Arithmetic subword complexity of automatic sequences, part I

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TU Wien

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Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad t_{22} = 1$$

$$(t(n))_{n\geq 0} = 0110100110010110100101100110 1001\dots$$

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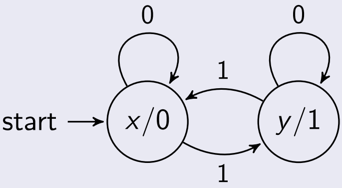

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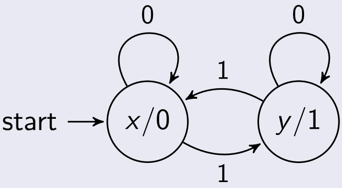

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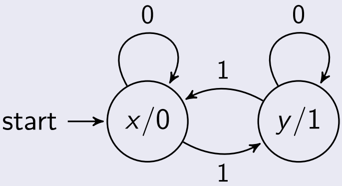

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Different Points of View I

$$(a(n))_{n\geq 0} = 01101001100101101001011001101001\ldots$$

Automaton (Computer Science)

Substitution (Dynamics)

Coding of the fixpoint of a substitution:

$$x \to xy \qquad x \mapsto 0$$
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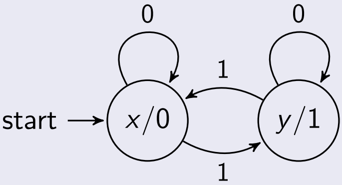

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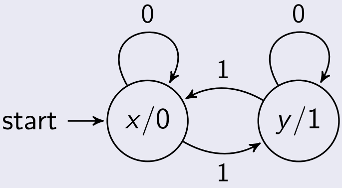

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Formal Power Series (Algebra)

Algebraicity over $\mathbf{F}_q(X)$.

$$t(X) := \sum_{n\geq 0} a(n)X^n$$

$$X + (1+X)^2 t(X) + (1+X)^3 t(X)^2 = 0$$

Finite Kernel

The *k-kernel* of a sequence $a(n)$ is defined as

$$\{(a(nk^\lambda + r))_{n\geq 0} : \lambda \geq 0, 0 \leq r < k^\lambda\}.$$

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Properties of Automatic Sequences

- Relatively easy to define (structured).
- Complex enough that interesting phenomena appear.
- Every subsequence $(a(xn+y))_{n\geq 0}$ along an arithmetic progression of an automatic sequence $\mathbf{a}$ is again automatic.

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Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = x \quad \forall q.$

Example

$\mathbf{w}_0 = 010.$

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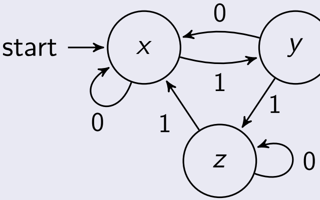

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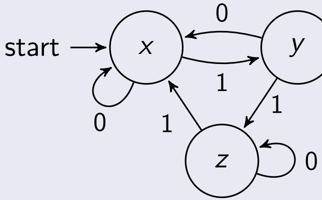

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Synchronizing Automata

Key Property

- Any word $\mathbf{w}$ containing $\mathbf{w}_0$ is also synchronizing.
- Most words are synchronizing.
- $\mathbf{a}$ can be approximated by periodic sequences:
  Let λ be large. Most words of length λ are synchronizing.
  $a(n) = a(n \bmod k^\lambda)$ if $n \bmod k^\lambda$ is synchronizing.

"Usual" Strategy

- Understand the problem for periodic sequences.
- Transfer the result to synchronizing automatic sequences.

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Subword Complexity

Let $\mathcal{A}$ be a finite alphabet and $\mathbf{u} = (u(n))_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$.

Definition (Subword Complexity)

The subword complexity of a sequence $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is defined by

$$p_{\mathbf{u}}(L) := \#\{\mathbf{w} \in \mathcal{A}^L : \exists k, (u(k), \ldots, u(k+L-1)) = \mathbf{w}\}.$$

$$p_{\mathbf{u}}(L) \leq |\mathcal{A}|^L$$

Subword complexity of automatic sequences

Let $\mathbf{a}$ be an automatic sequence. Then there exists $C > 0$ such that for all $L \in \mathbb{N}$

$$p_{\mathbf{a}}(L) \leq C \cdot L.$$

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Theorem (Avgustinovich, Fon-Der-Flaass and Frid; 2003)

- A certain class of invertible automatic sequences has maximal arithmetic subword complexity. (E.g. Thue-Morse sequence)
- Certain synchronizing automatic sequences have at most linear arithmetic subword complexity. (E.g. paperfolding sequence)

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Main Result

Definition (polynomial subword complexity)

Let $\mathbf{u}$ be a sequence over a finite alphabet $\mathcal{A}$.

$$p_{\mathbf{u}}^{\leq d}(L) := \#\{\mathbf{w} \in \mathcal{A}^L : \exists P \in \mathbb{Z}[x], P(\mathbb{N}) \subseteq \mathbb{N}, \deg P \leq d : u(P(i)) = \mathbf{w}(i) \text{ for } i = 0, \dots, L-1\}.$$

Theorem 1 (Deshouillers, Drmota, M., Shubin, Spiegelhofer; 2024+)

Let $a(n)$ be a synchronizing automatic sequence. Then for any $d \geq 1$

$$p_{\mathbf{a}}^{\leq d}(L) \leq \exp(o(L)).$$

Basically the same proof: there exist $c > 0, \eta > 0$ such that

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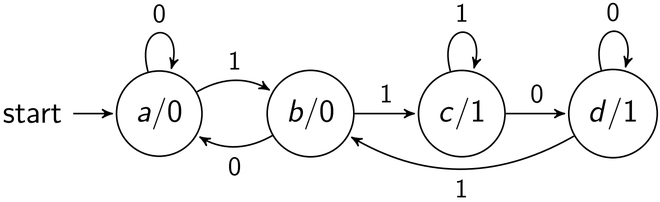
Numerical lower bounds

Paperfolding sequence:



Numerical lower bounds for $p_{pf}^{\leq d}(L)$:

L	1	2	3	4	5	6	7	8	9
$p_{pf}^{\leq 1}$	2	4	8	16	24	32	44	52	64
$p_{pf}^{\leq 2}$	2	4	8	16	32	64	128	256	400

L	10	11	12	13	14	15	16	17	18
$p_{pf}^{\leq 1}$	76	86	96	106	116	124	132	140	148
$p_{pf}^{\leq 2}$	600	828	1102	1504	1952	2352	2784	3334	?

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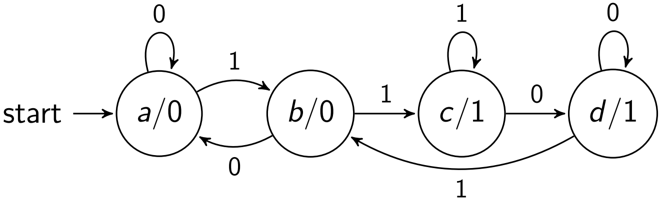

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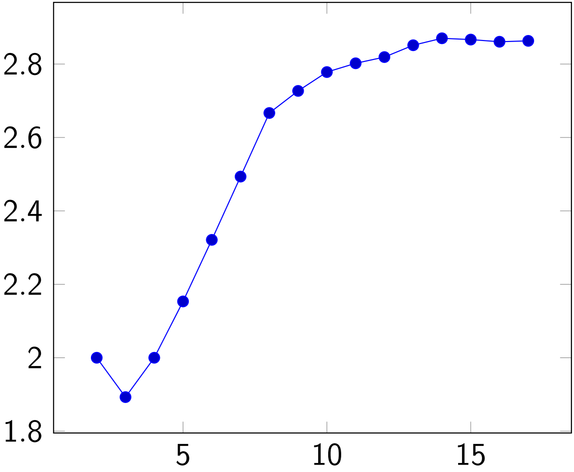

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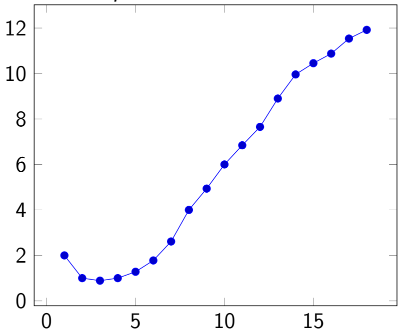

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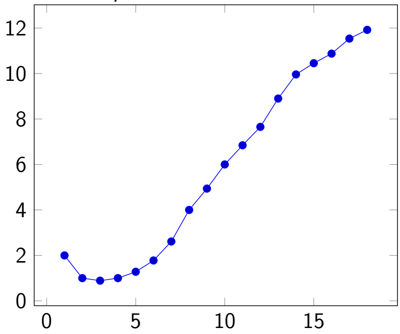

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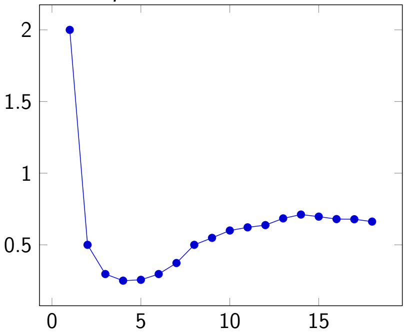

Can we do better?

Why did we consider the paper-folding sequence?

The number of words of length 2^ℓ that are not synchronizing equals 2 for all $\ell \in \mathbb{N}$! (In general: $O(2^{\ell(1-\eta)})$.)

Consider the period-doubling sequence, where only 1 word $(1 \dots 1)$ is not synchronizing:

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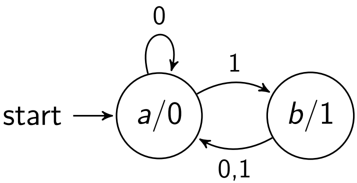

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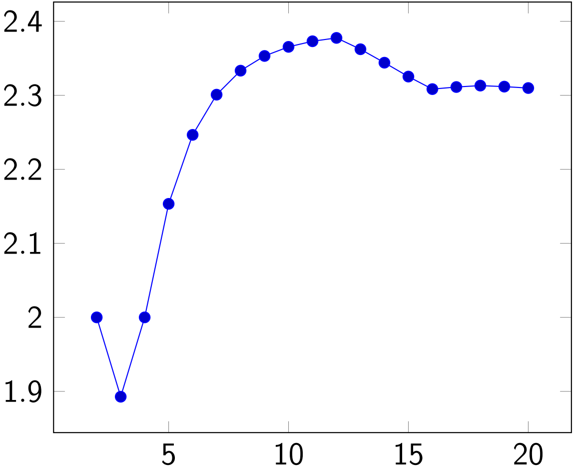

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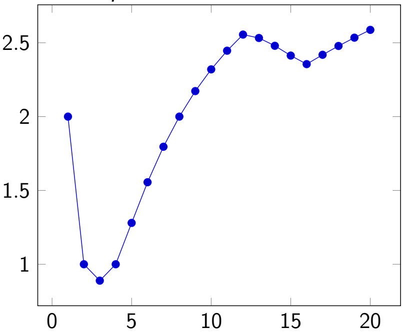

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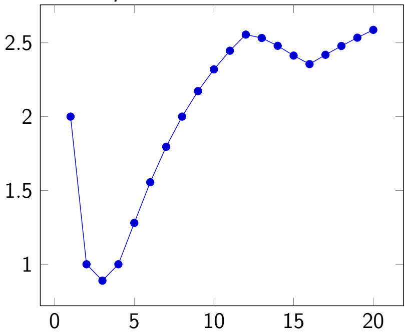

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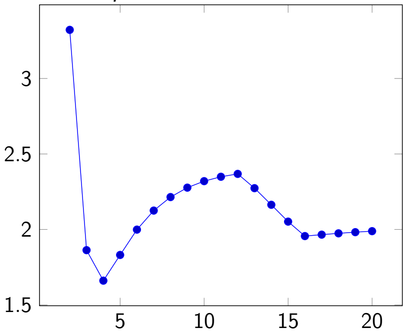
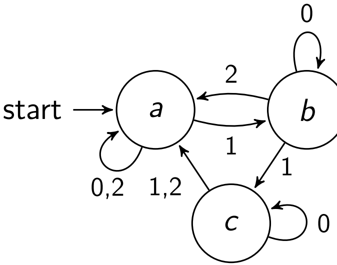


Does $p_a^{\leq 1}$ always grow at most polynomially?



The words that are not synchronizing belong to $\{0,1\}^*$.

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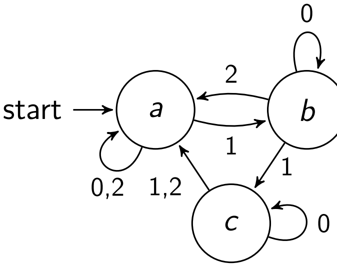

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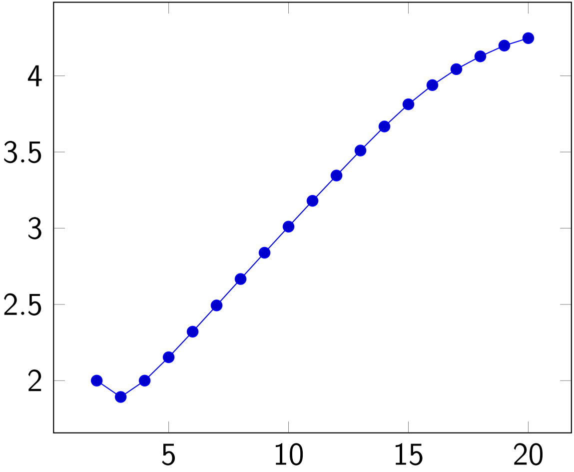

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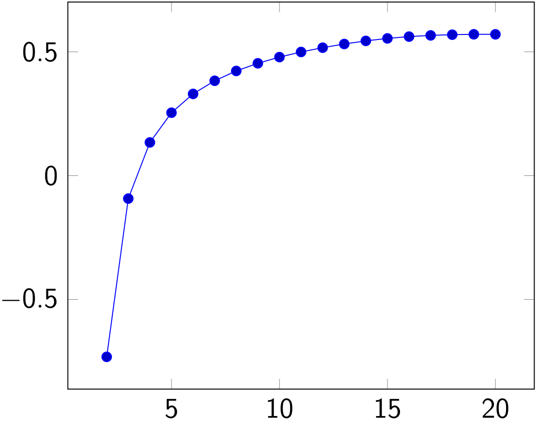

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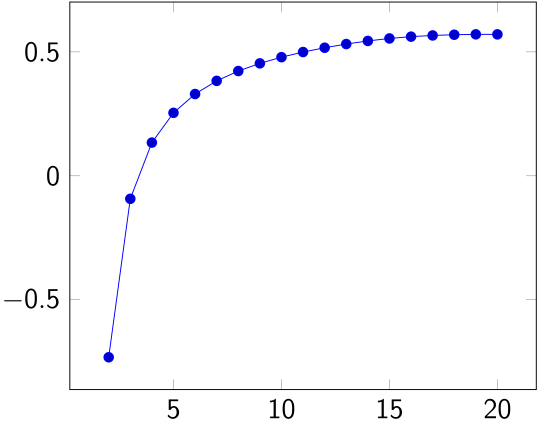

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Consequences of Theorem 1

Theorem 2 (Deshouillers, Drmota, M., Shubin, Spiegelhofer; 2024+)

Let $a(n)$ be a synchronizing automatic sequence. Then for any $c > 0$ the subword complexity of $a(\lfloor n^c \rfloor)$ grows sub-exponentially $(\exp(o(L)))$.

Remark: The same result holds for any function f with "nice" derivatives.

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Theorem (Drmota, Mauduit, Rivat; 2019)

Let **t** denote the Thue-Morse sequence. Then

$$p_{t(n^2)}(L) = 2^L.$$

Actually, the Thue-Morse sequence is normal along the squares.

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The same also holds for block-additive functions modulo m instead of the Thue-Morse sequence.

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Naive approach

- Let f be a m-periodic function. Then $p_f^{\leq d}(L) \leq m^{d+1}$.
- Approximate $a(n)$ by a k^λ-periodic function $f(n)$.
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Reductions

We study $(a(P(n)), a(P(n+1)), \dots, a(P(n+L-1)))$.

- Let $Q(\ell) = P(n+\ell)$. Study $(a(Q(0)), a(Q(1)), \dots, a(Q(L-1)))$.
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  $Q(\ell) = k^{\lambda_0}\left(z'_d \ell^d + \dots + z'_1 \ell + z'_0\right) + r$.
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Consider $Q'(\ell) = 5 \cdot 3^4 \cdot \ell$ modulo 6^4.

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- Problem: We still only hit few residue classes modulo k^λ. (E.g. $Q'(\ell) = 5 \cdot 3^\lambda \cdot \ell \bmod 6^\lambda$.)
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- Detection of digits: The digits of ℓ in base k between positions μ and λ coincide with the digits of $m < k^{\lambda-\mu}$ iff

$$\left\{\frac{\ell}{k^\lambda}\right\} \in \left[\frac{m}{k^{\lambda-\mu}}, \frac{m+1}{k^{\lambda-\mu}}\right).$$

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Example:

The gray area corresponds to the intersection of the strips $0.5 < x_0 < 1$, $0.5 < x_0 + x_1 < 1$ and $1 < x_0 + 2x_1 < 1.5$.

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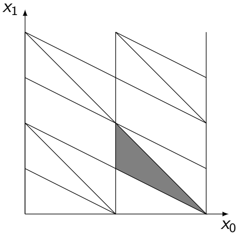

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- There are at most mL^{d+2} hyperplanes.
- There are at most $\sum_{i=0}^{d+1} \binom{mL^{d+2}}{i} \ll_d m^{d+1} L^{(d+1)(d+2)}$ regions.
- We have uniformly in m and L,

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Conclusion

- Synchronizing automatic sequences are easier to treat than "uniform" automatic sequences. (We can treat higher degrees.)
- However, questions about subword complexity are still difficult!

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Can we prove for some non-periodic automatic sequences that $p_a^{\leq d}(L)$ grows at most polynomially for all $d \in \mathbb{N}$?

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