Multiplicative automatic sequences

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Joint work with Jakub Konieczny and Mariusz Lemańczyk

Tuesday, May 19, 2020

Disjointedness of additive and multiplicative structures

Theorem (Solymosi - 2009)

For any finite set $A \subset \mathbb{R}$,

$$\max |A \cdot A|, |A + A| \gg |A|^{4/3 - o(1)}.$$

Conjecture (Chowla)

Let $\lambda(n) = (-1)^k$, where k is the number of prime factors of n. Then for all $a_1 < a_2 < \ldots < a_m$

$$\sum_{n \leq N} \lambda(n + a_1) \cdot \lambda(n + a_2) \cdots \lambda(n + a_m) = o(N).$$

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Sarnak Conjecture

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

Definition: A dynamical system is said to be determinist, if its topological entropy is 0.

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For every complex sequence $u = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system,

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Multiplicative functions

Definition (Multiplicative function)

A function $f : \mathbb{N} \to \mathbb{C}$ is called *(completely) multiplicative* if $f(nm) = f(n)f(m)$ for all n, m that are coprime (for all n, m)

Examples: μ, λ

Definition (Dirichlet character)

We call $\chi : \mathbb{Z} \to \mathbb{C}$ a Dirichlet character if

1. There exists $k > 0$ such that $\chi(n) = \chi(n + k)$ for all n.
2. If $\gcd(n, k) > 1$ then $\chi(n) = 0$; if $\gcd(n, k) = 1$ then $\chi(n) \neq 0$.
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Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u(22) = 1$$

$$\mathrm{u} = (u(n))_{n \geq 0} = 0110100110010110100101100110 1001 \ldots$$

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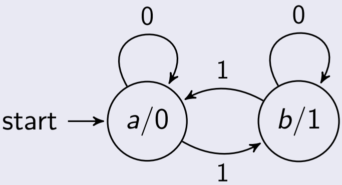

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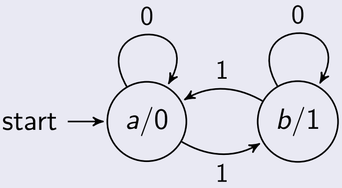

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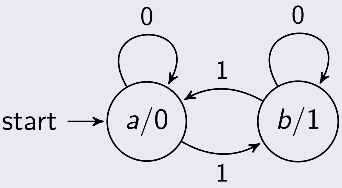

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Coding of the fixpoint of a substitution:

$$a \to ab \qquad a \mapsto 0$$
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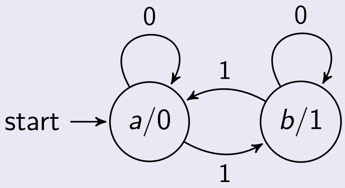

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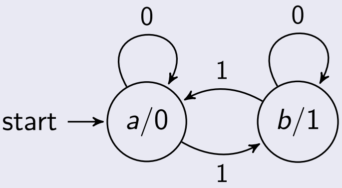

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Formal Power Series (Algebra)

Algebraicity over $F_q(X)$.

$$t(X) := \sum_{n\geq 0} u(n)X^n$$

$$X + (1+X)^2 t(X) + (1+X)^3 t(X)^2 = 0$$

Finite Kernel

The λ-*kernel* of a sequence $a(n)$ is defined as

$$\{(a(n\lambda^k + r))_{n\geq 0} : k \geq 0, 0 \leq r < \lambda^k\}.$$

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Being automatic in different bases

Question

Can a sequence be automatic in multiple bases?

Lemma

Let $\lambda, k \in \mathbb{N}$. A sequence is λ-automatic if and only if it is λ^k-automatic.

Proof works by considering the kernel.

Theorem (Cobham - 1972)

If a sequence $(a(n))_{n\geq 0}$ is both μ and λ automatic, where $\log(\mu)/\log(\lambda) \notin \mathbb{Q}$. Then $(a(n))_{n\geq 0}$ is eventually periodic.

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Simple examples and Properties

Lemma

Let $(a(n))_{n\geq 0}$ be eventually periodic. Then it is λ-automatic for every $\lambda \in \mathbb{N}$.

Proof: Follows from considering the λ-kernel.

Lemma

Let $a_1(n), a_2(n)$ be, λ-automatic sequences, then so is $(a_1(n) \cdot a_2(n))$.

Proof: We look at the corresponding λ-kernels:

$$\{(a_1(n\lambda^k + r) \cdot a_2(n\lambda^k + r) : k \in \mathbb{N}, 0 \leq r < \lambda^k\}$$
$$\subset \{(a_1(n\lambda^k + r) : k \in \mathbb{N}, 0 \leq r < \lambda^k\}$$
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Disjointedness of automatic and multiplicative sequences

Theorem (M. - 2017)

Any automatic sequence is orthogonal to the Möbius function. (Also true for the associated dynamical systems.) If the automatic sequence is primitive, then we also have a prime number theorem.

Theorem (Lemańczyk, M. - 2020?)

Let a be a primitive automatic sequence. Then it is orthogonal to any bounded, aperiodic, multiplicative function $u : \mathbb{N} \to \mathbb{C}$, i.e.

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Naive Question

Why not all bounded multiplicative functions?

Trivial counter-example: periodic sequences (e.g. Dirichlet characters).
Non-trivial counter-example: $a(n) = (-1)^{\nu_2(n)}$.

Definition (aperiodic sequence)

We call a sequence u aperiodic if for all $k, \ell \in \mathbb{N}$

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Let K be a field of characteristic 0, let $f : \mathbb{N} \to K$ be a multiplicative function, and its generating series $F(z) = \sum_{n \geq 1} f(n) z^n$ is algebraic over $K(z)$.

Then either f is finitely supported or there is a natural number k and a periodic multiplicative function $\chi : \mathbb{N} \to K$ such that $f(n) = n^k \chi(n)$ for all n.

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BBC-Conjecture

Conjecture (Bell, Bruin and Coons - 2012)

For any multiplicative automatic sequence $a : \mathbb{N} \to \mathbb{C}$ there exists an eventually periodic function $f : \mathbb{N} \to \mathbb{C}$ such that $f(p) = a(p)$ for all primes p.

Theorem/Corollary (Klurman, Kurlberg; Konieczny - 2019)

The conjecture is true. Moreover, there exists h, λ such that a is λ-automatic and coincides with χ on integers that are coprime to $h\lambda$, where χ is either zero or a Dirichlet character.

- χ is a Dirichlet character: *dense case*
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Result

Theorem 1 (Konieczny, Lemańczyk, M. - 2020+)

A sequence $a : \mathbb{N} \to \mathbb{C}$ is multiplicative and automatic if and only if there exists a prime p such that a is p-automatic and of the form

$$a(n) = f_1(\nu_p(n)) \cdot f_2(n/p^{\nu_p(n)}), \tag{1}$$

where f_1 is eventually periodic and f_2 is multiplicative, eventually periodic and vanishes at all multiples of p.

Previous Results

- Schlage-Puchta (2003): A criterion for multiplicative sequences to not be automatic.
- Coons (2010): Non-automaticity of special multiplicative functions
- Li (2017): completely multiplicative automatic sequences, nonvanishing prime numbers
- Allouche, Goldmakher (2018): completely multiplicative, never vanishing automatic sequences
- Li (2019): characterizing completely multiplicative automatic sequences
- Klurman, Kurlberg; Konieczny (2019): showed a stronger version of BBC-conjecture

Simple example

Lemma

Let $(a(n))_{n\geq 0}$ be multiplicative and p-automatic. Then

$$a(n) = a(p^{\nu_p(n)}) \cdot a(n/p^{\nu_p(n)}),$$

where $\alpha \mapsto a(p^\alpha)$ is eventually periodic.

Proof: The first part follows by multiplicativity.
As the p-kernel is finite, there exists $k_1, k_2 \in \mathbb{N}$ such that $a(np^{k_1}) = a(np^{k_2})$ for all $n \in \mathbb{N}$.
Choose $n = p^\alpha$.

Corollary

Theorem 1 is true for eventually periodic multiplicative sequences (for every p).

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f_1

Lemma

Let f_1 be eventually periodic with $f_1(0) = 1$. Then $a_1(n) = f_1(\nu_p(n))$ is p-automatic and multiplicative.

Proof: We consider again the p-kernel,

$$\{(f_1(\nu_p(np^k + r)))_{n \geq 0} : k \in \mathbb{N}, 0 \leq r < p^k\}$$
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Multiplicativity: If $(m, n) = 1$ then either $p \nmid m$ or $p \nmid n$. Thus, we have $\nu_p(mn) = \max(\nu_p(m), \nu_p(n))$ and $f_1(mn) = f_1(m)f_1(n)$.

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Lemma

Let f_1 be eventually periodic with $f_1(0) = 1$. Then $a_1(n) = f_1(\nu_p(n))$ is p-automatic and multiplicative.

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Let f_2 be eventually periodic. Then $a_2(n) = f_2(n/p^{\nu_p(n)})$ is p-automatic and multiplicative.

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&\quad = \{(f_2(n))_{n \geq 0}\} \cup \{(f_2(np^\ell + s))_{n \geq 0} : \ell \in \mathbb{N}, 0 < s < p^\ell\}.
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Let $(m, n) = 1$. Then also $(m/p^{\nu_p(m)}, n/p^{\nu_p(n)}) = 1$. Thus, a_2 is also multiplicative.

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Decomposing Dirichlet characters

Lemma

Let χ be a Dirichlet character of modulus $m = m_1 m_2$ where (m_1, m_2). Then $\chi = \chi_{m_1} \cdot \chi_{m_2}$, where $\chi_{m_i}(n)$ is a Dirichlet character of modulus m_i and $\chi_{m_i}(n) = \chi(n_i)$ with

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Corollary

Let χ be a Dirichlet character of modulus m. Then

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Dense case

Assumption: $\nu_p(h\lambda) = 1$ for all $p \mid h\lambda$!

Thus, $\chi = \prod_{p \mid h\lambda} \chi_p$.

Proposition

Let $a(n)$ be a dense multiplicative automatic sequence. Then

$$a(n) = \prod_{p \mid h\lambda} \chi_p\left(\frac{n}{p^{\nu_p(n)}}\right) \cdot \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}},$$

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Let $q \mid h$. Then

$$n \mapsto \chi_q\left(\frac{n}{q^{\nu_q(n)}}\right) \cdot \frac{a(q^{\nu_q(n)})}{\chi(\overline{q})^{\nu_q(n)}}$$

is periodic.

Proof:

- Show that χ_q is trivial.
- Show that $\gamma \mapsto a(q^{\nu_q(n)})/\chi(\overline{q})^{\gamma}$ is eventually constant.

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$$\nu_q(n\lambda^k + r_2) = \nu_q(n\lambda^k + r_1 + r_2 - r_1) = \nu_q(r_2 - r_1) = \beta.$$

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$$\chi_q(s) \cdot \frac{a(q^\gamma)}{\chi(\overline{q})^\gamma} = \chi_q\left(\frac{r_2 - r_1}{q^\beta}\right) \cdot \frac{a(q^\beta)}{\chi(\overline{q})^\beta},$$

for all $1 \leq s < q, \gamma$ large enough.
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Proposition 2

Let λ be composite and $p \mid \lambda$. Then

$$n \mapsto \chi_p\left(\frac{n}{p^{\nu_p(n)}}\right) \cdot \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}}$$

is periodic.

Proof: Similar to Proposition 1.
Choose $k \in \mathbb{N}$ and $0 < r_1 < r_2 < (\lambda/p)^k$ such that $r_1 \equiv r_2 \equiv 1 \bmod p, r_1 p^k \equiv r_2 p^k \equiv 1 \bmod h\lambda/p$ and

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We choose n such that

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Thus, we find for large enough γ (with $\beta = \nu_p(r_2 - r_1)$)

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Proof in the dense case

- $\lambda = p$: We have

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  The first two factors are in the form of (1) and the rest is periodic (i.e. admits a decomposition as in (1)).
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Tool for the sparse case

Let $u, v \in \{0, \ldots, \lambda - 1\}^*$ be words.
Then, uv denotes the concatenation and $v^k = v \ldots v$ the k-times concatenation.
For $u = (u_0 \ldots u_\ell)$,

$$[u]_\lambda = \sum_{i=0}^{\ell} u_i \lambda^{\ell-i}.$$

Pumping Lemma

Let f be a λ-automatic sequence. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exist words u, v, w over the alphabet $\{0, 1, \ldots, \lambda - 1\}$ (where v is non-empty) such that $[uvw]_\lambda = n$ and $f([uv^\ell w]_\lambda) = f(n)$ for all $\ell \geq 0$.

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An application

Theorem (Schützenberger)

Any automatic sequence that is only supported on primes is eventually zero.

Proof: We argue by contradiction assuming the existence of a λ-automatic sequence supported on $Q \subset \mathbb{P}$, $|Q| = \infty$.
Choose a large prime $p \in Q$. Then the pumping lemma implies that

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where y is a nonempty word such that $[uv^\ell w]_\lambda \in Q$ for $j \geq 0$.
Furthermore:

$$[uv^\ell w]_\lambda = [u]_\lambda \lambda^{\ell|v|+|w|} + [v]_\lambda \lambda^{|w|} \frac{\lambda^{\ell|v|} - 1}{\lambda^{|v|} - 1} + [w]_\lambda$$

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The sparse case

Claim

If a is sparse and $\alpha \mapsto a(p^\alpha)$ is not finitely supported. Then a is p-automatic.

Proof: We have $p \mid h\lambda$ as we are in the sparse case.
We only treat the case $p \mid \lambda$. Let i be large enough with $a(p^i) \neq 0$ and write $p^i = [uvw]_\lambda$. Then the pumping lemma implies that $a([uv^\ell w]_\lambda) \neq 0$ for all $\ell \geq 0$. We aim to show that $[uv^\ell w]_\lambda$ is again a power of p. Therefore, we need to control $[uv^\ell w]_\lambda \bmod h\lambda$.

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Proof in the sparse case

Let $P := \{p : \alpha \mapsto a(p^\alpha)$ is not finitely supported$\}$ which is contained in the set of prime divisors of $h\lambda$.

- $|P| = 0$: a is finitely supported - trivial.
- $|P| \geq 2$: a is eventually periodic by Gelfond´s Theorem - trivial.
- $|P| = 1$: We write

$$\begin{aligned} a(n) &= a(p^{\nu_p(n)}) \cdot a(n/p^{\nu_p(n)}) \\ &= f_1(\nu_p(n)) \cdot f_2(n/p^{\nu_p(n)}). \end{aligned}$$

It remains to note that f_2 is multiplicative and eventually equal to zero. Furthermore, f_1 is eventually periodic as a is p-automatic.

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$$\begin{aligned} a(n) &= a(p^{\nu_p(n)}) \cdot a(n/p^{\nu_p(n)}) \\ &= f_1(\nu_p(n)) \cdot f_2(n/p^{\nu_p(n)}). \end{aligned}$$

It remains to note that f_2 is multiplicative and eventually equal to zero. Furthermore, f_1 is eventually periodic as a is p-automatic.

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Proof in the sparse case

Let $P := \{p : \alpha \mapsto a(p^\alpha) \text{ is not finitely supported}\}$ which is contained in the set of prime divisors of $h\lambda$.

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- $|P| \geq 2$: a is eventually periodic by Gelfond´s Theorem - trivial.
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