, \\ \{j_1, \dots, j_k\} = \{1, \dots, k\}}} \ell_{i_{j_1}} \dots \ell_{i_{j_m}} \otimes \ell_{i_{j_{m+1}}} \dots \ell_{i_{j_k}} \\ \varepsilon(\emptyset) &= 1 \\ \varepsilon(\ell_{i_1} \dots \ell_{i_k}) &= 0 \quad \forall k \geq 1 \\ S(\ell_{i_1} \dots \ell_{i_k}) &= (-1)^k \ell_{i_1} \dots \ell_{i_k} \end{aligned}$$

The antipode equality can be proved analogously to the previous example. Note that the product of any splitting of the word $\ell_{i_1} \dots \ell_{i_k}$ inevitably results in the same word (as the letters are sorted in the multiplication steps). The computation of the antipode thus comes down to a combinatorial calculation of the different ways to split, which can again be proved inductively.

The thus defined **symmetric algebra** is a graded, commutative and cocommutative Hopf algebra. We have used the notation

$$\ell_1^{i_1} \dots \ell_n^{i_n} := \underbrace{\ell_1 \dots \ell_1}_{i_1 \text{ times}} \dots \underbrace{\ell_n \dots \ell_n}_{i_n \text{ times}},$$

which gives rise to the observation that the symmetric algebra \mathcal{S} is isomorphic to the algebra of polynomials in n commuting variables $\mathbb{K}[x_1, \dots, x_n]$. //

Example 4.21: Let the alphabet be unordered again. For this final example, we define the composition rule as any **shuffle** which mixes letters of the two words to be composed while preserving their relative order. In equation form, this leads to

$$l_{i_1} \dots l_{i_m} \star l_{i_{m+1}} \dots l_{i_{m+k}} := \bigoplus_{\substack{\sigma(1) < \dots < \sigma(m), \\ \sigma(m+1) < \dots < \sigma(m+k)}} \{(l_{i_{\sigma(1)}} \dots l_{i_{\sigma(m)}}, l_{i_{\sigma(m+1)}} \dots l_{i_{\sigma(m+k)}})\},$$

where the index set of the multiset sum on the right-hand side runs over all permutations σ of $\{1, \dots, m+k\}$ which preserve the relative order of $\{1, \dots, m\}$ and $\{m+1, \dots, m+k\}$, respectively. This composition rule is obviously nonmonoidal.

One compatible decomposition rule is cutting a words into two parts and exchanging the order of the parts, as in

$$\langle l_{i_1} \dots l_{i_k} \rangle = \bigoplus_{j=0}^k \{(l_{i_{j+1}} \dots l_{i_k}, l_{i_1} \dots l_{i_j})\}.$$

These definitions give us the following maps for the Hopf algebra structure:

$$\begin{aligned} l_{i_1} \dots l_{i_m} \diamond l_{i_{m+1}} \dots l_{i_{m+k}} &= \sum_{\substack{\sigma(1) < \dots < \sigma(m), \\ \sigma(m+1) < \dots < \sigma(m+k)}} l_{i_{\sigma(1)}} \dots l_{i_{\sigma(m)}} l_{i_{\sigma(m+1)}} \dots l_{i_{\sigma(m+k)}} \\ \eta(k) &= k\emptyset \\ \Delta(l_{i_1} \dots l_{i_k}) &= \sum_{j=0}^k l_{i_{j+1}} \dots l_{i_k} \otimes l_{i_1} \dots l_{i_j} \\ \varepsilon(\emptyset) &= 1 \\ \varepsilon(l_{i_1} \dots l_{i_k}) &= 0 \quad \forall k \geq 1 \\ S(l_{i_1} \dots l_{i_k}) &= (-1)^k l_{i_k} \dots l_{i_1} \end{aligned}$$

The antipode's simple structure in this case is a result of the shuffle product's properties. The **shuffle algebra** which we have constructed is a commutative Hopf algebra which is not cocommutative. (Had we not exchanged the order of the two parts in the definition of the decomposition rule, the Hopf algebra would also be cocommutative.) //

[Cro06] offers a complete treatment of possible Hopf algebra structures on the combinatorial class of words formed by combining any of three multiplication and three comultiplication maps (though only four of the nine possible combinations actually yield Hopf algebras).

4.7.2 Graphs

The following two examples come from a different field of combinatorics, namely graph theory.

We consider the class \mathcal{G} of unlabelled undirected graphs without isolated vertices (representing such a graph as a collection of vertices V and the edges E by which they are connected). For the following, we represent a graph as a map

$$\Gamma: E \longrightarrow V^{(2)}$$

determining which vertices the edges are attached to (with $V^{(2)}$ representing the set of unordered pairs of vertices). As the size of a graph we use the number of its edges $|\Gamma| := |E|$.

Example 4.22: Let us now consider a simple composition rule: Given two graphs Γ_1, Γ_2 , we define the composition as the disjoint union of the graphs

$$\Gamma_1 \star \Gamma_2 := \Gamma_1 \cup \Gamma_2,$$

which means that we simply put the two graphs Γ_1 and Γ_2 side by side. We formally represent the union graph by a map

$$\Gamma_1 \cup \Gamma_2: E_1 \cup E_2 \longrightarrow V_1^{(2)} \cup V_2^{(2)}$$

which fulfils

$$\begin{aligned} (\Gamma_1 \cup \Gamma_2)|_{E_1} &= \Gamma_1 \\ (\Gamma_1 \cup \Gamma_2)|_{E_2} &= \Gamma_2. \end{aligned}$$

This composition rule fulfils conditions (C1)–(C4). (The unit element is the void graph \emptyset .)

Consider an arbitrary edge subset $L \subseteq E$ for a graph $\Gamma \in \mathcal{G}$. L induces a subgraph

$$\Gamma|_L: L \longrightarrow V^{(2)}$$

by restricting Γ to L and all the vertices in the image $\Gamma(L)$. On the other hand, the remainder of the edges $R := E \setminus L$ also defines a subgraph $\Gamma|_R$. If we now consider all ordered partitions of the edges E into disjoint subsets L, R (which means $L \cup R = E$), we gain pairs of edge-wise disjoint subgraphs $(\Gamma|_L, \Gamma|_R)$. This allows the following decomposition rule:

$$\langle \Gamma \rangle := \bigsqcup_{L \cup R = E} \{(\Gamma|_L, \Gamma|_R)\}$$

The decomposition rule fulfils conditions (D1)–(D5), and together with the composition rule, conditions (E1)–(E2) also hold. We thus arrive at the following Hopf algebra structure:

$$\begin{aligned} \Gamma_1 \diamond \Gamma_2 &= \Gamma_1 \cup \Gamma_2 \\ \eta(k) &= k\emptyset \\ \Delta(\Gamma) &= \sum_{L \cup R = E} \Gamma_L \otimes \Gamma_R \\ \varepsilon(\emptyset) &= 1 \\ \varepsilon(\Gamma) &= 0 \quad \forall \Gamma \neq \emptyset \\ S(\Gamma) &= \sum_{\substack{L_1 \cup \dots \cup L_n = E, \\ L_1 \neq \emptyset, \dots, L_n \neq \emptyset}} (-1)^n \Gamma|_{L_1} \cup \dots \cup \Gamma|_{L_n} \end{aligned}$$

This **algebra of graphs** is a graded Hopf algebra which is both commutative and co-commutative. //

A special kind of graph is the **rooted tree**, which is an undirected simple graph, connected and acyclic, with one distinguished vertex, the root. Let \mathcal{T} denote the combinatorial class of rooted trees. A **rooted forest** is, naturally, a collection of rooted trees; the combinatorial class of rooted forests \mathcal{F} is thus given via the multiset sum by

$$\mathcal{F} := \mathbf{MSet}(\mathcal{T}).$$

This time, we use the number of vertices for the size of a rooted tree or a rooted forest.

Example 4.23: As the composition rule on rooted forests, we again use the disjoint union of graphs

$$\Gamma_1 \star \Gamma_2 := \Gamma_1 \cup \Gamma_2;$$

we remark that $\langle \mathcal{F}, \star \rangle$ is a commutative monoid with the rooted trees \mathcal{T} as a generating class.

For the decomposition rule, we first introduce the concept of proper subtrees. Given a rooted tree $\tau \in \mathcal{T}$, a **proper subtree** is a subtree τ^p which has the same root as τ ; the empty rooted tree \emptyset is also considered to be a proper subtree. If we now have a rooted tree τ and choose a proper subtree τ^p , we can take all vertices of $\tau \setminus \tau^p$ and the edges that connect them and consider them a rooted forest τ^c .

A graphic interpretation of this process is to “cut off” branches of the rooted tree τ to arrive at τ^p ; the “branches” that have been “cut off” then form the rooted forest τ^c .

We now define decomposition on rooted trees as a splitting of any rooted tree τ into the pair (τ^p, τ^c) consisting of a proper subtree τ^p and the remaining rooted forest τ^c . Formulated as an equation, we have

$$\langle \tau \rangle := \bigsqcup_{\substack{\tau^p \subseteq \tau, \\ \tau^p \text{ proper}}} \{(\tau^p, \tau^c)\},$$

where τ^c is the rooted forest which “complements” τ^p to form τ . Clearly, this decomposition rule is not cocommutative. As the rooted trees \mathcal{T} generate the rooted forests \mathcal{F} , we again can extend the decomposition rule to any rooted forest $\Gamma = \tau_1 \cup \dots \cup \tau_n \in \mathcal{F}$ by

$$\langle \Gamma \rangle = \bigsqcup_{\substack{\tau_1^p \subseteq \tau_1, \dots, \tau_n^p \subseteq \tau_n, \\ \tau_1^p, \dots, \tau_n^p \text{ proper}}} \{(\tau_1^p \cup \dots \cup \tau_n^p, \tau_1^c \cup \dots \cup \tau_n^c)\},$$

which can also be interpreted as “trimming branches” off the whole rooted forest Γ and collecting them in the second component $\Gamma^c := \tau_1^c \cup \dots \cup \tau_n^c$ while keeping the rooted trees which remain in the first component $\Gamma^p := \tau_1^p \cup \dots \cup \tau_n^p$. Using this notation, we can also write

$$\langle \Gamma \rangle = \bigsqcup_{\substack{\Gamma^p \subseteq \Gamma, \\ \Gamma^p \text{ proper}}} \{(\Gamma^p, \Gamma^c)\}$$

for the decomposition rule.

Composition and decomposition rules fulfil the necessary conditions; using the usual construction, this results in the following Hopf algebra:

$$\begin{aligned}
\Gamma_1 \diamond \Gamma_2 &= \Gamma_1 \cup \Gamma_2 \\
\eta(k) &= k\emptyset \\
\Delta(\Gamma) &= \sum_{\substack{\Gamma^p \subseteq \Gamma, \\ \Gamma^p \text{ proper}}} \Gamma^p \otimes \Gamma^c \\
\varepsilon(\emptyset) &= 1 \\
\varepsilon(\Gamma) &= 0 \quad \forall \Gamma \neq \emptyset \\
S(\emptyset) &= \emptyset \\
S(\Gamma) &= - \sum_{\substack{\Gamma^p \subseteq \Gamma, \\ \Gamma^p \text{ proper}}} S(\Gamma^p) \cup \Gamma^c
\end{aligned}$$

In this case, we have used Lemma 4.18 for the definition of the antipode.

The result is the **algebra of rooted forests**, a graded Hopf algebra which is commutative but not cocommutative. //

Chapter 5

Enumeration and Discrete Methods

[JR82] discuss a wide variety of occurrences of coalgebras and bialgebras in discrete methods and enumeration problems, building on a general framework of incidence coalgebras and reduced incidence coalgebras, akin to the concept of incidence algebras. We consider some of their results in this chapter.

5.1 Definitions

Definition 5.1: An **equivalence relation** \sim on a set M is a relation (a subset of $M \times M$) which is

- reflexive ($m \sim m$ for all $m \in M$),
- symmetric ($m \sim n$ implies $n \sim m$) and
- transitive ($m \sim n \sim o$ implies $m \sim o$).

A **partial ordering relation** \leq on a set P is a relation which is

- reflexive,
- antisymmetric ($p \leq q$ and $q \leq p$ imply $p = q$) and
- transitive.

A set P with a partial ordering relation \leq is a **partially ordered set** or **poset**. If we have $p \leq q$ in P , then we define the **segment** or **interval**

$$[p, q] := \{x \in P \mid p \leq x \leq q\}.$$

We denote the set of all segments of P by $\text{Seg}(P)$. A poset $\langle P, \leq \rangle$ is called **locally finite** if all segments $[p, q]$ of P are finite.

A **lattice** is a poset in which the minimum and maximum (or meet and join, denoted by \vee and \wedge , respectively) of two elements are always defined. ∇

We require one further algebraic concept.

Definition 5.2: Let $\langle C, \Delta, \varepsilon \rangle$ be a \mathbb{K} -coalgebra and $D \trianglelefteq C$ a subspace of C . D is called a **subcoalgebra** if $\Delta(D) \subseteq D \otimes D$, and a **coideal** if

$$\begin{aligned}\Delta(D) &\subseteq D \otimes C + C \otimes D \\ \varepsilon(D) &= \{0\}.\end{aligned}$$

If \sim is an equivalence relation on a basis $\{c_i \mid i \in I\}$ of C such that the subspace J spanned by

$$\{c - c' \mid c \sim c'\}$$

is a coideal, then the quotient space C/\sim can be given a \mathbb{K} -coalgebra structure by using the structure maps of C on the classes of C/\sim . We call the resulting \mathbb{K} -coalgebra the **quotient coalgebra** C modulo J . ▽

5.2 Section Coefficients

We first require the concept of section coefficients, a natural generalization of binomial coefficients. For this, let I be a set; then the **section coefficients** $(i \mid j, k)$ of I are a way of specifying and counting the number of possible “sections” (cuttings) of an element i into an ordered pair of pieces (j, k) (with $j, k \in I$). **Multisection coefficients** $(i \mid j, k, \ell)$ similarly count the number of “sections” of i into an ordered triple of pieces (j, k, ℓ) .

There are two different ways of arriving at $(i \mid j, k, \ell)$:

- cut i into two pieces (x, ℓ) and then cut x into two pieces (j, k) or
- cut i into two pieces (j, y) and then cut y into two pieces (k, ℓ) .

A natural restriction thus is to demand that the result be the same; one also expects the number of different sections (j, k) of an element i to be finite. These considerations give rise to the following definition:

Definition 5.3: Let I be a set. **Section coefficients** are a map

$$(\cdot \mid \cdot, \cdot): I \times I \times I \longrightarrow \mathbb{Z}: (i, j, k) \longmapsto (i \mid j, k)$$

which fulfils

$$\#\{(j, k) \in I \times I \mid (i \mid j, k) \neq 0\} < \infty$$

for all $i \in I$ (finiteness) as well as

$$(i \mid j, k, \ell) := \sum_x (i \mid x, \ell)(x \mid j, k) = \sum_y (i \mid j, y)(y \mid k, \ell)$$

for all $i, j, k, \ell \in I$ (a kind of coassociativity). This definition immediately allows extension to more general multisection coefficients of the form $(i \mid j_1, j_2, \dots, j_n)$. ∇

We require in the following that there also exist a function $\varepsilon: I \longrightarrow \mathbb{K}$ satisfying

$$\begin{aligned} \sum_j (i \mid j, k) \varepsilon(j) &= \delta_{ik} \\ \sum_k (i \mid j, k) \varepsilon(k) &= \delta_{ij}. \end{aligned}$$

Definition 5.4: Let $\langle I, + \rangle$ be a commutative semigroup. The section coefficients are called **bisection coefficients** if they fulfil

$$(i + j \mid p, q) = \sum_{\substack{p_1 + p_2 = p, \\ q_1 + q_2 = q}} (i \mid p_1, q_1)(j \mid p_2, q_2).$$

This means that “cutting” $i + j$ into two pieces (p, q) is the same as separately cutting i and j into two pieces (p_1, q_1) and (p_2, q_2) , respectively, and combining the results. ∇

Remark 5.5: We note the obvious similarities to the previous chapter on composition and decomposition rules.

Example 5.6: The **binomial coefficients** are section coefficients on $\langle \mathbb{N}, + \rangle$, defined by

$$(n \mid j, k) := \begin{cases} \frac{n!}{j!k!} & \text{if } j + k = n, \\ 0 & \text{otherwise.} \end{cases}$$

These well-known coefficients, usually denoted by $\binom{n}{j}$, count the ways a set with n elements can be divided into two disjoint sets of sizes j and $k = n - j$. The binomial coefficients fulfil the finiteness condition, as

$$\#\{(j, k) \in \mathbb{N} \times \mathbb{N} \mid (n \mid j, k) \neq 0\} = \#\{(j, k) \in \mathbb{N} \times \mathbb{N} \mid j + k = n\} = \#\{0, \dots, n\} = n + 1.$$

The coassociativity condition is also satisfied, since for $j + k + \ell = n$ we have

$$(n \mid j, k, \ell) = \frac{n!}{j!k!\ell!} = \frac{n!}{j!(k + \ell)!} \frac{(k + \ell)!}{k!\ell!} = \frac{n!}{(j + k)!\ell!} \frac{(j + k)!}{j!k!}.$$

The **Vandermonde identity**

$$(i + j \mid p, q) = \binom{i + j}{p} = \sum_{p_1 + p_2 = p} \binom{i}{p_1} \binom{j}{p_2} = \sum_{\substack{p_1 + p_2 = p, \\ q_1 + q_2 = q}} (i \mid p_1, q_1)(j \mid p_2, q_2)$$

finally shows that the binomial coefficients are bisection coefficients. \parallel

Lemma 5.7: Assume we have section coefficients and a function $\varepsilon: I \longrightarrow \mathbb{K}$ as above; assume further that there is an element $0 \in I$ which fulfils

$$(i \mid 0, j) = (i \mid j, 0) = \delta_{ij}.$$

Then we can derive a \mathbb{K} -coalgebra C as follows:

- For each $i \in I$, define a variable x_i .
- Define C as the free \mathbb{K} -vector space with basis $\{x_i \mid i \in I\}$.
- Define Δ on the basis as

$$\Delta(x_i) := \sum_{j,k \in I} (i \mid j, k) x_j \otimes x_k.$$

- Define ε on the basis as

$$\varepsilon(x_i) := \delta_{i0}.$$

$\langle C, \Delta, \varepsilon \rangle$ is thus a \mathbb{K} -coalgebra, with the coassociativity assured by the coassociativity condition on the section coefficients. C is cocommutative if and only if

$$(i \mid j, k) = (i \mid k, j)$$

for all $i, j, k \in I$.

Lemma 5.8: *With the same assumptions as above, additionally let the section coefficients be bisection coefficients. Then C allows a \mathbb{K} -bialgebra structure.*

PROOF. Define the product on the basis as $\mu(x_i, x_j) = x_{i+j}$. Then we have (again using the notation $\bar{\mu} = (\mu \otimes \mu) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1})$, as in Example 3.25)

$$\begin{aligned} \bar{\mu}(\Delta(x_i), \Delta(x_j)) &= \bar{\mu} \left(\sum_{p_1, q_1} (i \mid p_1, q_1) x_{p_1} \otimes x_{q_1}, \sum_{p_2, q_2} (j \mid p_2, q_2) x_{p_2} \otimes x_{q_2} \right) \\ &= \sum_{p, q} \sum_{\substack{p_1+p_2=p, \\ q_1+q_2=q}} (i \mid p_1, q_1) (j \mid p_2, q_2) x_{p_1+p_2} \otimes x_{q_1+q_2} \\ &= \sum_{p, q} (i+j \mid p, q) x_p \otimes x_q = \Delta(x_{i+j}) = \Delta(x_i x_j). \end{aligned}$$

With x_0 as the unit element, it is easy to see that Δ and ε are \mathbb{K} -algebra morphisms and that $\langle C, \mu, x_0, \Delta, \varepsilon \rangle$ is a \mathbb{K} -bialgebra. ■

5.3 Incidence Coalgebras for Posets

In the following, we assume that $\text{char}(\mathbb{K}) = 0$.

Definition 5.9: Let $\langle P, \leq \rangle$ be a locally finite poset. Consider the set of functions

$$\mathcal{F}(P) := \{f: P \times P \longrightarrow \mathbb{K} \mid f(p, q) = 0 \text{ for all } p, q \text{ with } p \not\leq q\};$$

we define the sum of two functions f, g and the scalar multiple kf of a function f (with $k \in \mathbb{K}$) as usual. In addition, we define a convolution product as follows:

$$(f * g)(p, q) := \sum_{x \in P} f(p, x) g(x, q)$$

It is clear that the summands in the definition are nonzero only for $x \in [p, q]$. As the poset P is locally finite, the sum is also finite. Furthermore, the associativity of the product thus defined is evident. The neutral element with respect to the product is

$$\delta(p, q) := \delta_{pq} = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

We call the resulting \mathbb{K} -algebra $\langle \mathcal{A}(P), *, \delta \rangle$ the **incidence algebra** of P over \mathbb{K} . ∇

Remark 5.10: We will not study incidence algebras further; they are defined here only to show the analogy to incidence coalgebras, which we will discuss in the following.

Definition 5.11: Let $\langle P, \leq \rangle$ be a locally finite poset. The **incidence coalgebra** $\mathcal{C}(P)$ of P over \mathbb{K} is defined as the free \mathbb{K} -vector space spanned by the indeterminates

$$\{[p, q] \mid p, q \in P, p \leq q\}.$$

This means that the set of indeterminates is $\text{Seg}(P)$. The structure maps are defined by

$$\begin{aligned} \Delta([p, q]) &:= \sum_{p \leq x \leq q} [p, x] \otimes [x, q] \\ \varepsilon([p, q]) &:= \delta_{pq} \end{aligned}$$

for all $p, q \in P$. Deriving the section coefficients from this definition, we have

$$([p, q] \mid [x, y], [z, w]) = \delta_{px} \delta_{yz} \delta_{qw},$$

that is, the section coefficients are 1 if and only if they are of the form

$$([p, q] \mid [p, x], [x, q])$$

and 0 otherwise. It is clear from the definition that Δ is coassociative and it is straightforward to see that $\langle \mathcal{C}(P), \Delta, \varepsilon \rangle$ is indeed a \mathbb{K} -coalgebra. ∇

Remark 5.12: $\mathcal{C}(P)$ is cocommutative if and only if the partial order \leq is trivial, that is, if

$$p \leq q \iff p = q.$$

For our applications, the full incidence coalgebras are not required; instead, a quotient coalgebra of $\mathcal{C}(P)$ often suffices (as is frequently the case for enumeration problems). These are derived by using admissible equivalence relations on $\text{Seg}(P)$.

Definition 5.13: An equivalence relation \sim on $\text{Seg}(P)$ is called **order compatible** if the subspace J spanned by

$$\{[p, q] - [r, s] \mid [p, q] \sim [r, s]\}$$

is a coideal. ▽

Definition 5.14: Let $\langle P, \leq \rangle$ be a locally finite poset and \sim an order compatible equivalence relation on $\text{Seg}(P)$. Then the quotient coalgebra of $\mathcal{C}(P)$ isomorphic to the quotient space $\mathcal{C}(P)/\sim$ is called **reduced incidence coalgebra**.

The nonempty equivalence classes of $\mathcal{C}(P)/\sim$ are called **types** (denoted by $\text{type}([x, y])$) and the set comprising them is denoted by $\text{Typ}(P)$. ▽

The reduced incidence coalgebra can be viewed as the vector space spanned by the variables

$$\{x_\alpha \mid \alpha \in \text{Typ}(P)\}.$$

Considering such a reduced incidence coalgebra, we derive a collection of section coefficients $(\alpha \mid \beta, \gamma)$ via

$$(\alpha \mid \beta, \gamma) := \#\{x \in P \mid p, q \in P, x \in [p, q], \text{type}([p, q]) = \alpha, \text{type}([p, x]) = \beta, \text{type}([x, q]) = \gamma\}$$

by adding up the section coefficients in the full incidence coalgebra according to the types of intervals. The comultiplication map then is given by

$$\Delta(x_\alpha) = \sum_{\beta, \gamma \in \text{Typ}(P)} (\alpha \mid \beta, \gamma) x_\beta \otimes x_\gamma.$$

5.4 Standard Reduced Incidence Coalgebras

Definition 5.15: Let $\langle P, \leq \rangle$ be a locally finite poset and define \cong by

$$[p, q] \cong [r, s] \iff [p, q] \text{ is isomorphic to } [r, s].$$

We remark (without proof) that \cong is order compatible. The reduced incidence coalgebra of $\mathcal{C}(P)$ by \cong is called the **standard reduced incidence coalgebra**. ▽

We now consider a few examples of standard reduced incidence coalgebras.

Example 5.16: The divided power coalgebra from Example 2.5 c) is isomorphic to a standard reduced incidence coalgebra.

Consider the lattice \mathbb{N} of natural numbers with natural ordering. The incidence coalgebra $\mathcal{C}(\mathbb{N})$ is spanned by all segments $[k, \ell]$ with the comultiplication

$$\Delta([k, \ell]) := \sum_{k \leq i \leq \ell} [k, i] \otimes [i, \ell].$$

The divided power coalgebra then is the standard reduced incidence coalgebra of $\mathcal{C}(\mathbb{N})$, as the types of these segments are determined by their size, yielding

$$\Delta(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}.$$

//

Example 5.17: Let \mathbb{N}^\times be the lattice of positive integers ordered by divisibility, that is,

$$m \leq n \iff m \mid n \iff \frac{n}{m} \in \mathbb{N}^\times.$$

The **Dirichlet coalgebra** \mathcal{D} is given by using

$$[k, \ell] \sim [m, n] \iff \frac{\ell}{k} = \frac{n}{m}$$

as the order compatible equivalence relation on the segments. Alternatively, the Dirichlet coalgebra can be defined as the \mathbb{K} -vector space spanned by the variable n^x , $n \in \mathbb{N}^\times$, with structure maps defined by

$$\begin{aligned} \Delta(n^x) &:= \sum_{pq=n} p^x \otimes q^x \\ \varepsilon(n^x) &:= \delta_{1n} \end{aligned}$$

for all $n \in \mathbb{N}^\times$.

Using the natural algebraic structure given by

$$\mu(m^x, n^x) = (mn)^x,$$

it is clear that \mathcal{D} is not a bialgebra: Consider

$$\begin{aligned} \Delta(2^x) &= 1^x \otimes 2^x + 2^x \otimes 1^x \\ \Delta(4^x) &= 1^x \otimes 4^x + 2 \cdot 2^x \otimes 2^x + 4^x \otimes 1^x \\ \Delta(8^x) &= 1^x \otimes 8^x + 2^x \otimes 4^x + 4^x \otimes 2^x + 8^x \otimes 1^x, \end{aligned}$$

therefore

$$\begin{aligned} \mu(\Delta(2^x), \Delta(4^x)) &= \mu(1^x \otimes 2^x + 2^x \otimes 1^x, 1^x \otimes 4^x + 2 \cdot 2^x \otimes 2^x + 4^x \otimes 1^x) \\ &= (1^x \otimes 8^x + 2 \cdot 2^x \otimes 4^x + 4^x \otimes 2^x) + (2^x \otimes 4^x + 2 \cdot 4^x \otimes 2^x + 8^x \otimes 1^x) \\ &= 1^x \otimes 8^x + 3 \cdot 2^x \otimes 4^x + 3 \cdot 4^x \otimes 2^x + 8^x \otimes 1^x, \end{aligned}$$

which contradicts

$$\Delta(\mu(m^x, n^x)) = \mu(\Delta(m^x), \Delta(n^x)).$$

However, the comultiplication is an algebra map as long as m, n are coprime, that is,

$$\Delta(\mu(m^x, n^x)) = \mu(\Delta(m^x), \Delta(n^x))$$

as long as $\gcd(m, n) = 1$.

The standard reduced incidence coalgebra in this case is a subcoalgebra of the Dirichlet coalgebra \mathcal{D} . Let $[k, \ell], [m, n]$ be two segments and let

$$\begin{aligned}\frac{\ell}{k} &= p_1^{\alpha_1} \cdot \dots \cdot p_r^{\alpha_r} \\ \frac{n}{m} &= q_1^{\beta_1} \cdot \dots \cdot q_s^{\beta_s}\end{aligned}$$

be the respective prime factorizations. The segments $[k, \ell]$ and $[m, n]$ are isomorphic if and only if the multisets $\{\alpha_i \mid i = 1, \dots, r\}$ and $\{\beta_j \mid j = 1, \dots, s\}$ are identical. This means that $[k, \ell] \cong [m, n]$ if and only if there is a multiplicative bijective map π on the prime numbers \mathbb{P} such that

$$\pi\left(\frac{\ell}{k}\right) = \pi\left(\frac{n}{m}\right).$$

Alternatively, for

$$n = \prod_{p_i \in \mathbb{P}} p_i^{\gamma_i}$$

define

$$\begin{aligned}\lambda_k &:= \#\{p_i \in \mathbb{P} \mid \gamma_i = k\} \\ \text{shape}(n) &:= (\lambda_1, \lambda_2, \dots),\end{aligned}$$

that is, λ_k is the number of distinct primes in the prime factorization of n whose exponent is exactly k . Then we have

$$[k, \ell] \cong [m, n] \iff \text{shape}\left(\frac{\ell}{k}\right) = \text{shape}\left(\frac{n}{m}\right).$$

//

Example 5.18: Consider a vector space of countable dimension over $\text{GF}(q)$, and let V be the lattice of all finite-dimensional subspaces, ordered by inclusion. The standard reduced incidence coalgebra of $\mathcal{C}(V)$ is gained via

$$[R, S] \cong [T, U] \iff \dim S - \dim R = \dim U - \dim T.$$

The resulting coalgebra is the **Eulerian coalgebra** \mathcal{E} . Its section coefficients count the number of subspaces of dimension k contained in a subspace of dimension n and are given by the **Gaussian coefficients**

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

where

$$[n]_q! := (1-q)(1-q^2) \cdot \dots \cdot (1-q^n).$$

The Eulerian coalgebra thus is the vector space $\mathbb{K}[x]$ with structure maps defined by

$$\begin{aligned}\Delta(x^n) &:= \sum_{k=0}^n \binom{n}{k}_q x^k \otimes x^{n-k} \\ \varepsilon(x^n) &:= \delta_{0n}\end{aligned}$$

for all $n \in \mathbb{N}$. It is clear from the definition that \mathcal{E} is cocommutative. //

5.5 Boolean and Binomial Coalgebras

Definition 5.19: The **Boolean poset** \mathbb{B} is the set of all finite sets of positive integers, ordered by inclusion. The minimum element in this set is the empty set \emptyset . By taking the incidence coalgebra spanned by all segments $[M, N]$ with comultiplication defined by

$$\Delta([M, N]) := \sum_{M \subseteq X \subseteq N} [M, X] \otimes [X, N],$$

we get the **Boolean incidence coalgebra** $\mathcal{C}(B)$. ▽

Definition 5.20: Consider the \mathbb{K} -vector space spanned by all finite sets of positive integers with structure maps

$$\begin{aligned}\Delta(A) &:= \sum_{A_1 \sqcup A_2 = A} A_1 \otimes A_2 \\ \varepsilon(A) &:= \delta_{A\emptyset}\end{aligned}$$

for all $A \subseteq \mathbb{N}^*$, $\#A < \infty$. This \mathbb{K} -vector space is a \mathbb{K} -coalgebra, the **Boolean coalgebra** \mathcal{B} . This coalgebra is isomorphic to the reduced Boolean incidence coalgebra obtained by using the order compatible equivalence relation defined by

$$[K, L] \sim [M, N] \iff L \setminus K = N \setminus M.$$

Therefore, each set $A \in \mathcal{B}$ represents the equivalence class of all segments $[X, Y]$ such that $Y \setminus X = A$. ▽

Definition 5.21: Let s be a positive integer. The \mathbb{K} -vector space $\mathbb{K}[x_1, \dots, x_s]$ with structure maps

$$\begin{aligned}\Delta(x_1^{n_1} \cdots x_s^{n_s}) &:= \sum_{i_1 \leq n_1, \dots, i_s \leq n_s} \binom{n_1}{i_1} \cdots \binom{n_s}{i_s} x_1^{i_1} \cdots x_s^{i_s} \otimes x_1^{n_1-i_1} \cdots x_s^{n_s-i_s} \\ \varepsilon(x_1^{n_1} \cdots x_s^{n_s}) &:= \delta_{0n_1} \cdots \delta_{0n_s}\end{aligned}$$

for all $n_1, \dots, n_s \in \mathbb{N}$ is a coalgebra, the **binomial coalgebra** \mathcal{B}_s . \mathcal{B}_s is cocommutative by definition. ▽

Lemma 5.22: *The binomial coalgebras are isomorphic to reduced Boolean incidence coalgebras by suitable order compatible equivalence relations \sim .*

PROOF. We first consider $s = 1$. $\mathcal{B}_1 = \mathbb{K}[x]$ is gained by setting

$$[K, L] \sim_1 [M, N] \iff \#(L \setminus K) = \#(N \setminus M).$$

In fact, this is the standard reduced incidence coalgebra. (The section coefficients in this case are $\binom{n}{k}$.)

For $s = 2$, we set

$$[K, L] \sim_2 [M, N] \iff \begin{cases} \#\{i \in L \setminus K \mid i \text{ odd}\} = \#\{j \in N \setminus M \mid j \text{ odd}\} \\ \#\{i \in L \setminus K \mid i \text{ even}\} = \#\{j \in N \setminus M \mid j \text{ even}\}, \end{cases}$$

and for general s we set

$$[K, L] \sim_s [M, N] \iff \begin{cases} \#\{i \in L \setminus K \mid i \equiv 0 \pmod{s}\} = \#\{j \in N \setminus M \mid j \equiv 0 \pmod{s}\} \\ \#\{i \in L \setminus K \mid i \equiv 1 \pmod{s}\} = \#\{j \in N \setminus M \mid j \equiv 1 \pmod{s}\} \\ \vdots \\ \#\{i \in L \setminus K \mid i \equiv s-1 \pmod{s}\} = \#\{j \in N \setminus M \mid j \equiv s-1 \pmod{s}\}. \end{cases}$$

A quick calculation shows that these reduced Boolean incidence coalgebras by \sim_s are precisely the binomial coalgebras \mathcal{B}_s . ■

Remark 5.23: With the natural product and unit element the binomial coalgebras are in fact cocommutative \mathbb{K} -bialgebras, and with the antipode defined by

$$S(x_i) := -x_i$$

for all $i \in \{1, \dots, s\}$ they are even \mathbb{K} -Hopf algebras. We remark the similarities between the binomial Hopf algebra \mathcal{B}_1 and the divided power Hopf algebra in Example 3.25.

Furthermore, we note that heuristically $\mathcal{B}_\infty := \lim_{s \rightarrow \infty} \mathcal{B}_s = \mathcal{B}$.

5.6 Polynomial Sequences

Definition 5.24: A polynomial sequence $(p_n(x))_{n \in \mathbb{N}}$ is called **of binomial type** if

$$\begin{aligned} \deg p_n(x) &= n \\ p_n(x+y) &= \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(y) \end{aligned}$$

for all $n \in \mathbb{N}$. ▽

We want to restate the second condition in the notation of bialgebras. Consider the polynomial ring $\mathbb{K}[x, y]$; it is isomorphic to $\mathbb{K}[x] \otimes \mathbb{K}[x]$ by the map defined via

$$\begin{aligned} x &\longmapsto x \otimes 1 \\ y &\longmapsto 1 \otimes x, \end{aligned}$$

which by linearity leads to

$$\begin{aligned} q(x) &\longmapsto q(x) \otimes 1 \\ r(y) &\longmapsto 1 \otimes r(x) \end{aligned}$$

for any polynomials $q(x), r(y)$. Therefore, translated into bialgebraic terminology, the second condition reads

$$p_n(x \otimes 1 + 1 \otimes x) = \sum_{k=0}^n \binom{n}{k} p_k(x) \otimes p_{n-k}(x)$$

for all $n \in \mathbb{N}$.

Lemma 5.25: *Let $\mathbb{K}[x]$ be endowed with the structure of the binomial coalgebra. A polynomial sequence $(p_n(x))_{n \in \mathbb{N}}$ is of binomial type if and only if it is the image of the polynomial sequence $(x^n)_{n \in \mathbb{N}}$ under an invertible coalgebra map $p: \mathbb{K}[x] \longrightarrow \mathbb{K}[x]$.*

PROOF. Let p be an invertible coalgebra map. Define $p_n(x) := p(x^n)$. As $\mathbb{K}[x]$ is a bialgebra, we have

$$\begin{aligned} (\Delta \circ p)(x^n) &= \Delta(p_n(x)) = p_n(\Delta(x)) = p_n(x \otimes 1 + 1 \otimes x) \\ ((p \otimes p) \circ \Delta)(x^n) &= (p \otimes p) \left(\sum_{k=0}^n x^k \otimes x^{n-k} \right) = \sum_{k=0}^n \binom{n}{k} p_k(x) \otimes p_{n-k}(x), \end{aligned}$$

and as p is a coalgebra map, this means

$$p_n(x \otimes 1 + 1 \otimes x) = (\Delta \circ p)(x^n) = ((p \otimes p) \circ \Delta)(x^n) = \sum_{k=0}^n \binom{n}{k} p_k(x) \otimes p_{n-k}(x).$$

Therefore the polynomial sequence $(p_n(x))_{n \in \mathbb{N}}$ is of binomial type.

Conversely, if we have a polynomial sequence $(p_n(x))_{n \in \mathbb{N}}$ of binomial type and define

$$p: \mathbb{K}[x] \longrightarrow \mathbb{K}[x]: x^n \longmapsto p_n(x),$$

then $\deg p_n(x) = n$ assures the invertibility of p and the second condition makes it a coalgebra map. ■

Example 5.26: The following polynomial sequences are of binomial type:

- $(x^n)_{n \in \mathbb{N}}$;
- $(x^{\underline{n}})_{n \in \mathbb{N}}$ (the “lower factorials”), where $x^{\underline{n}} := x \cdot (x-1) \cdot \dots \cdot (x-n+1)$ (read: “ x to the n falling”);

- $(A_n(x; a))_{n \in \mathbb{N}}$ (the Abel polynomials), where $A_n(x; a) := x(x - an)^{n-1}$;
- $(L_n(x))_{n \in \mathbb{N}}$ (the Laguerre polynomials), where

$$L_n(x) := \frac{e^x}{n!} \frac{\partial^n}{\partial x^n} (e^{-x} x^n);$$

- $(G_n(x; a, b))_{n \in \mathbb{N}}$ (the Gould polynomials), where

$$G_n(x; a, b) := \frac{x}{x - an} \left(\frac{x - an}{n} \right)^n;$$

- $(T_n(x))_{n \in \mathbb{N}}$ (the Touchard polynomials), where

$$T_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k k^n}{k!}.$$

//

Remark 5.27: Polynomial sequences of binomial type are also characterized by a number of other conditions (Bell polynomials, convolution identities, generating functions, ...).

Definition 5.28: A polynomial sequence $(p_A(x))_{A \subseteq M}$ indexed by the finite subsets of a set M is called **of Boolean type** if

$$\begin{aligned} \deg p_A(x) &= \#A \\ p_A(x + y) &= \sum_{A_1 \cup A_2 = A} p_{A_1}(x) p_{A_2}(y) \end{aligned}$$

for all finite $A \subseteq M$. Similar to the lemma on the characterization of polynomial sequences of binomial type, $p_A(x)$ is of Boolean type if and only if it is the image of A under a coalgebra map p from the Boolean incidence coalgebra of M to $\mathbb{K}[x]$. ∇

Example 5.29: Let G be a graph. The **chromatic polynomial** $\chi_G(x)$ counts the number of vertex colourings (assignments of colours to the vertices $V(G)$ such that no two vertices connected by an edge share a colour) of G with x colours. Given a subset W of the vertices $V(G)$, we consider the subgraph H with vertex set W and edge set consisting of all edges in $E(V)$ which connect two vertices in W . Similarly, we derive a second subgraph $G \setminus H$ by considering all vertices in $V(G) \setminus W$ and the edges in $E(G)$ connecting them.

It is then straightforward to see that

$$\chi_G(x + y) = \sum_H \chi_H(x) \chi_{G \setminus H}(y),$$

since every vertex colouring of G with $x + y$ different colours can be decomposed into vertex colourings of H and $G \setminus H$ with x and y colours, respectively, and every pair of vertex colourings of H and $G \setminus H$ with x and y colours, respectively, can be joined to form a vertex colouring of G with $x + y$ colours. Thus the chromatic polynomials of a graph G are of Boolean type. //

5.7 Umbral Calculus

In conclusion, we give a brief summary of some of the main concepts of the theory of umbral calculus.

Definition 5.30: The **umbral algebra** \mathcal{U} is an algebra structure on the linear functionals on $\mathbb{K}[x]$, with structure maps defined on the basis by

$$\begin{aligned}\mu(L_1, L_2)(x^n) &:= \sum_{k=0}^n L_1(x^k) L_2(x^{n-k}) \\ \eta(x^n) &:= \delta_{0n}.\end{aligned}$$

▽

Definition 5.31: The **shift operator** E^a is defined as

$$E^a(p(x)) := p(x + a).$$

A **shift-invariant operator** T is a linear operator fulfilling

$$T \circ E^a = E^a \circ T.$$

We call the set of all shift-invariant operators \mathcal{S} ; it is evidently an algebra (with composition as its multiplication map and the identity map as its unit element). ▽

Lemma 5.32: *The umbral algebra \mathcal{U} is isomorphic to the algebra of shift-invariant operators \mathcal{S} . The isomorphism is given by the map*

$$\xi: \mathcal{U} \longrightarrow \mathcal{S}: L \longmapsto Q,$$

where Q is defined on the basis of $\mathbb{K}[x]$ by

$$Q(x^n) := \sum_{k=0}^n \binom{n}{k} L(x^k) x^{n-k}.$$

Definition 5.33: An **umbral operator** U is a coalgebra isomorphism on \mathcal{B}_1 ; that means it fulfils

$$\Delta(U(x^n)) = \sum_{k=0}^n \binom{n}{k} U(x^k) \otimes U(x^{n-k}).$$

▽

Remark 5.34: In fact, for an umbral operator U the polynomial sequence $(U(x^n))_{n \in \mathbb{N}}$ is of binomial type.

Definition 5.35: A **delta functional** L is a functional on $\mathbb{K}[x]$ such that $L(1) = 0, L(x) \neq 0$. Given a delta functional, there are two related polynomial sequences of binomial type: the **associated polynomial sequence** $(p_n(x))_{n \in \mathbb{N}}$ defined by

$$L^k(p_n(x)) = n! \delta_{kn}$$

and the **conjugate polynomial sequence** $(q_n(x))_{n \in \mathbb{N}}$ defined by

$$q_n(x) := \sum_{k=0}^n L^k(x^n) \frac{x^k}{k!}.$$

Conversely, every polynomial sequence of binomial type is the associated sequence and the conjugate sequence, respectively, of unique delta functionals L and \tilde{L} . ∇

Definition 5.36: Given a delta functional L and its associated sequence $(p_n(x))_{n \in \mathbb{N}}$, the shift-invariant operator Q belonging to L is called a **delta operator**. ∇

Lemma 5.37: *With the above definitions, we have*

$$Q(p_n(x)) = n p_{n-1}(x).$$

PROOF. Using the definition of Q and $p_n(x)$ by L , we have

$$Q(p_n(x)) = \sum_{k=0}^n \binom{n}{k} L(p_k(x)) p_{n-k}(x) = \sum_{k=0}^n \binom{n}{k} k! \delta_{k1} p_{n-k}(x) = n p_{n-1}(x).$$

■

This result is analogous to the classical result of applying the ordinary differential operator D (with $D(p(x)) = p'(x)$) to the polynomial sequence $(x^n)_{n \in \mathbb{N}}$. This analogy allows the generalization of a number of formulae from classical calculus to delta operators. Taylor's formula, for example, has a generalization in the form

$$f(x+a) = \sum_{n \in \mathbb{N}} \frac{p_n(a)}{n!} Q^n(f(x)).$$

Example 5.38: The delta operator for the polynomial sequence $(x^n)_{n \in \mathbb{N}}$ is the difference operator Δ defined by

$$\Delta(p(x)) := p(x+1) - p(x).$$

//

Definition 5.39: Given a delta operator Q and its associated sequence $(p_n(x))_{n \in \mathbb{N}}$, there is an invertible operator P such that $Q = DP$. P is called the **transfer operator** of $(p_n(x))_{n \in \mathbb{N}}$. ∇

This allows the formulation of the first important result of umbral calculus:

Theorem 5.40: *Let Q be a delta operator, $(p_n(x))_{n \in \mathbb{N}}$ its associated sequence and P the invertible operator such that $Q = DP$. Then the transfer formula*

$$p_n(x) = x \cdot P^{-n}(x^{n-1})$$

holds.

Definition 5.41: Let X be the operator defined by

$$X(p(x)) := x \cdot p(x);$$

then the **Pincherlé derivative** Q' of an operator Q is defined as

$$Q' := Q \circ X - X \circ Q.$$

If Q is shift-invariant, then so is Q' . ▽

Finally, we will use the theorem above to formulate another way of calculating the associated sequence of a delta operator.

Theorem 5.42: *Let Q be a delta operator, $(p_n(x))_{n \in \mathbb{N}}$ its associated sequence and Q' its Pincherlé derivative. Then the recurrence formula*

$$p_n(x) = x \cdot (Q')^{-1}(p_{n-1}(x))$$

holds.

Remark 5.43: For an extensive treatment of the theory of umbral calculus, see [KOR73] and [RR78].

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Glossary

symbol	meaning	symbol	meaning
\mathbb{N}	natural numbers, $\{0, 1, 2, 3, \dots\}$	Ring	rings
\mathbb{Z}	integers, $\{\dots, -2, -1, 0, 1, 2, \dots\}$	CommRing	commutative rings
\mathbb{N}^\times	positive integers, $\{1, 2, 3, \dots\}$	Field	fields
\mathbb{P}	prime numbers	${}_R\mathbf{Mod}$	left R -modules
δ_{jk}	Kronecker symbol	\mathbf{Mod}_R	right R -modules
1	identity map	End	endomorphisms
$R (S)$	ring	BilMap	bilinear maps
$\mathbb{K} (\mathbb{L})$	field	Alg$_{\mathbb{K}}$	\mathbb{K} -algebras
0	neutral element in rings/fields	CoAlg$_{\mathbb{K}}$	\mathbb{K} -coalgebras
1	unit element in rings/fields	BiAlg$_{\mathbb{K}}$	\mathbb{K} -bialgebras
char	characteristic of a ring/field	HopfAlg$_{\mathbb{K}}$	\mathbb{K} -Hopf algebras
$M (N)$	module	MSet	multiset construction
$V (W)$	vector space		
span	linear span of a set		
M^*/V^*	dual module/vector space of M/V		
\otimes	tensor product		
$A (B)$	algebra		
μ	multiplication map		
η	unit map		
o	neutral element in modules/algebras		
e	unit element in algebras		
$C (D)$	coalgebra		
Δ	comultiplication map		
ε	counit map		
B	bialgebra		
$*$	convolution product		
H	Hopf algebra		
S	antipode		
\mathcal{C}	combinatorial class		

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