All automatic sequences fulfill the Sarnak conjecture

Clemens Müllner

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Sarnak Conjecture

A sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \le N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture, 2009)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Results

- Constant sequences
- Periodic sequences
- Quasiperiodic sequences $f(n) = F(\alpha n \bmod 1)$ - Davenport
- Nilsequences - Green and Tao
- Horocycle Flows - Bourgain, Sarnak and Ziegler
- Bounded depth circuits - Green
- Some special examples/classes of automatic sequences

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Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

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Automatic Sequences

Definition

Let E be a finite set and σ a k-uniform morphism such that $\sigma(E) \subseteq E^k$. Then if $\mathbf{w}$ is a fixed point of σ, i.e. $\sigma(\mathbf{w}) = \mathbf{w}$, then $\mathbf{w}$ is a k-automatic sequence.

Example (Thue-Morse)

$E = \{0, 1\}$
$\sigma(0) = 01$
$\sigma(1) = 10$

$$01101001100101101001011001101001\ldots$$

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Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

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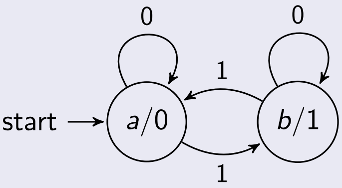

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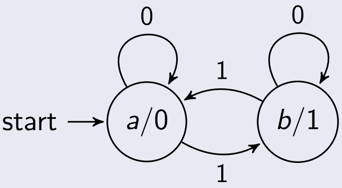

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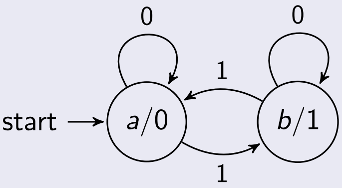

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Fixpoint of the following substitution:

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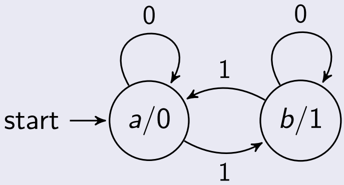

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Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the Sarnak Conjecture

Theorem 2

Let $A = (Q', \Sigma, \delta', q'_0, \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q'_0, 0) = q'_0$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathbb{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are "usually" uniformly distributed.

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Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

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$\mathbf{w}_0 = 010.$

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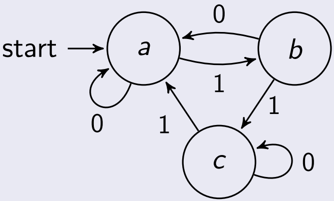

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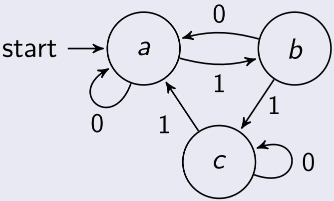

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Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

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exists. Furthermore, the densities for the following subsequences exist

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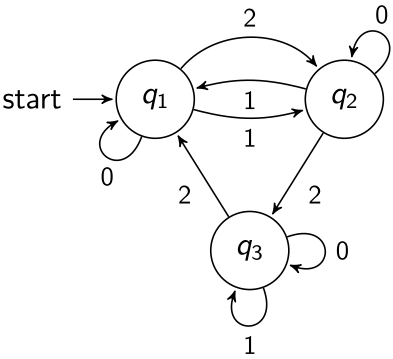
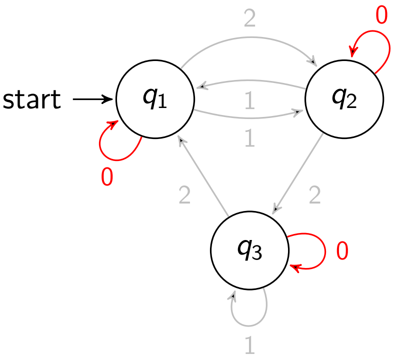
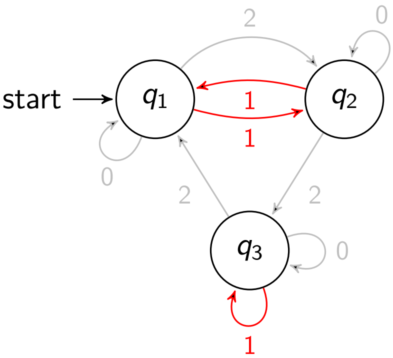
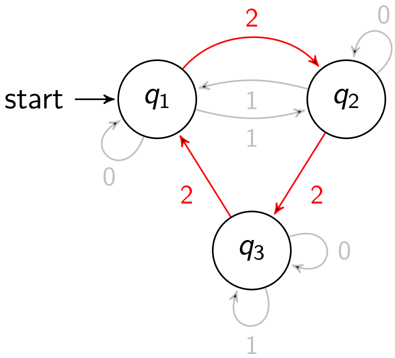

2
0
start
q_1
1
q_2
1
0
2
2
q_3
0
1



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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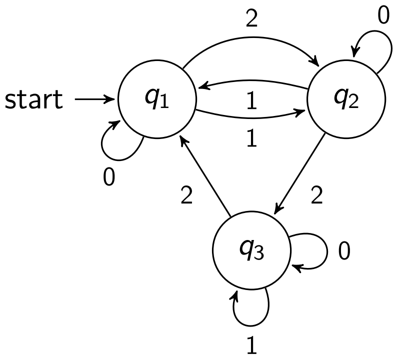

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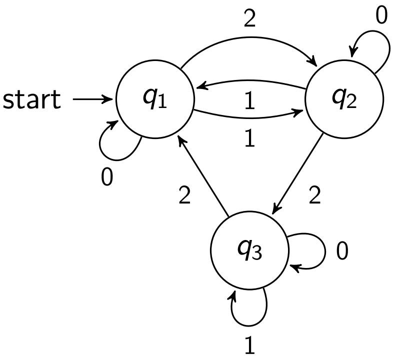
$$11 = (102)_3 : \qquad M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



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$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

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Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

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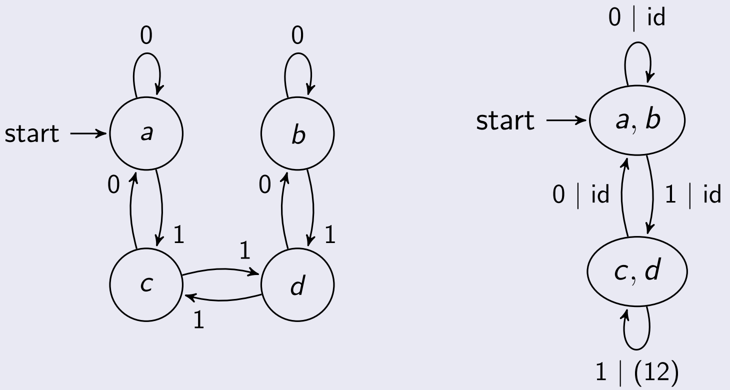

Theorem [Mauduit + Rivat, Tao]

The Rudin-Shapiro Sequence is orthogonal to the Möbius function.

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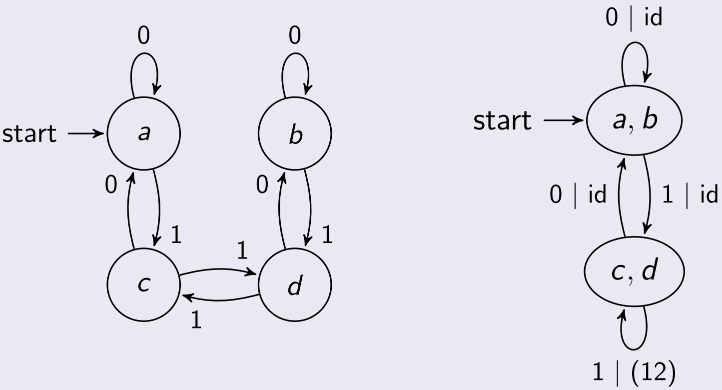

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Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

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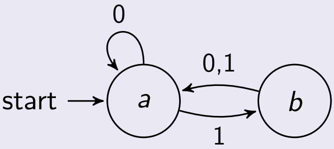

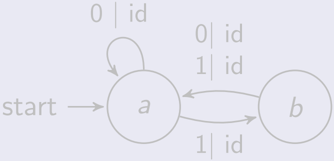

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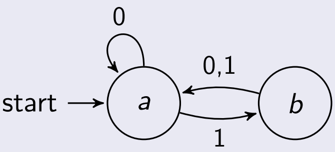

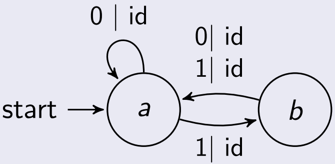

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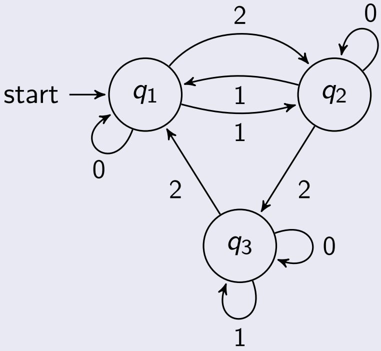

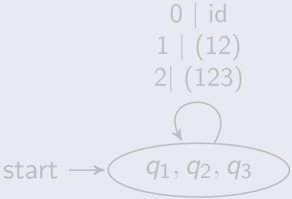

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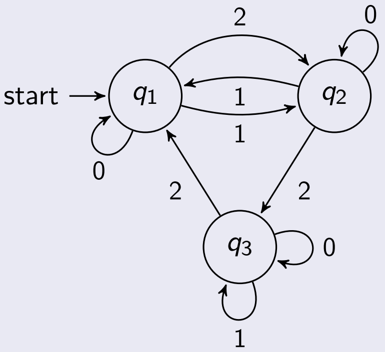

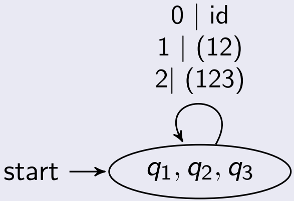

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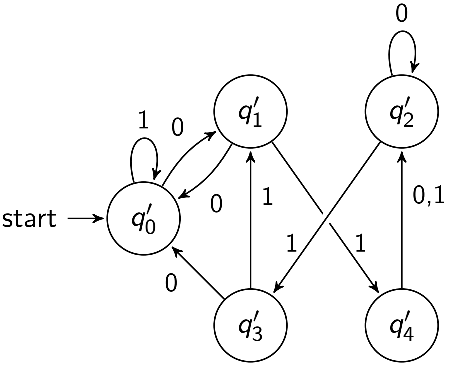

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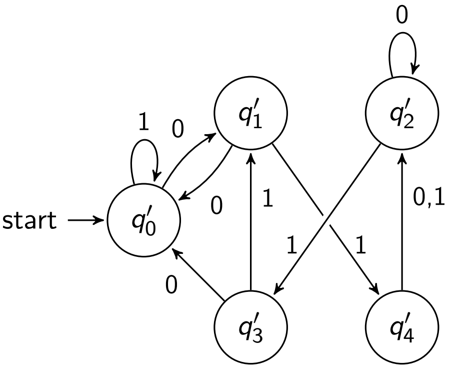

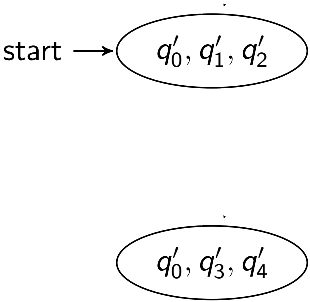

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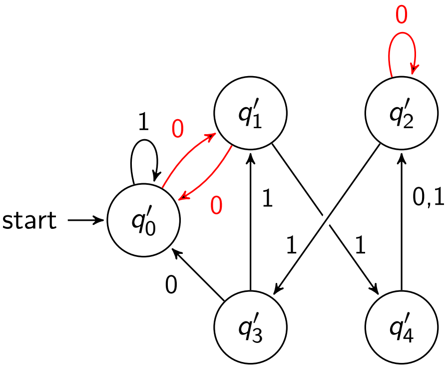

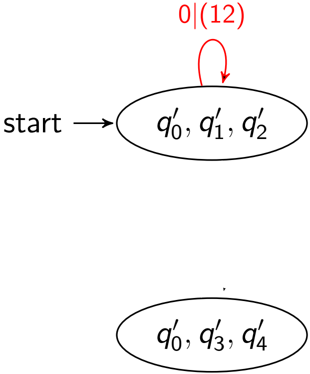

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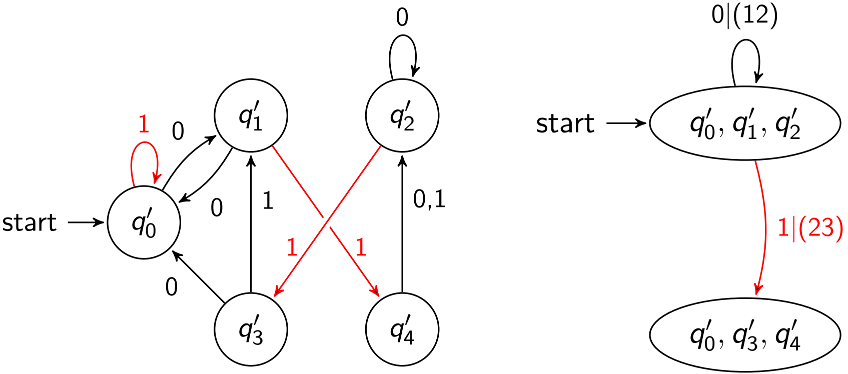

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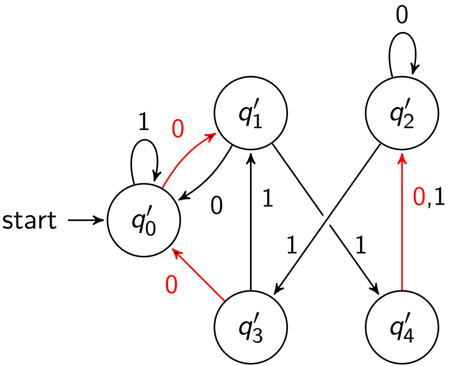

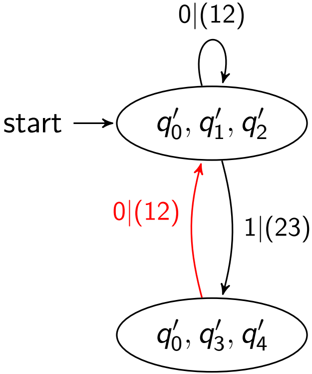

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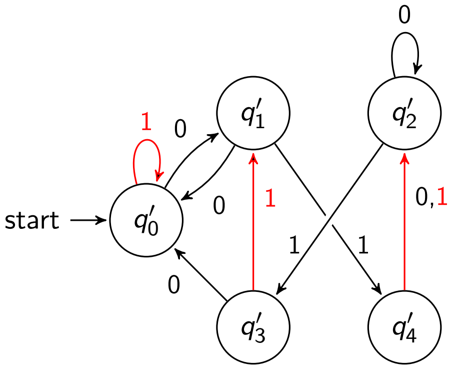

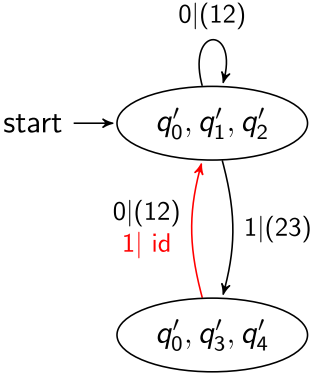

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Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

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Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

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Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a finite group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k \times k}$.

Lemma

Let f be a continuous function from G to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

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Lemma

Suppose that

$$\sum_{\substack{n<N \\ \cdots}} D(T(n))\mu(n)a_n = o(N)$$

holds for all irreducible unitary representations of G. Then $\mathbf{u} = (u_n)_{n\geq 0}$ is orthogonal to $\mu(n)$.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

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One has to work more carefully to extract the main term.

The actual frequencies can be made explicit.

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