Substitution dynamical systems in the context of Sarnak's conjecture

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Möbius function

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence **u** is **orthogonal to the Möbius function** $\mu(n)$ if

$$\lim N \to \infty \frac{1}{N} \sum_{n \le N} \mu(n) u_n = 0.$$

Old Heuristic - Mobius Randomness Law

Any "reasonably defined (easy)"bounded sequence independent of μ is orthogonal to μ.

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Orthogonality to μ

Results

- Constant sequences $\Leftrightarrow$ PNT
- Periodic sequences $\Leftrightarrow$ PNT in arithmetic Progressions
- Quasiperiodic sequences $f(n) = F(\alpha n \bmod 1)$ - Davenport
- Nilsequences - Green and Tao
- Horocycle Flows - Bourgain, Sarnak and Ziegler
- Dynamical systems with discrete spectrum

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Definition

A dynamical system is said to be deterministic, if its topological entropy is 0.

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Chowla Conjecture

Conjecture (Chowla)

Let $0 \leq a_1 < a_2 < \ldots < a_t$ and $k_1, k_2, \ldots, k_t$ in $\{1, 2\}$ not all even, then as $N \to \infty$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu^{k_1}(n + a_1)\mu^{k_2}(n + a_2) \cdots \mu^{k_t}(n + a_t) = 0.$$

Theorem (Sarnak)

The Chowla Conjecture implies the Sarnak Conjecture.

Theorem (Tao)

The logarithmic version of the Sarnak Conjecture implies the logarithmic version of the Chowla Conjecture.

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Theorem (Tao, Tao - Teräväinen)

The logarithmic version of the Chowla conjecture is true for $t = 2$ and for t odd.

Theorem (Frantzikinakis, Host)

The logarithmic version of the Sarnak conjecture is true if the dynamical system has countable many ergodic components.

Literature: „Sarnak conjecture: What's new?" (Ferenczi - Kulaga Przymus - Lemanczyk)

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Substitutive Sequences

Substitutive (Morphic) sequences

Let $\mathcal{A}$ be a finite set and θ a substitution (morphism) such that $\theta : \mathcal{A} \to \mathcal{A}^*$. Then if $\mathbf{w}$ is a fixed point of θ, i.e. $\theta(\mathbf{w}) = \mathbf{w}$, then $\pi(\mathbf{w})$ is a *substitutive sequence*, where π is a code.

Automatic sequnces

If the substitution θ is of constant length k, i.e. $\theta : \mathcal{A} \to \mathcal{A}^k$, then we call a fixed point $\mathbf{w}$ *k-automatic*.

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The Thue-Morse sequence

The Thue-Morse substitution

$$\theta(a) = ab \qquad \pi(a) = 0$$
$$\theta(b) = ba \qquad \pi(b) = 1$$

$$\theta^0(a) = a$$
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Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n \geq 0} = 01101001100101101001011001101001\dots$$

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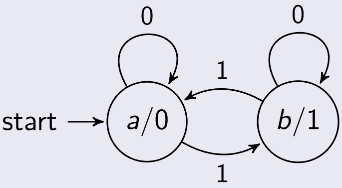

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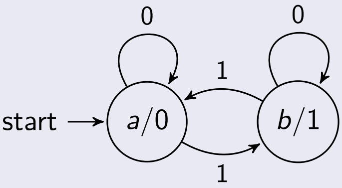

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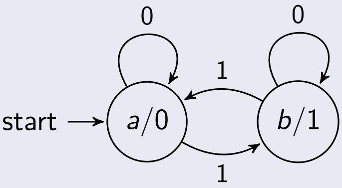

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$(u_n)_{n\geq 0} = 01101001100101101001011001101001\ldots$

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Fixpoint of the following substitution (+ code):

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Formal Power Series

Algebraicity over $\mathbf{F}_q(X)$.

$$t(X) := \sum_{n\geq 0} a_n X^n$$

$$X + (1+X)^2 t(X) + (1+X)^3 t(X)^2 = 0$$

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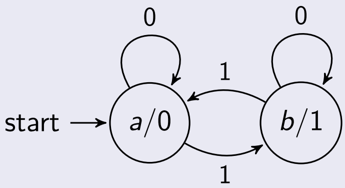

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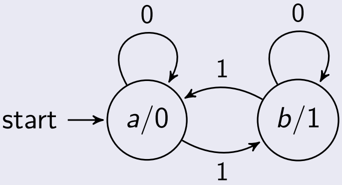

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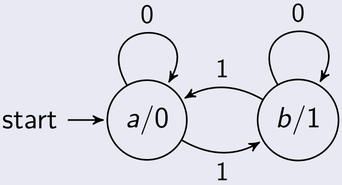

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Properties

- For every automatic sequence **u** there exists the logarithmic density

$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$

- The subword complexity p_k of an automatic sequence is (at most) linear.The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

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Theorem 1 (M., 2017)

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the Sarnak Conjecture

Theorem 2 (M., 2017)

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

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Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example

$\mathbf{w}_0 = 010.$

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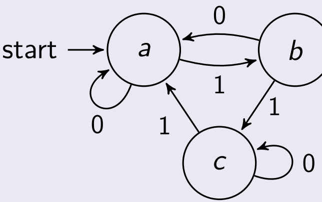

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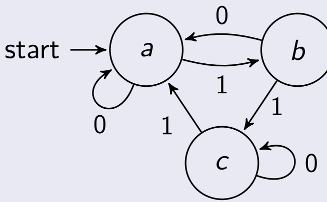

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Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

$$dens(\mathbf{u}, \alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{1}_{[u_n = \alpha]}$$

exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p \in \mathcal{P}}$
- $(u_{P(n)})_{n \in \mathbb{N}}$

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then* $\mathbf{u} = (u_n)_{n > 0}$ *is orthogonal to the Möbius function* $\mu(n)$.

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Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

$$dens(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=\alpha]}$$

exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p\in\mathcal{P}}$
- $(u_{P(n)})_{n\in\mathbb{N}}$

Theorem (Deshouillers + Drmota + M.)

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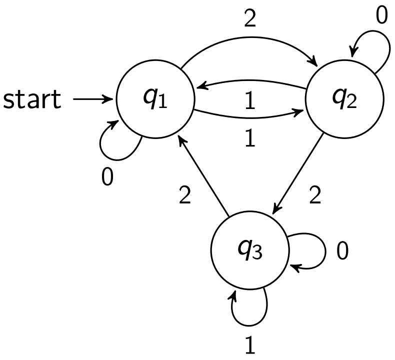
Transition Matrices



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

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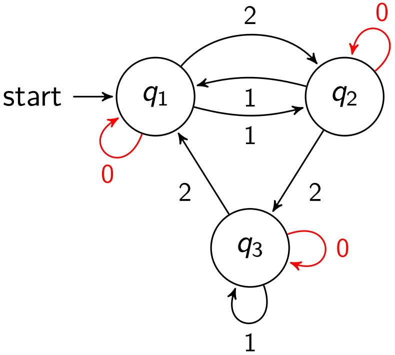

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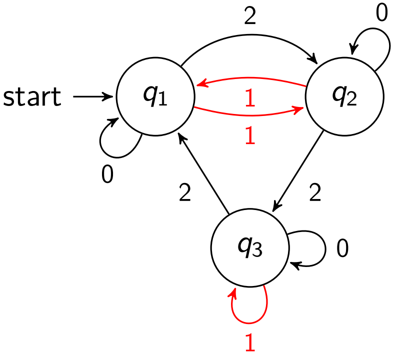

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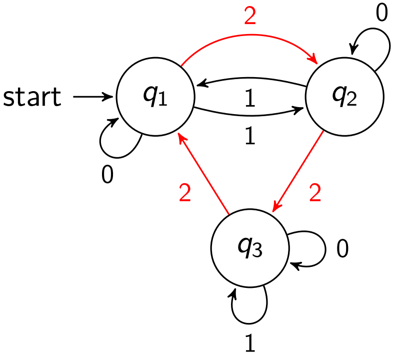

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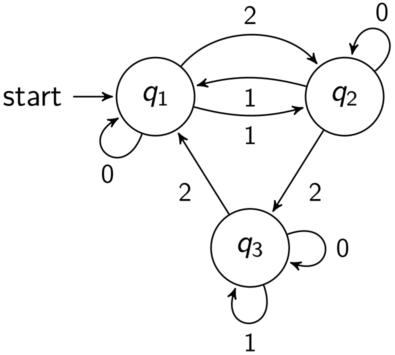

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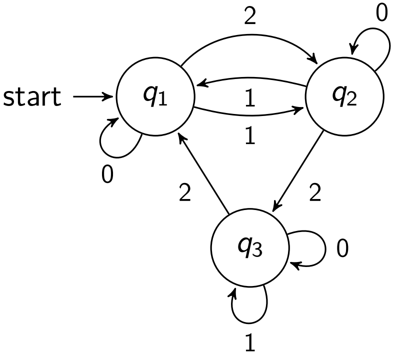

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Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

M is primitive iff there exists $m \geq 0$ such that for every $a, b \in Q$ there exists $\mathbf{w} \in \Sigma^m$ such that $\delta(a, \mathbf{w}) = b$.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the frequencies

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Results for Invertible Automata

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

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Group extension of automaton (GEA)

Let $A = (Q, \Sigma, \delta, q_0)$ be an automaton. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, G, \lambda)$ a *group extension of A* if we “attach to each transition $\delta(q, a)$ a permutation $\lambda(q, a) \in G$“.

Efficient GEA

We call a GEA *efficient* if

- A is a synchronizing automaton.
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Examples

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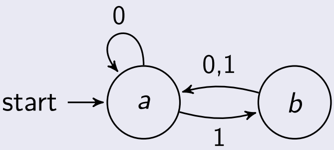

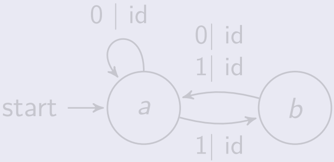

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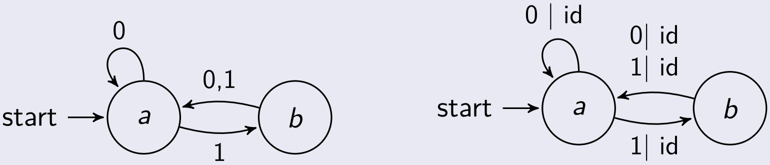

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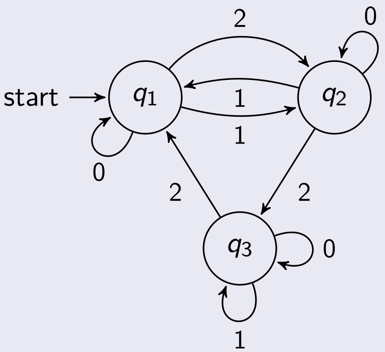

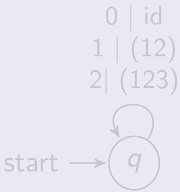

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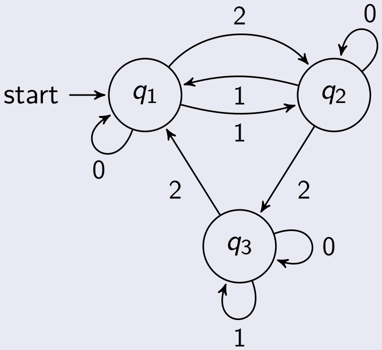

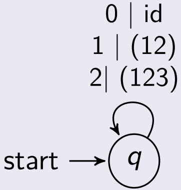

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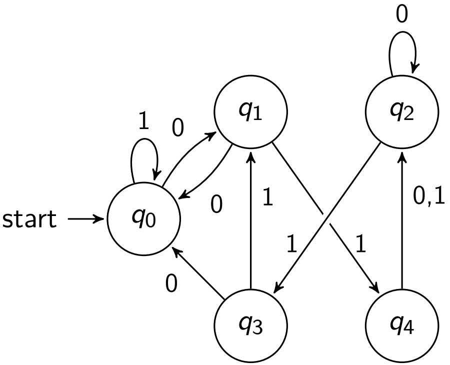

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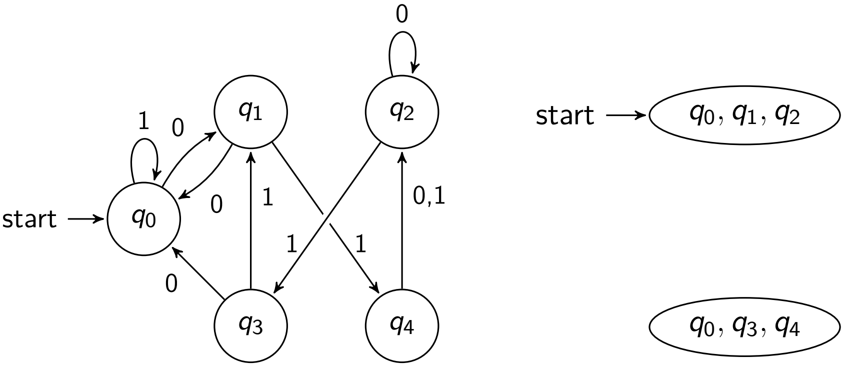

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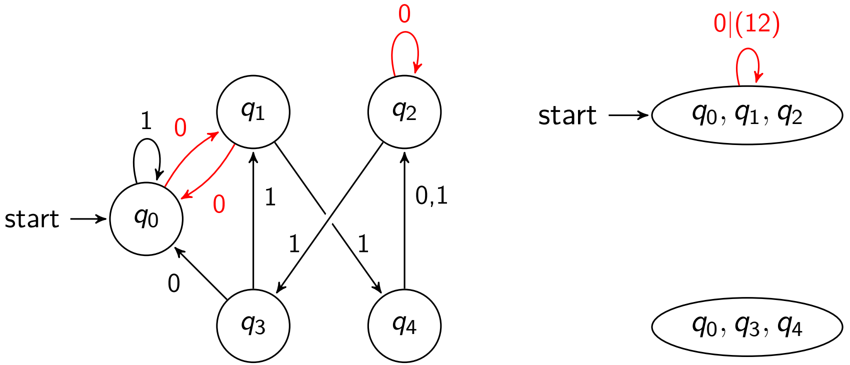

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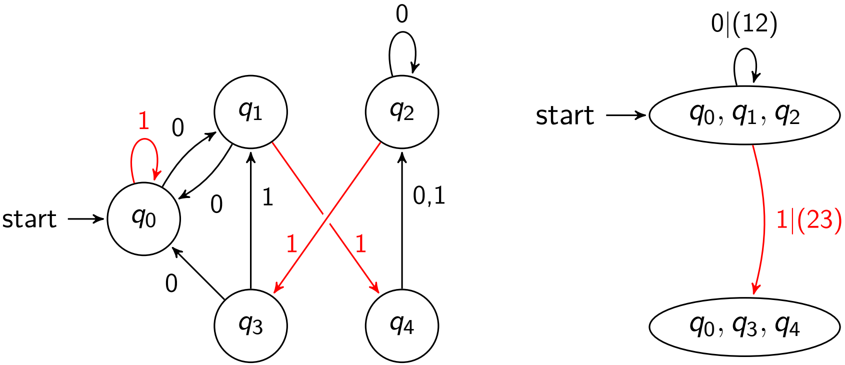

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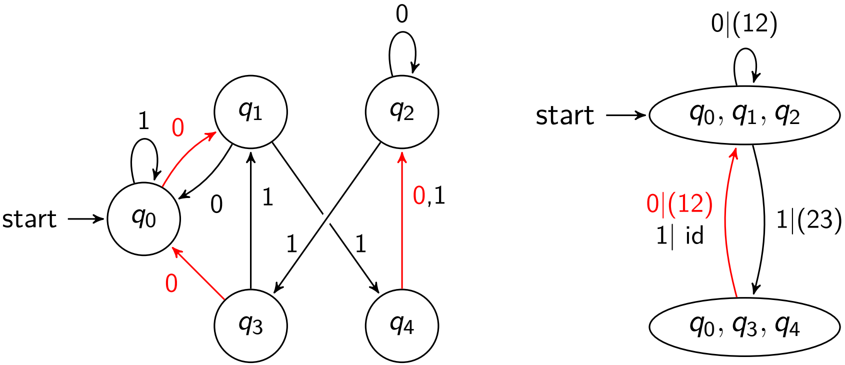

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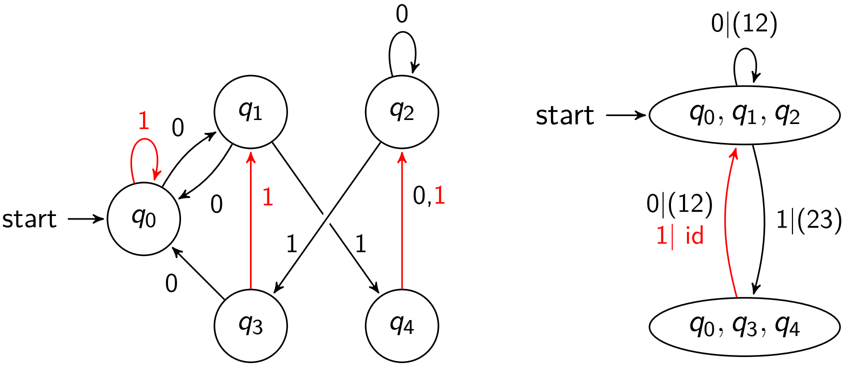

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Lemma

Let A be a strongly connected automaton and G_A its realization as a EGEA. Then,

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Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a finite group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k\times k}$.

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Let f be a continuous function from G to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

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Lemma

Suppose that

$$\sum_{\substack{n<N \\ \dots}} D(T(n))\mu(n) = o(N)$$

holds for all irreducible unitary representations of G. Then $\mathbf{u} = (u_n)_{n\geq 0}$ is orthogonal to $\mu(n)$.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta\lambda}.$$

Carry Property: the contribution of high digits and the contribution of low digits are „independent".

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Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

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Ideas for the proof

Vaughan method: Estimating

$$S_I(\theta) = \sum_m \left| \sum_{\substack{n \\ mn \in I}} f(mn)\,\mathrm{e}(\theta mn) \right|$$

$$S_{II}(\theta) = \sum_m \sum_n a_m b_n f(mn)\,\mathrm{e}(\theta mn)$$

provides estimates for

$$\sum_{n<N} \mu(n) f(n), \qquad \sum_{n<N} \Lambda(n) f(n)$$

Ideas for the proof

- Van-der-Corput inequality + Carry property

$$\left|\sum_{n\leq N} x_n\right|^2 \leq \frac{N+H-1}{N} \sum_{|h|\leq H} \left|\sum_{n\leq N} x_n \overline{x_{n+h}}\right|.$$

- Use analytic methods to detect digits.

$$n = (v0w)_2 \Leftrightarrow \left\{\frac{n}{2^{|0w|}}\right\} \in \left[0, \frac{1}{2}\right).$$

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Dynamical systems associated to a substitution

Subshift $(X_{\mathbf{u}}, S)$ related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ sequence on a finite alphabet $\mathcal{A}$

$S\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X_{\mathbf{u}} = \overline{\{S^k(\mathbf{u}) : k \geq 0\}} \subset \mathcal{A}^{\mathbb{N}}$

Subshift (X_θ, S) associated to θ

Let θ be a primitive substitution.

For any fixed point $\mathbf{w}$, we define $(X_\theta, S) = (X_{\mathbf{w}}, S)$.

- (X_θ, S) is uniquely ergodic.
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Automatic sequences

Column number

Let θ be a primitive substitution with constant length k.
Write $\theta^n(a)$ for $a \in \mathcal{A}$ below each other.
Then $c(\theta)$ is the minimal number of symbols in a column.

$$c(\theta) = \min_{n,\ell} \#\{\theta^n(a)_\ell : a \in \mathcal{A}\}.$$

Lemma

(X_θ, S) is isomorphic to a $c(\theta)$ point extension of the k-adic odometer (H_k, R).

$$H_k = \varprojlim_n \mathbb{Z}/k^n\mathbb{Z}$$

R is the addition by 1 (with carry).

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Synchronizing automatic sequences

Lemma

An automatic sequence is synchronizing iff the corresponding substitution θ has column number 1.

Eigenvalues (Host)

Let θ be a length k substitution. Then the eigenvalues of (X_θ, S) are exactly $\exp(2\pi i \frac{\ell}{k^n})$ for $\ell, n \in \mathbb{N}$.

Discrete Spectrum (Host)

If $c(\theta) = 1$, then (X_θ, S) has discrete spectrum - i.e. any continuous function (e.g. a code) can be approximated by a linear combination of eigenfunctions.

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Efficient group extension automata

Proposition (Lemanczyk + M.)

Let A be an automaton and G_A its group extension automaton with corresponding substitutions θ and θ_G. Then (X_θ, S) is a topological factor of (X_{θ_G}, S).

Let η be the "synchronizing part" of θ_G. Then (X_{θ_G}, S) is isomorphic to a finite group extension of (X_η, S'), i.e. there exists a measurable $\varphi : X_\eta \to G$ such that (X_{θ_G}, S) is isomorphic to $(X_\eta \times G, S'_\varphi)$ where

$$S'_\varphi(x, g) = (S'x, \varphi(x)g).$$

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Results 2

Theorem (Lemanczyk + M,)

Every primitive automatic sequence $a(n)$ is orthogonal to any bounded, aperiodic and multiplicative sequence $m : \mathbb{N} \to \mathbb{C}$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} a(n)m(n) = 0.$$

Decomposition of functions

Let $f \in C(X_{\theta_G}, S)$.

We can decompose $f = f_1 + f_2$, where

- f_1 can be approximated by periodic functions.
- f_2 is orthogonal to the L^2 -space of (H_k, R).

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The DKBSZ Criterion

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Let x_n be a bounded sequence such that for large enough primes $p \neq q$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} x_{pn}\overline{x_{qn}} = 0.$$

Then, for any multiplicative function $m(n)$ we have

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Joinings of dynamical systems

We aim to study

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f_2(S^{pn}(x))\overline{f_2(S^{qn}(x))}.$$

Consider

$$\rho = \lim_{\ell\to\infty} \frac{1}{N_\ell} \sum_{n\leq N_\ell} \delta_{(S^p\times S^q)^n(x,x)}.$$

ρ is $(S^p \times S^q)$ invariant and projects to ergodic measures for S^p and S^q

$\Rightarrow$ it is a joining of (X_{θ_G}, S^p) and (X_{θ_G}, S^q).

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Recall $S = R_\varphi$, where (H_k, R) is the k-adic odometer.

- $\rho|_{H_k \times H_k}$ is a graph joining of (H_k, R^p) and (H_k, R^q) via W.
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- $$\begin{aligned} &\frac{1}{N_\ell} \sum_{n \leq N_\ell} f_2(S^{pn}x)\overline{f_2(S^{qn}x)} \\ &\to \int_{X_{\theta_G} \times X_{\theta_G}} f_2 \otimes \overline{f_2}\, d\rho \\ &= \int_{H_k \times H_k} \mathbb{E}(f_2 \otimes \overline{f_2} | H_k \times H_k)\, d\rho|_{H_k \times H_k} \\ &= \int_{H_k} \mathbb{E}(f_2 | H_k) \cdot \overline{\mathbb{E}(F|H_k) \circ W}\, dm_{H_k} = 0 \end{aligned}$$

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Zeckendorf Representation

Fibonacci numbers

$F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}},$$

where, φ is the golden ratio.

Zeckendorf Representation (Lekkerkerker)

Every positive integer n admits a unique representation

$$n = \sum_{i \geq 2} \varepsilon_i(n) F_i,$$

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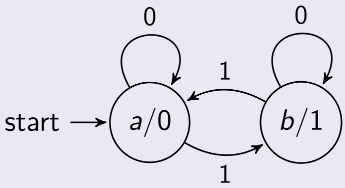
Thue-Morse



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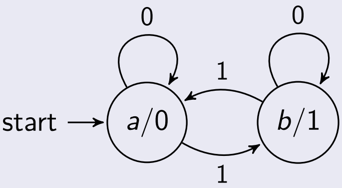

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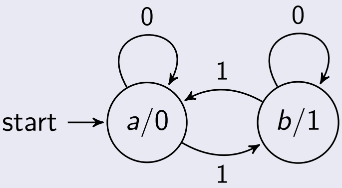

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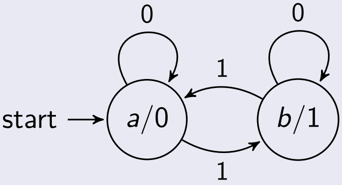

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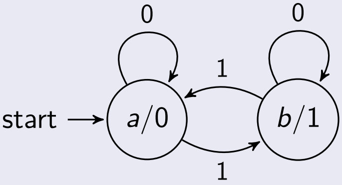

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Theorem (Drmota, M., Spiegelhofer 2022+)

Let z denote the sum-of-digits function in the Zeckendorf representation. Then for real θ,

$$\sum_{p \le x} \mathrm{e}(\theta z(p)) \ll (\log x)^{c_1} x^{1-c_2\|\theta\|^2},$$

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We have

$$\#\{p \le x : z(p) = k\} = c\mathcal{N}(\mu(x), \sigma(x)) + ErrorTerm,$$

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$$\begin{aligned} a &\mapsto ab \\ b &\mapsto c \\ c &\mapsto cd \\ d &\mapsto a. \end{aligned}$$

This gives the sequence $(-1)^{s_\varphi(n)}$ under the coding $\tau(a) = \tau(d) = 1, \tau(b) = \tau(c) = -1$.

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A Morphism

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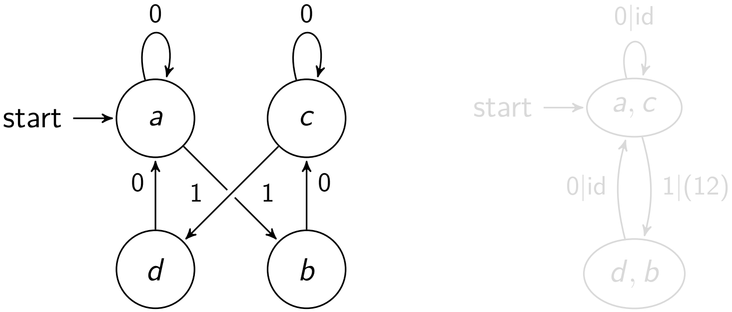
This gives the sequence $(-1)^{s_\varphi(n)}$ under the coding $\tau(a) = \tau(d) = 1, \tau(b) = \tau(c) = -1$.

This is one of the first examples of a substitution with non-constant length to be orthogonal to the Möbius function.

Fibonacci Thue-Morse

We use as input the Zeckendorf representation of n, i.e. $\varepsilon_k(n), \ldots, \varepsilon_0(n)$:



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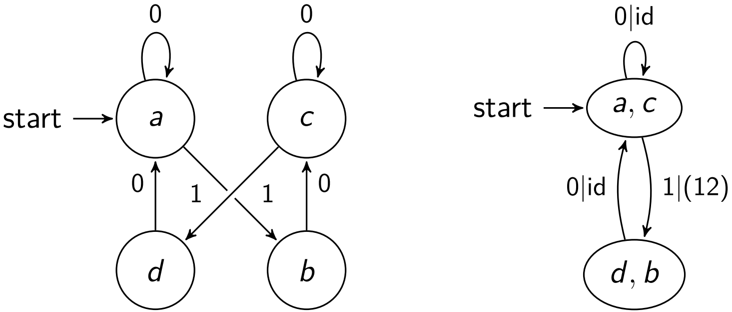

Adapted Techniques (Zeckendorf sum-of-digits)

- Carry Property:
  Addition behaves relatively nicely with respect to digits.
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Least significant digits in Zeckendorf base

Lemma

The least significant digits of n in Zeckendorf base are $(w_r, w_{r-1}, \ldots, w_2)$ iff $n\varphi \in I_w + \mathbb{Z}$. I_w contains $\varphi \sum_{k=2}^{r} F_k w_k$.

$-1/\varphi^2$ 0 $1/\varphi$

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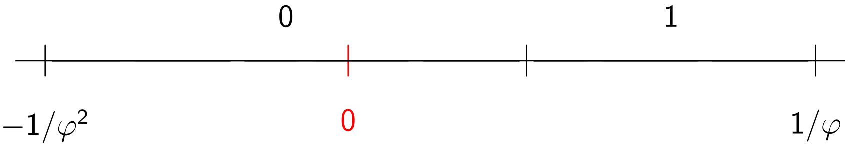

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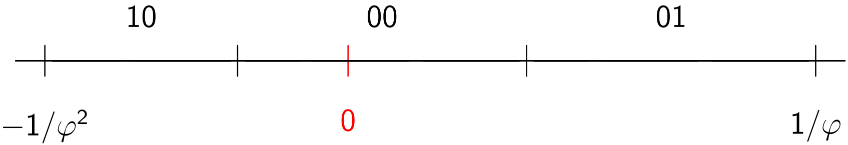

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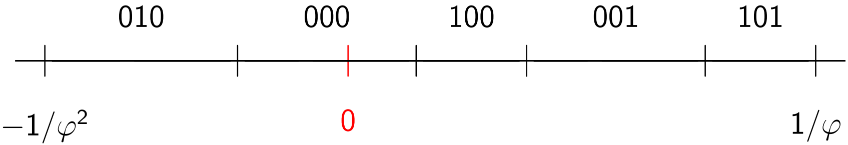

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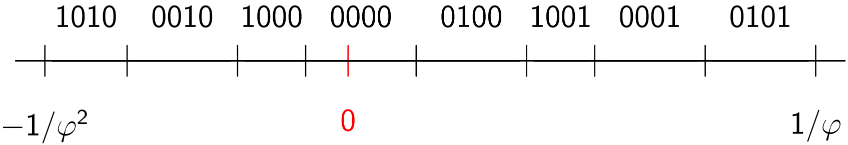

Digits in the middle of the expansion

Lemma

If $\varepsilon_j(n) = b$, then

$$\left(\left\{ \frac{n}{\varphi^{j+2}} \right\}, \left\{ \frac{n}{\varphi^{j+3}} \right\} \right) \in (A_b \mod 1) + O(\varphi^{-}j).$$