Subsequences of Automatic Sequences

Clemens Müllner

Wednesday, November 6, 2019

Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

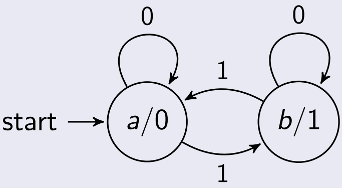
$$\mathbf{u} = (u_n)_{n \geq 0} = 01101001100101101001011001101001\dots$$

Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

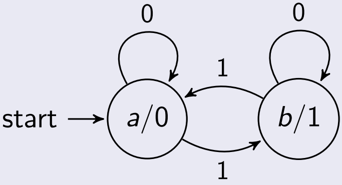
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001 \dots$$

Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

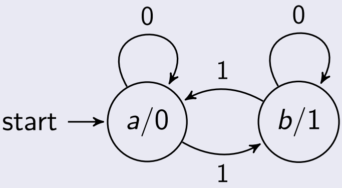
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001\ldots$$

Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110100 1\dots$$

Properties of Automatic Sequences

- For every primitive automatic sequence $\mathbf{u}$ there exists the density
$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\le n\le N} \mathbf{1}_{[u_n=a]}.$$
- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\le n\le N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$
- The subword complexity of an automatic sequence is (at most) linear.
- Every subsequence $(u_{an+b})_{n\ge0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\ge0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Properties of Automatic Sequences

- For every primitive automatic sequence $\mathbf{u}$ there exists the density
$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}.$$
- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$
- The subword complexity of an automatic sequence is (at most) linear.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Properties of Automatic Sequences

- For every primitive automatic sequence **u** there exists the density
$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}.$$
- For every automatic sequence **u** there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$
- The subword complexity of an automatic sequence is (at most) linear.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Properties of Automatic Sequences

- For every primitive automatic sequence **u** there exists the density
$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}.$$
- For every automatic sequence **u** there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$
- The subword complexity of an automatic sequence is (at most) linear.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \leq N : s_q(nr + l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \leq N : s_q(nr+l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \leq N : s_q(nr + l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \leq N : s_q(nr + l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \leq N : s_q(nr + l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \leq N : s_q(nr + l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Gelfond

Theorem (Gelfond)

Let $q, m, r, l, a \in \mathbb{N}$ with $(m, q-1) = 1$. Then we have

$$\frac{\#\{n \le N : s_q(nr + l) \equiv a \bmod m\}}{N} = \frac{1}{m} + O_q(N^{-\lambda} r).$$

Gelfond Problems

1. The joint distribution of the sum-of-digits in different bases.
   - Besinau (1972), Kim (1999)
2. The distribution of the sum-of-digits of primes.
   - Mauduit, Rivat (2010)
3. The distribution of the sum-of-digits of polynomials.
   - Squares: Mauduit, Rivat (2009)

Thue-Morse sequence along squares

Conjecture (Allouche and Shallit, 2003)

Every block $B \in \{0,1\}^k, k \geq 1$, appears in $(t(n^2))_{n\geq 0}$.

Resolved by Moshe (2007)

But what can be said about the frequency of a given block?

Theorem (Drmota + Mauduit + Rivat, 2019)

The sequence $(t(n^2))_{n\geq 0}$ is normal, i.e. every block of length k appears with asymptotic frequency 2^{-k}.

Thue-Morse sequence along squares

Conjecture (Allouche and Shallit, 2003)

Every block $B \in \{0,1\}^k, k \geq 1$, appears in $(t(n^2))_{n\geq 0}$.

Resolved by Moshe (2007)

But what can be said about the frequency of a given block?

Theorem (Drmota + Mauduit + Rivat, 2019)

The sequence $(t(n^2))_{n\geq 0}$ is normal, i.e. every block of length k appears with asymptotic frequency 2^{-k}.

Thue-Morse sequence along squares

Conjecture (Allouche and Shallit, 2003)

Every block $B \in \{0,1\}^k, k \geq 1$, appears in $(t(n^2))_{n\geq 0}$.

Resolved by Moshe (2007)
But what can be said about the frequency of a given block?

Theorem (Drmota + Mauduit + Rivat, 2019)

The sequence $(t(n^2))_{n\geq 0}$ is normal, i.e. every block of length k appears with asymptotic frequency 2^{-k}.

Thue-Morse sequence along squares

Conjecture (Allouche and Shallit, 2003)

Every block $B \in \{0,1\}^k, k \geq 1$, appears in $(t(n^2))_{n\geq 0}$.

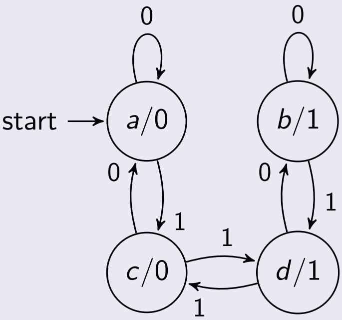
Resolved by Moshe (2007)
But what can be said about the frequency of a given block?

Theorem (Drmota + Mauduit + Rivat, 2019)

The sequence $(t(n^2))_{n\geq 0}$ is normal, i.e. every block of length k appears with asymptotic frequency 2^{-k}.

A more general method

Rudin-Shapiro sequence

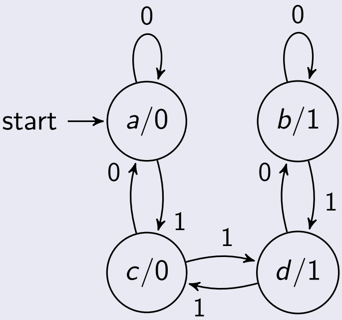


$r(n)$ "counts" 11 in the digital expansion of n.

The old approach does not work directly.

A more general method

Rudin-Shapiro sequence

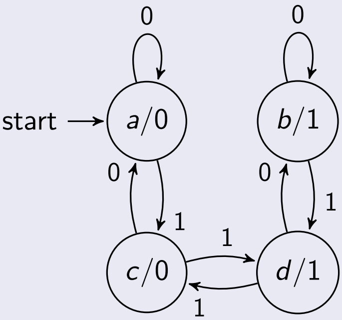


$r(n)$ "counts" 11 in the digital expansion of n.

The old approach does not work directly.

A more general method

Rudin-Shapiro sequence



$r(n)$ "counts" 11 in the digital expansion of n.

The old approach does not work directly.

A general method

Fourier Property

A function $f : \mathbb{N} \to \mathbb{U}$ has the Fourier property if the Fourier transform is uniformly small, e.g. $\exists \eta > 0$, s.t.

$$\left| \frac{1}{k^\lambda} \sum_{m < k^\lambda} f(mk^\alpha) e(-mt) \right| \leq k^{-\eta\lambda}.$$

Carry Property

The contribution of high digits and the contribution of low digits are „independent“.

A general method

Fourier Property

A function $f : \mathbb{N} \to \mathbb{U}$ has the Fourier property if the Fourier transform is uniformly small, e.g. $\exists \eta > 0$, s.t.

$$\left| \frac{1}{k^\lambda} \sum_{m < k^\lambda} f(mk^\alpha) e(-mt) \right| \leq k^{-\eta\lambda}.$$

Carry Property

The contribution of high digits and the contribution of low digits are „independent“.

A general method

Theorem (Mauduit + Rivat, 2015)

If a sequence satisfies a (sufficently strong) Fourier Property and the Carry Property, then one knows how it behaves along primes.

Proposition (Mauduit + Rivat, 2015)

The function $e(\alpha f_{11}(n) + \theta n)$ satisfies for $\alpha \notin \mathbb{Z}$ the Fourier Property and the Carry Property, where $f_{11}(n)$ denotes the number of 11 in the binary expansion of n.

Corollary (Mauduit + Rivat, 2015)

$$\lim_{N\to\infty} \frac{\#\{0 \leq p < N : r(p) = 0\}}{\pi(N)} = \frac{1}{2}.$$

A general method

Theorem (Mauduit + Rivat, 2015)

If a sequence satisfies a (sufficently strong) Fourier Property and the Carry Property, then one knows how it behaves along primes.

Proposition (Mauduit + Rivat, 2015)

The function $e(\alpha f_{11}(n) + \theta n)$ satisfies for $\alpha \notin \mathbb{Z}$ the Fourier Property and the Carry Property, where $f_{11}(n)$ denotes the number of 11 in the binary expansion of n.

Corollary (Mauduit + Rivat, 2015)

$$\lim_{N\to\infty} \frac{\#\{0 \leq p < N : r(p) = 0\}}{\pi(N)} = \frac{1}{2}.$$

A general method

Theorem (Mauduit + Rivat, 2015)

If a sequence satisfies a (sufficently strong) Fourier Property and the Carry Property, then one knows how it behaves along primes.

Proposition (Mauduit + Rivat, 2015)

The function $e(\alpha f_{11}(n) + \theta n)$ satisfies for $\alpha \notin \mathbb{Z}$ the Fourier Property and the Carry Property, where $f_{11}(n)$ denotes the number of 11 in the binary expansion of n.

Corollary (Mauduit + Rivat, 2015)

$$\lim_{N\to\infty} \frac{\#\{0 \leq p < N : r(p) = 0\}}{\pi(N)} = \frac{1}{2}.$$

A general method

Theorem (Mauduit + Rivat, 2015)

If a sequence satisfies a (sufficently strong) Fourier Property and the Carry Property, then one knows how it behaves along squares.

Corollary (Mauduit + Rivat, 2018)

$$\lim_{N\to\infty} \frac{\#\{0 \leq n < N : r(n^2) = 0\}}{N} = \frac{1}{2}.$$

A general method

Theorem (Mauduit + Rivat, 2015)

If a sequence satisfies a (sufficently strong) Fourier Property and the Carry Property, then one knows how it behaves along squares.

Corollary (Mauduit + Rivat, 2018)

$$\lim_{N\to\infty} \frac{\#\{0 \leq n < N : r(n^2) = 0\}}{N} = \frac{1}{2}.$$

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example

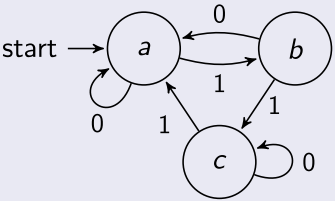
$\mathbf{w}_0 = 010.$

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example



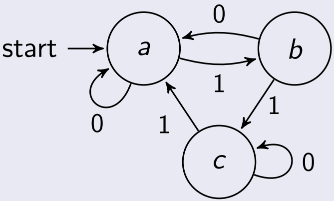
$\mathbf{w}_0 = 010.$

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example



$\mathbf{w}_0 = 010.$

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

$$\delta(\alpha) = \lim_{N\to\infty} \frac{1}{N} \#\{n \le N : u_n = \alpha\}$$

exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p\in\mathcal{P}}$
- $(u_{P(n)})_{n\in\mathbb{N}}$

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then* $\mathbf{u} = (u_n)_{n>0}$ *is orthogonal to the Möbius function* $\mu(n)$.

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

$$\delta(\alpha) = \lim_{N\to\infty} \frac{1}{N}\#\{n \le N : u_n = \alpha\}$$

exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p\in\mathcal{P}}$
- $(u_{P(n)})_{n\in\mathbb{N}}$

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then* $\mathbf{u} = (u_n)_{n>0}$ *is orthogonal to the Möbius function* $\mu(n)$.

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then for every α the density

$$\delta(\alpha) = \lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : u_n = \alpha\}$$

exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p\in\mathcal{P}}$
- $(u_{P(n)})_{n\in\mathbb{N}}$

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ is orthogonal to the Möbius function $\mu(n)$.

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

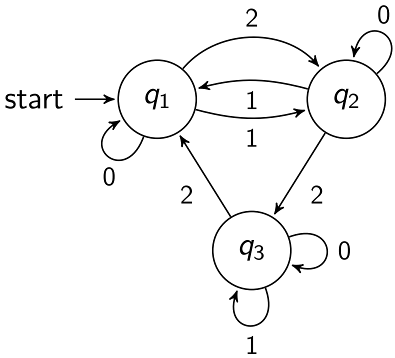
$$\delta(\alpha) = \lim_{N\to\infty} \frac{1}{N} \#\{n \le N : u_n = \alpha\}$$

exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p\in\mathcal{P}}$
- $(u_{P(n)})_{n\in\mathbb{N}}$

Theorem (Deshouillers + Drmota + M.)

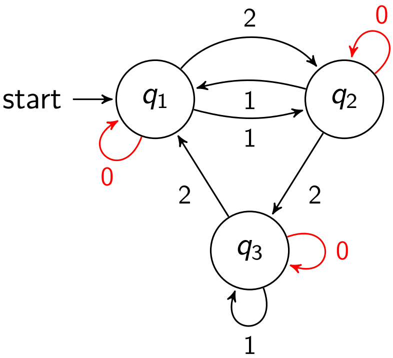
Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then* $\mathbf{u} = (u_n)_{n>0}$ *is orthogonal to the Möbius function* $\mu(n)$*.*



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

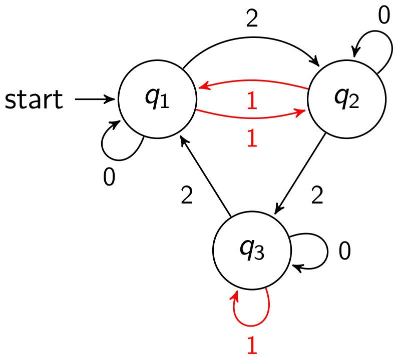
$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

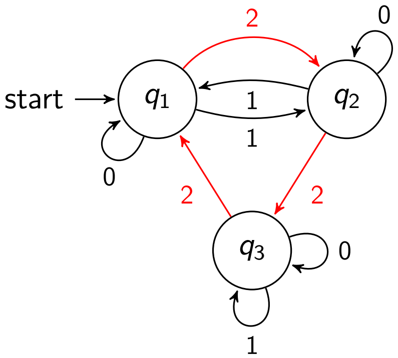
$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

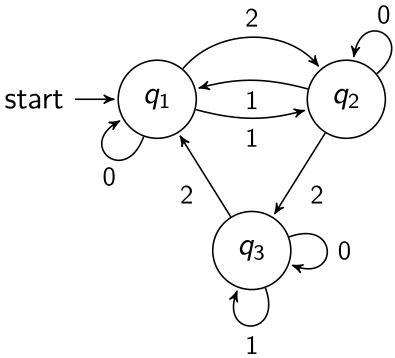
$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

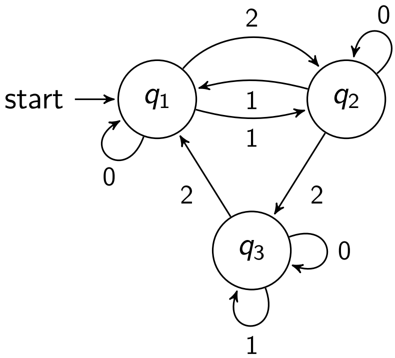
$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n \geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p \in \mathcal{P}}$ exists.

Theorem [Drmota + Morgenbesser]

The frequency of each letter of the subsequence $(u_{n^2})_{n \geq 0}$ exists.

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

Theorem [Drmota + Morgenbesser]

The frequency of each letter of the subsequence $(u_{n^2})_{n\geq 0}$ exists.

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n \geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p \in \mathcal{P}}$ exists.

Theorem [Drmota + Morgenbesser]

The frequency of each letter of the subsequence $(u_{n^2})_{n \geq 0}$ exists.

Definition

An automaton is called invertible if all transition matrices $M_0, \dots, M_{k-1}$ are invertible and if $M = M_0 + \dots + M_{k-1}$ is primitive.

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

Theorem [Drmota + Morgenbesser]

The frequency of each letter of the subsequence $(u_{n^2})_{n\geq 0}$ exists.

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

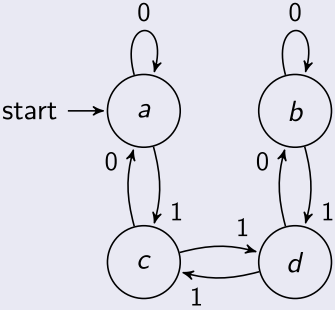
The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

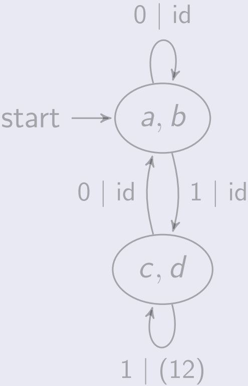
Theorem [Drmota + Morgenbesser]

The frequency of each letter of the subsequence $(u_{n^2})_{n\geq 0}$ exists.

Representation of automatic sequences

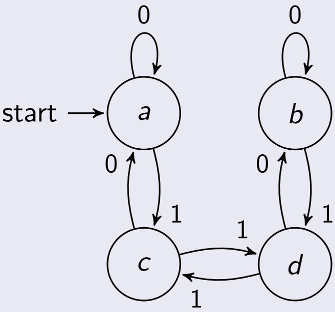
Example (Rudin-Shapiro)

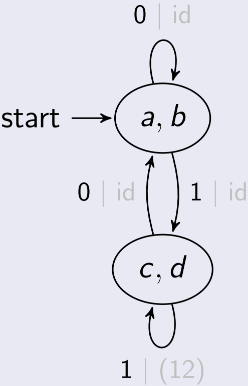




Representation of automatic sequences

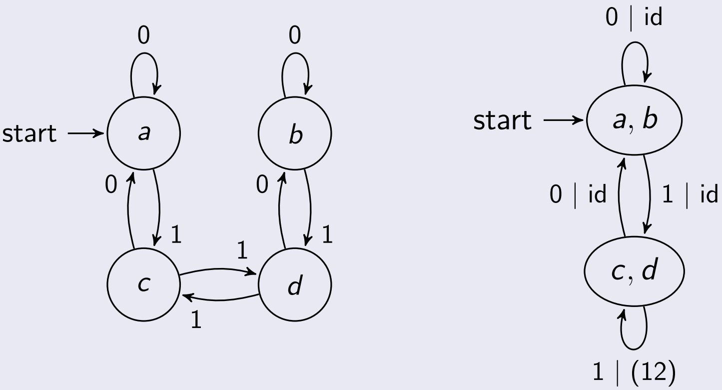
Example (Rudin-Shapiro)





Representation of automatic sequences

Example (Rudin-Shapiro)



Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

Theorem (M., 2017)

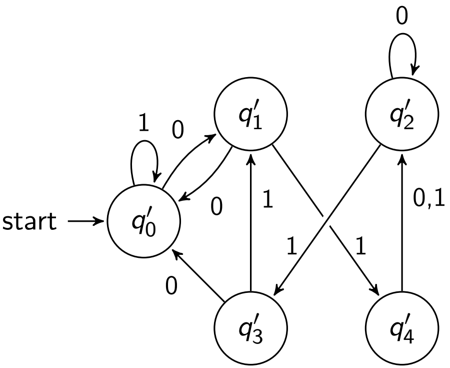
For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

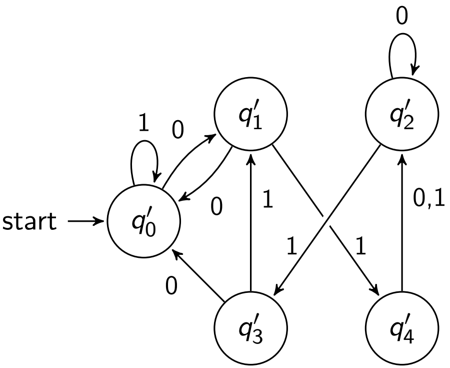
Example:

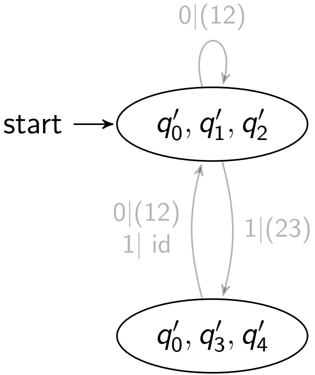


Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

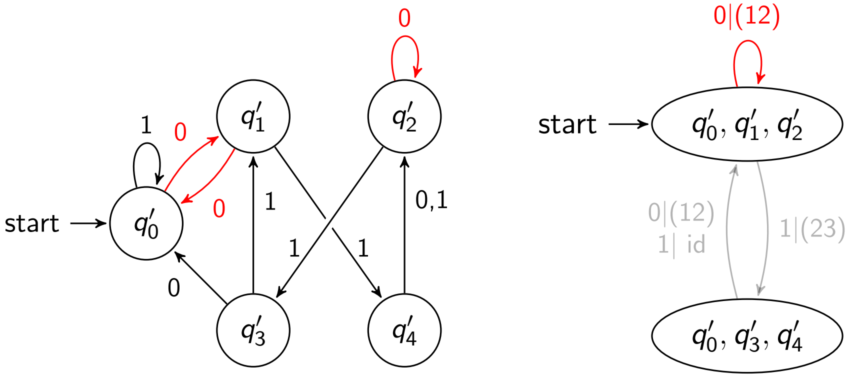




Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

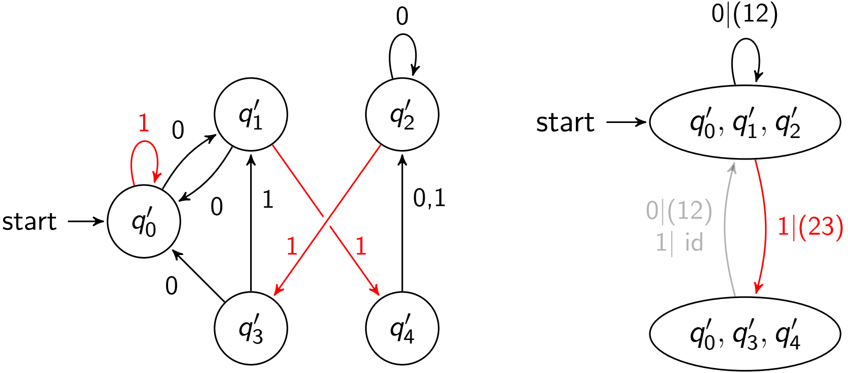
Example:



Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

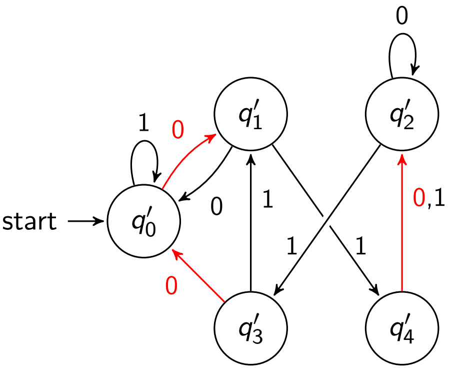
Example:

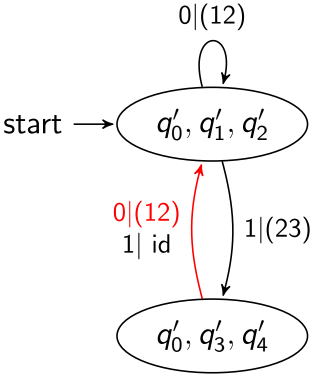


Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

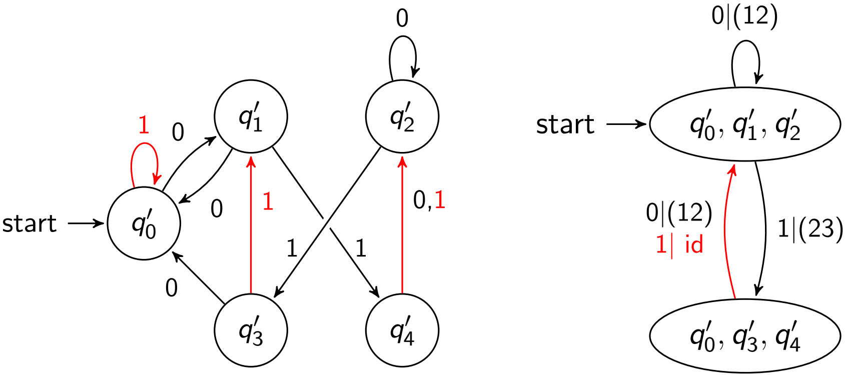




Theorem (M., 2017)

For every primitive automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:



Representation of automatic sequences

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a primitive automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

Representation of automatic sequences

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \\ \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a primitive automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

Using the structure

Theorem (M., 2017)

For any "non-trivial" unitary representation D we have a generalized Fourier Property and a generalized Carry Property for $D(T(n))$.

Theorem (M., 2017)

Let $u(n)$ be a primitive automatic sequence. Then

$$\lim_{N\to\infty} \frac{\#\{0 \leq p < N : u(p) = \alpha\}}{\pi(N)}$$

exists for all α (and can be computed).

Theorem (Byszewski, Konieczny, M.; 2019+)

For any "non-trivial" unitary representation D, $D(T(n))$ is Gower's uniform of any order.

Using the structure

Theorem (M., 2017)

For any "non-trivial" unitary representation D we have a generalized Fourier Property and a generalized Carry Property for $D(T(n))$.

Theorem (M., 2017)

Let $u(n)$ be a primitive automatic sequence. Then

$$\lim_{N\to\infty} \frac{\#\{0 \leq p < N : u(p) = \alpha\}}{\pi(N)}$$

exists for all α (and can be computed).

Theorem (Byszewski, Konieczny, M.; 2019+)

For any "non-trivial" unitary representation D, $D(T(n))$ is Gower's uniform of any order.

Using the structure

Theorem (M., 2017)

For any "non-trivial" unitary representation D we have a generalized Fourier Property and a generalized Carry Property for $D(T(n))$.

Theorem (M., 2017)

Let $u(n)$ be a primitive automatic sequence. Then

$$\lim_{N\to\infty} \frac{\#\{0 \leq p < N : u(p) = \alpha\}}{\pi(N)}$$

exists for all α (and can be computed).

Theorem (Byszewski, Konieczny, M.; 2019+)

For any "non-trivial" unitary representation D, $D(T(n))$ is Gower's uniform of any order.

Sarnak Conjecture for automatic sequences

Theorem (M., 2017)

Let $u(n)$ be a complex-valued automatic sequence.
Then we have

$$\sum_{n \leq N} u(n)\mu(n) = o(N).$$

Theorem(Lemanczyk + M., 2018)

Let $u(n)$ be a complex-valued primitive automatic sequence. Then we have for any aperiodic bounded multiplicative sequence $m(n)$,

$$\sum_{n \leq N} u(n)m(n) = o(N).$$

Sarnak Conjecture for automatic sequences

Theorem (M., 2017)

Let $u(n)$ be a complex-valued automatic sequence.
Then we have

$$\sum_{n \leq N} u(n)\mu(n) = o(N).$$

Theorem(Lemanczyk + M., 2018)

Let $u(n)$ be a complex-valued primitive automatic sequence. Then we have for any aperiodic bounded multiplicative sequence $m(n)$,

$$\sum_{n \leq N} u(n)m(n) = o(N).$$

Other subsequences

Mauduit and Rivat (1995, 2005), Spiegelhofer(2014,2017, 2018+)
$1 < c < 2$:

$$\lim_{N\to\infty} \frac{\#\{0 \le n < N : t_{\lfloor n^c \rfloor} = 0\}}{N} = \frac{1}{2},$$

Theorem (M. + Spiegelhofer, 2017)

Suppose that $1 < c < 3/2$. Then the sequence $(t_{\lfloor n^c \rfloor})$ is normal.

Other subsequences

Mauduit and Rivat (1995, 2005), Spiegelhofer(2014,2017, 2018+)
$1 < c < 2$:

$$\lim_{N\to\infty} \frac{\#\{0 \leq n < N : t_{\lfloor n^c \rfloor} = 0\}}{N} = \frac{1}{2},$$

Theorem (M. + Spiegelhofer, 2017)

Suppose that $1 < c < 3/2$. Then the sequence $(t_{\lfloor n^c \rfloor})$ is normal.

Other subsequences

Theorem (Deshouillers, Drmota and Morgenbesser, 2012)

Let u_n be a k-automatic sequence (on an alphabet $\mathcal{A}$) and

$$1 < c < 7/5.$$

Then for each $a \in \mathcal{A}$ the asymptotic density $dens(u_{\lfloor n^c \rfloor}, a)$ of a in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of a in u_n exists and we have

$$dens(u_{\lfloor n^c \rfloor}, a) = dens(u_n, a).$$

Upcoming results

Theorem (Adamczewski + Drmota + M., 2019+)

Let $u(n)$ be a primitive automatic sequence. Then

$$\lim_{N\to\infty} \frac{\#\{0 \leq n < N : u(n^2) = \alpha\}}{N}$$

exists for all α (and can be computed).

Upcoming results

Theorem (Adamczewski + Drmota + M., 2019+)

Let (n_ℓ) be a regularly varying sequence such that for any primitive automatic sequence $u(n)$ the densities along n_ℓ exist.
Then, for any automatic sequence $v(n)$ exist the logarithmic densities along n_ℓ.

Theorem (Adamczewski + Drmota + M., 2019+)

Let $u(n)$ be an automatic sequence, then the logarithmic densities exist along primes and squares.

Upcoming results

Theorem (Adamczewski + Drmota + M., 2019+)

Let (n_ℓ) be a regularly varying sequence such that for any primitive automatic sequence $u(n)$ the densities along n_ℓ exist.
Then, for any automatic sequence $v(n)$ exist the logarithmic densities along n_ℓ.

Theorem (Adamczewski + Drmota + M., 2019+)

Let $u(n)$ be an automatic sequence, then the logarithmic densities exist along primes and squares.