Möbius Orthogonality for the Zeckendorf Sum-of-Digits Function

Clemens Müllner

Friday, May 26, 2017

Joint work with Michael Drmota and Lukas Spiegelhofer

Möbius function

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \leq N} \mu(n) u_n = o\Big(\sum_{n \leq N} |u_n|\Big) \qquad (N \to \infty).$$

Old Heuristic - Mobius Randomness Law

Any "reasonably defined (easy)"bounded sequence independent of μ is orthogonal to μ.

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Orthogonality to μ

Results

- Constant sequences $\Leftrightarrow$ PNT
- Periodic sequences $\Leftrightarrow$ PNT in arithmetic Progressions
- Quasiperiodic sequences $f(n) = F(\alpha n \bmod 1)$ - Davenport
- Nilsequences - Green and Tao
- Horocycle Flows - Bourgain, Sarnak and Ziegler
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Sarnak Conjecture

Definition

A dynamical system is said to be deterministic, if its topological entropy is 0.

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Chowla Conjecture

Conjecture (Chowla)

Let $0 \leq a_1 < a_2 < \ldots < a_t$ and $k_1, k_2, \ldots, k_t$ in $\{1, 2\}$ not all even, then as $N \to \infty$

$$\sum_{n \leq N} \mu^{k_1}(n + a_1)\mu^{k_2}(n + a_2) \cdots \mu^{k_t}(n + a_t) = o(N).$$

Theorem (Sarnak)

The Chowla Conjecture implies the Sarnak Conjecture.

Theorem (Tao)

The logarithmic version of the Sarnak Conjecture implies the logarithmic version of the Chowla Conjecture.

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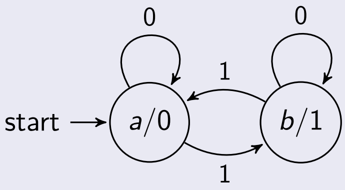

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

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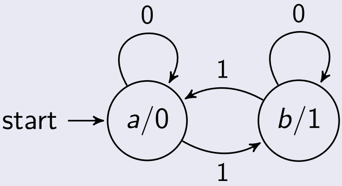

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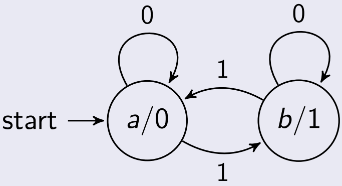

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Theorem (M., 2016)

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the Sarnak Conjecture

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Under suitable (weak) conditions one also gets a Prime Number Theorem for automatic sequence.

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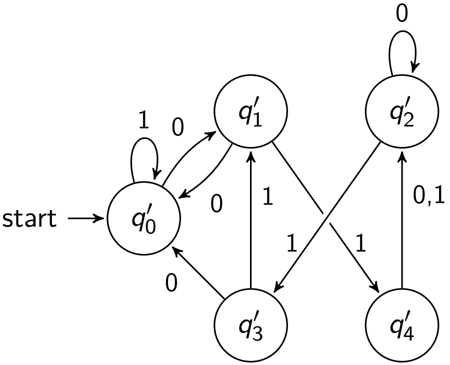

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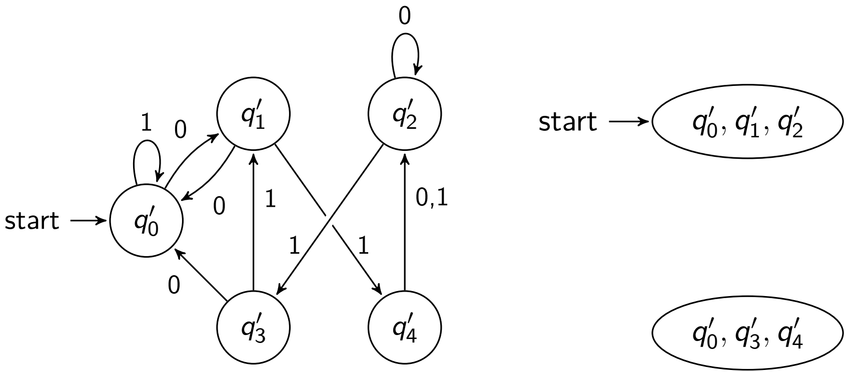

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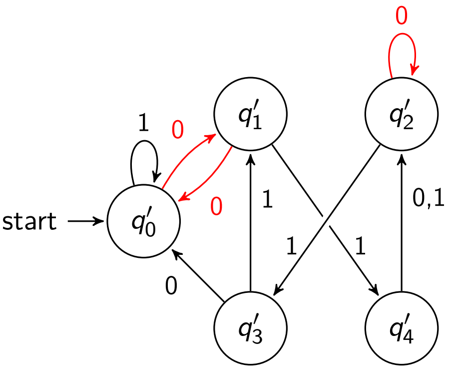

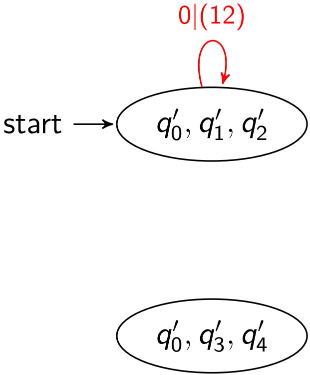

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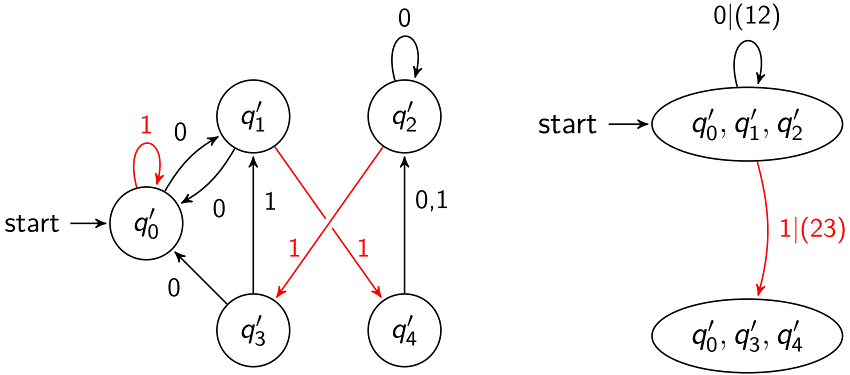

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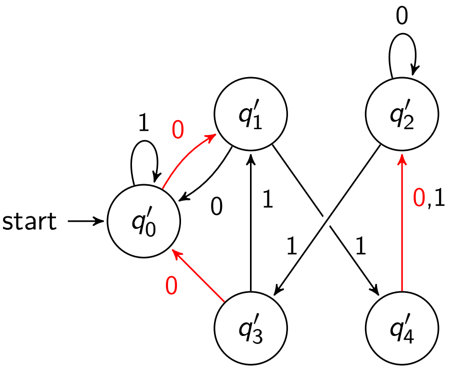

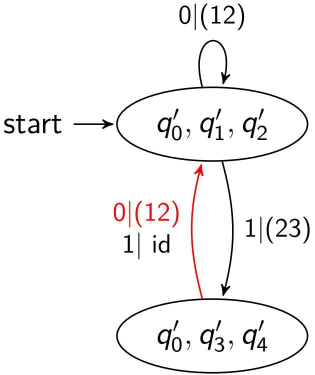

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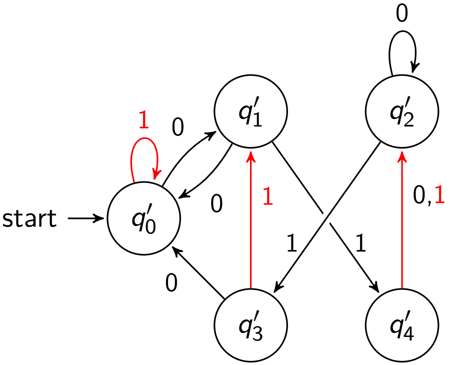

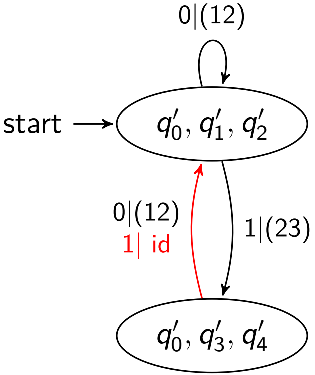

Techniques

Use and adopt a framework of Mauduit and Rivat developed for the Rudin-Shapiro sequence.

- Carry Property: The contribution of high and low digits is „independent".
- Fourier Property:
  We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha)e(mt) \right\| \leq ck^{-\eta\lambda}.$$

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Zeckendorf Representation

Fibonacci numbers

$F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}},$$

where, φ is the golden ratio.

Zeckendorf Representation

Every positive integer n admits a unique representation

$$n = \sum_{i \geq 2} \varepsilon_i(n) F_i,$$

where, $\varepsilon_i(n) \in \{0, 1\}$ and $\varepsilon_i = 1 \Rightarrow \varepsilon_{i+1} = 0$.

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We denote by

$$s_\varphi(n) = \sum_{i \geq 2} \varepsilon_i(n)$$

the Zeckendorf sum-of-digits function.

We note that $s_\varphi(n)$ is the least k such that n is the sum of k Fibonacci numbers.

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Main Result

Theorem (Drmota, M., Spiegelhofer, 2017)

Let $s_\varphi(n)$ be the Zeckendorf sum-of-digits function and $m(n)$ a bounded multiplicative function. Then we have

$$\sum_{n<N} (-1)^{s_\varphi(n)} m(n) = o(N) \qquad (N \to \infty).$$

This implies that the Zeckendorf sum-of-digits function is orthogonal to the Möbius function.

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Fixpoint of a Substitution

Definition

Let E be a finite set and σ a k-uniform morphism such that $\sigma(E) \subseteq E^k$. Then if $\mathbf{w}$ is a fixed point of σ, i.e. $\sigma(\mathbf{w}) = \mathbf{w}$, then $\mathbf{w}$ is a k-automatic sequence.

Example (Thue-Morse)

$E = \{0, 1\}$
$\sigma(0) = 01$
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A Morphism

$$\begin{aligned} a &\mapsto ab \\ b &\mapsto c \\ c &\mapsto cd \\ d &\mapsto a. \end{aligned}$$

This gives the sequence $(-1)^{s_\varphi(n)}$ under the coding $\tau(a) = \tau(d) = 1, \tau(b) = \tau(c) = -1.$

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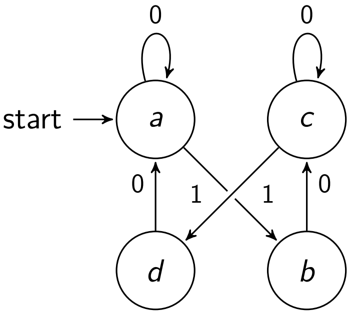
A DFAO

We use as input the Zeckendorf representation of n, i.e. $\varepsilon_k(n), \ldots, \varepsilon_0(n)$:



Plan of the Proof

- Use the Kátai Criterion to reduce the problem to

$$\sum_{n \le N} (-1)^{s_\varphi(pn)+s_\varphi(qn)} = o(N),$$

for all different primes p, q.

- Use a generating function approach and "quasi-additivity" of $(-1)^{s_\varphi(pn)+s_\varphi(qn)}$ to reduce this to:

$$s_\varphi(pn_0) \not\equiv s_\varphi(qn_0) \bmod 2 \qquad (1)$$

for some n_0.

- Show (1).

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Then for all bounded multiplicative functions $m(n)$ it follows that

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Quasi Additivity

Definition

We say that n_1 and n_2 are *r-separated at position k* if $\varepsilon_i(n_1) = 0$ for $i \geq k - r$ and $\varepsilon_i(n_2) = 0$ for $i \leq k + r$.

Example:

$$n_1 = 4 \Rightarrow 0000101$$
$$n_2 = 29 \Rightarrow 1010000$$

Definition

We say that n_1 and n_2 are *r-separated at position k* if $\varepsilon_i(n_1) = 0$ for $i \geq k - r$ and $\varepsilon_i(n_2) = 0$ for $i \leq k + r$.

Example:

$$n_1 = 4 \Rightarrow 0000101$$
$$n_2 = 29 \Rightarrow 1010000$$

Quasi Additivity

Definition (for integer base by Kropf, Wagner)

We call a function $f(n)$ quasi-additive (with respect to the Zeckendorf expansion) if there exists $r \geq 0$ such that

- $$f(n_1 + n_2) = f(n_1) + f(n_2)$$

  for all integers n_1, n_2 that are r separated.
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Lemma

Let $q > p \geq 2$ and $f(n) = s_\varphi(pn) + s_\varphi(qn)$. Then $f(n)$ is quasi-additive with respect to the Zeckendorf expansion.

Proof (Sketch):

It suffices to work with $s_\varphi(mn)$ as the sum of quasi-additive functions is again quasi-additive.

Choose r such that $\varphi^{r-1} < m$.

$n_1 < F_{k-r} \Rightarrow mn_1 < F_k$.

$\varepsilon_i(n_2) = 0 \forall i < k + r \Rightarrow \varepsilon_i(mn_2) = 0 \forall i < k$.

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Generating Functions Approach

Let f be a quasi-additive function and

$$H(x,z) := \sum_{k\geq 3} x^k \sum_{F_{k-1}\leq n<F_k} z^{f(n)}.$$

Note that

$$[x^k]H(x,-1) = \sum_{F_{k-1}\leq n<F_k} (-1)^{s_\varphi(pn)+s_\varphi(qn)}.$$

Let $\mathcal{B}$ be the set of integers n whose Zeckendorf expansion ends with exactly r zeros and that can not be decomposed into positive, r-separated summands. Let

$$B(x,z) = \sum_{n\in\mathcal{B}} x^{\ell(n)} z^{f(n)},$$

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Thus by decomposing n into parts belonging to $\mathcal{B}$, we find

$$\begin{aligned} H(x,z) &= \frac{1}{1-x}\frac{1}{1-B(x,z)\frac{x^{2r+1}}{1-x}}B'(x,z) \\ &= \frac{B'(x,z)}{1-x-x^{2r+1}B(x,z)}. \end{aligned}$$

The dominant singularity of $H(x,1)$ is at $x_0 = \frac{1}{\varphi}$.
This is due to the fact that $x = x_0$ is a solution for

$$x + x^{2r+1}B(x,1) = 1.$$

It suffices to show that there exists no solution in $|x| < x_0 + \varepsilon$ for

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The key point is to find n_1, n_2 such that

$$s_\varphi(pn_1) + s_\varphi(pn_2) \equiv s_\varphi(p(n_1 + n_2)) \bmod 2$$
$$s_\varphi(qn_1) + s_\varphi(qn_2) \equiv s_\varphi(q(n_1 + n_2)) + 1 \bmod 2.$$

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