Normal Subsequences of Automatic Sequences

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Monday, February 2, 2019

Normal Sequences

Let $\mathcal{A}$ be a finite alphabet with b elements and $\mathbf{u} = (u_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$.

Definition

Let $a \in \mathcal{A}$ and $\mathbf{w} = (w_0, \ldots, w_{\ell-1}) \in \mathcal{A}^{\ell}$.

$$N_{\mathbf{u}}(a, n) := \#\{k \leq n : u_k = a\}$$
$$N_{\mathbf{u}}(\mathbf{w}, n) := \#\{k \leq n : u_k = w_0, \ldots, u_{k+\ell-1} = w_{\ell-1}\}.$$

Definition (Subword Complexity)

The subword complexity of a sequence $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is defined by

$$p_{\mathbf{u}}(n) := \#\{\mathbf{w} \in \mathcal{A}^n : \exists k, N_{\mathbf{u}}(\mathbf{w}, k) \geq 1\}.$$

$$p_{\mathbf{u}}(n) \leq |\mathcal{A}|^n$$

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Definition (Simple Normality)

We say that $\mathbf{u}$ is *simply normal in base b* if for every $a \in \mathcal{A}$

$$\lim_{n\to\infty} \frac{N_{\mathbf{u}}(a,n)}{n} = \frac{1}{b}.$$

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Examples

- Almost every sequence $\mathbf{u}$ is normal (1909).
- Champernowne (1933): The sequence $0123456789101112131415\ldots$ is normal in base 10.
- Copeland-Erdös (1946): The sequence $235711131719232931\ldots$ is normal in base 10.

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Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001 \dots$$

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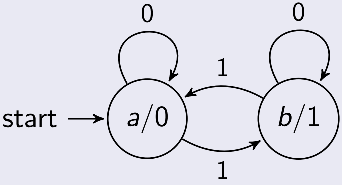

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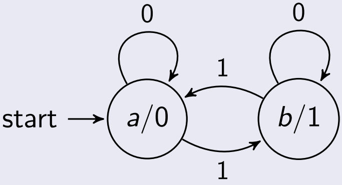

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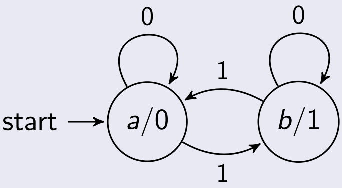

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Examples of Automatic Sequences

- Periodic sequences.
- q-additive function modulo m: $u_n = f(n) \bmod m$

$$f(n) = \sum_{j \geq 0} f(\varepsilon_j(n)) \text{ and } f(0) = 0.$$

- q-block-additive function modulo m: $u_n = f(n) \bmod m$

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Properties of Automatic Sequences

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N \to \infty} \frac{1}{log(N)} \sum_{1 \leq n \leq N} \frac{1}{n} \mathbf{1}_{[u_n = a]}.$$
- The subword complexity p_k of an automatic sequence is (at most) linear.
- Every subsequence $(u_{an+b})_{n \geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n \geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

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General Idea

- Start with an automatic sequence u_n that is uniformly distributed on the output alphabet.
- Consider a relatively sparse subsequence u_{n_k} that has the same asymptotic frequencies. (The size of the gaps needs to increase sufficiently fast.)
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Thue-Morse sequence along Piatetski-Shapiro sequence $\lfloor n^c \rfloor$

Thue-Morse sequence $(t_n)_{n\geq 0}$:
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Mauduit and Rivat (1995, 2005), Spiegelhofer(2014,2017, 2018+)
$1 < c < 2$:

$$\#\{0 \leq n < N : t_{\lfloor n^c \rfloor} = 0\} \approx \frac{N}{2},$$

that is, $(t_{\lfloor n^c \rfloor})_{n\in\mathbb{N}}$ is simply normal in base 2.

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Subsequences along $\lfloor n^c \rfloor$

Theorem (Deshouillers, Drmota and Morgenbesser, 2012)

Let u_n be a k-automatic sequence (on an alphabet $\mathcal{A}$) and

$$1 < c < 7/5.$$

Then for each $a \in \mathcal{A}$ the asymptotic density $dens(u_{\lfloor n^c \rfloor}, a)$ of a in the subsequence $u_{\lfloor n^c \rfloor}$ exists if and only if the asymptotic density of a in u_n exists and we have

$$dens(u_{\lfloor n^c \rfloor}, a) = dens(u_n, a).$$

Thue-Morse sequence along squares

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Solution of a Conjecture of Gelfond (1968).

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Theorem (M., 2017+)

Let u_n be a k-automatic sequence (on an alphabet $\mathcal{A}$) generated by a strongly connected automaton such that a initial state is mapped to itself under 0. Then for each $a \in \mathcal{A}$ the asymptotic density

$$dens(u_{n^2}, a)$$

exists (and can be computed).

Thue-Morse sequence along primes

Thue-Morse sequence $(t_n)_{n\geq 0}$:
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exists, where p_n denotes the n-th prime number (and can be computed).

Sarnak Conjecture for automatic sequences

Theorem (M., 2016)

Let u_n be a complex-valued automatic sequence.
Then we have

$$\sum_{n \leq N} u_n \mu(n) = o(N),$$

where $\mu(n)$ denotes the Möbius function.

This generalizes several results by Dartyge and Tenenbaum (Thue-Morse); Mauduit and Rivat (Rudin-Shapiro); Tao (Rudin-Shapiro); Drmota (invertible); Ferenczi, Kulaga-Przymus, Lemanczyk, and Mauduit (invertible); Deshoulliers, Drmota and M. (synchronizing).

Thue-Morse sequence along squares

$p_k^{(2)}$ ... subword complexity of $(t_{n^2})_{n\geq 0}$.

Conjecture (Allouche and Shallit, 2003)

$$p_k^{(2)} = 2^k$$

Equivalently: every block $B \in \{0,1\}^k, k \geq 1$, appears in $(t_{n^2})_{n\geq 0}$.

(Moshe, 2007): $p_k^{(2)} = 2^k$.
But what can be said about the frequency of a given block?

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Theorem (Drmota + Mauduit + Rivat, 2013+)

The sequence (t_{n^2}) is normal.

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Suppose that $1 < c < 3/2$. Then the sequence $(t_{\lfloor n^c \rfloor})$ is normal.

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Normal subsequences

Theorem (M., 2018+)

Let $f(n)$ be a block-additive function and $u_n = f(n) \bmod m$ an automatic sequence which is uniformly distributed on the alphabet $\{0, \ldots, m-1\}$ along arithmetic subsequences.
Then the sequence $(u_{\lfloor n^c \rfloor})_{n \geq 0}$ is normal for all c with $1 < c < 4/3$.
Furthermore, $(u_{n^2})_{n \geq 0}$ is normal.

Conjecture (Drmota)

Suppose that $c > 1$ and $c \notin \mathbb{Z}$. Then for every automatic sequence u_n (on an alphabet $\mathcal{A}$) the asymptotic density $dens(u_{\lfloor n^c \rfloor}, a)$ of $a \in \mathcal{A}$ in the subsequence $(u_{\lfloor n^c \rfloor})$ exists if and only if the asymptotic density of a in u_n exists and we have up to periodic behavior

$$\lim_{N\to\infty} \#\{n < N, u_{\lfloor n^c \rfloor} = b_0, \ldots, u_{\lfloor (n+k-1)^c \rfloor} = b_{k-1}\}$$
$$= dens(u_n, b_0) \cdots dens(u_n, b_{k-1})$$

for every $k \geq 1$ and for all $b_0, \ldots, b_{k-1} \in \mathcal{A}$.

Conjecture (Drmota)

Let $P(x)$ be a positive integer valued polynomial and u_n an automatic sequence generated by a strongly connected automaton. Then, for every $a \in \mathcal{A}$ the densities $\delta_a = dens(u_{P(n)}, a)$ exists and we have (up to periodic behavior)

$$\lim_{N\to\infty} \#\{n < N, u_{P(n)} = b_0, \ldots, u_{P(n+k-1)} = b_{k-1}\}$$
$$= \delta_{b_0} \cdots \delta_{b_{k-1}}$$

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Let u_n be an automatic sequence and $\phi(n)$ a positive sequence such that $\phi(n)/n$ is non-decreasing.

What can be said about $u_{\lfloor \phi(n) \rfloor}$?

- We cannot expect general results for exponentially growing sequences $\phi(n)$.
- If $\phi(n) = an + b$ with integers a, b. Then $u_{\phi(n)}$ is again an automatic sequence.
- If $\phi(n) = n \log_2(n)$ then $t_{\lfloor \varphi(n) \rfloor}$ behaves like the Thue-Morse sequence t_n, but the density for blocks of length 2 does not exist. (Deshouillers + Drmota + Morgenbesser (2012))

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General Strategy

- Rewrite the statement in terms of exponential sums. E.g. $dens(t_{n^2}, 0) = 1/2$ holds if

$$\left|\sum_{n\leq N} \mathrm{e}\left(\frac{s_2(n^2)}{2}\right)\right| = o(N),$$

where $\mathrm{e}(x) = exp(2\pi ix)$.

- Use independence of „high“ and „low“ digits.
- Statement involving the discrete Fourier transform

$$F_\lambda(h,\alpha) = \frac{1}{2^\lambda}\sum_{u<2^\lambda} \mathrm{e}(\alpha s_2(u) - hu2^{-\lambda}).$$

- Recursive structure:

$$|F_\lambda(h,1/2)| \leq 2^{-\eta m}\left|F_{\lambda-m}(h,1/2)\right|.$$

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Representation of automatic sequences

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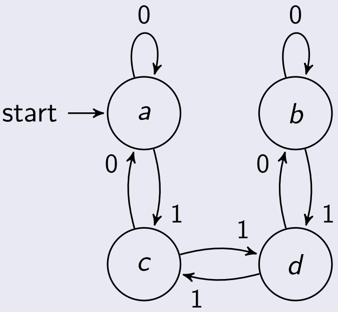

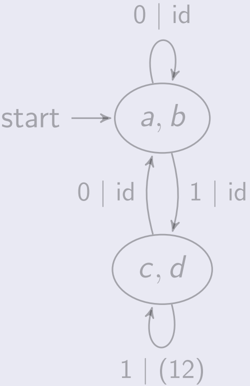

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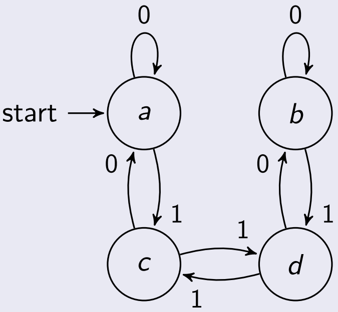

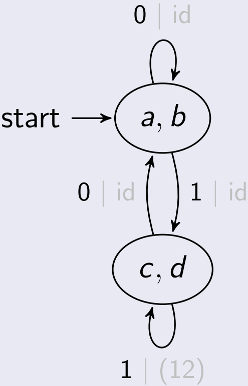

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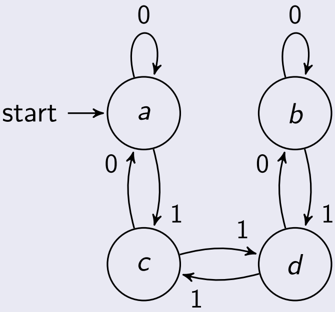

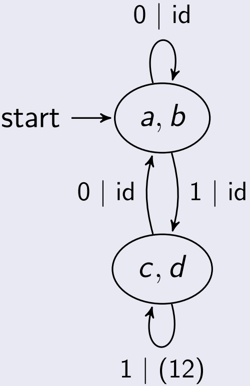

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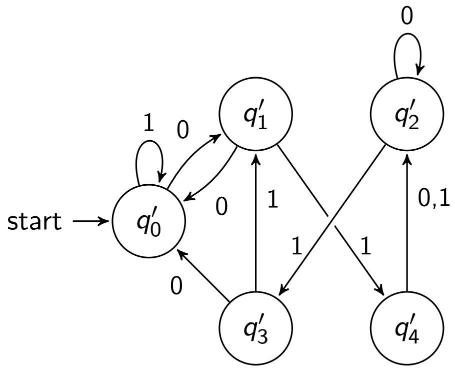

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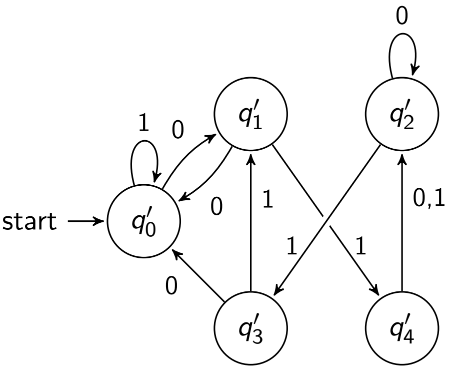

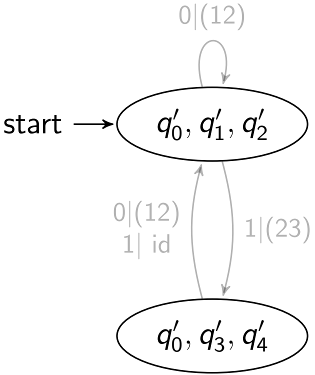

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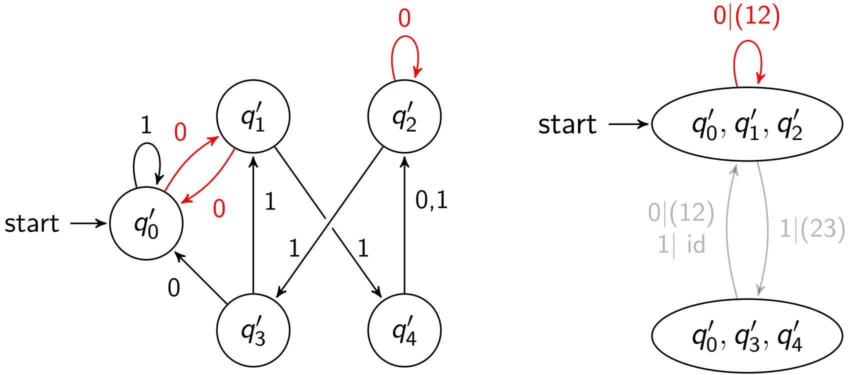

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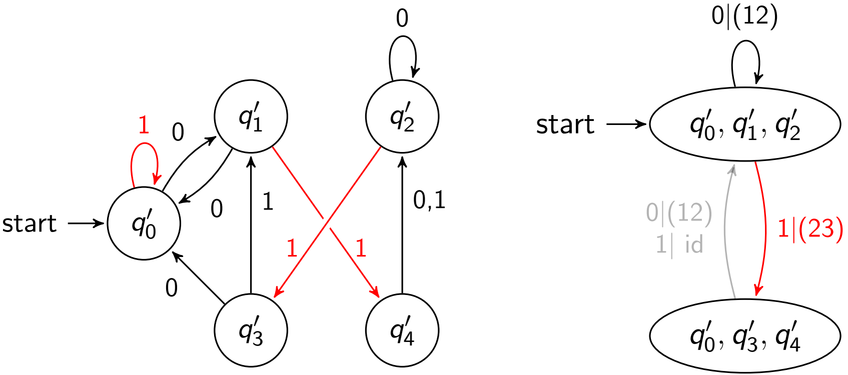

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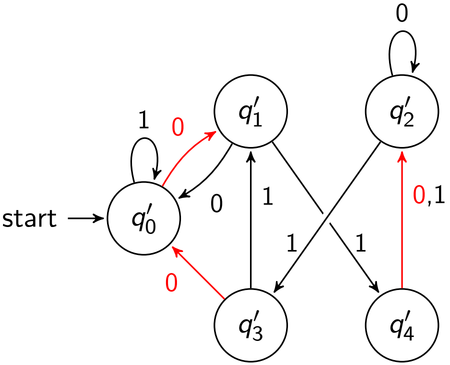

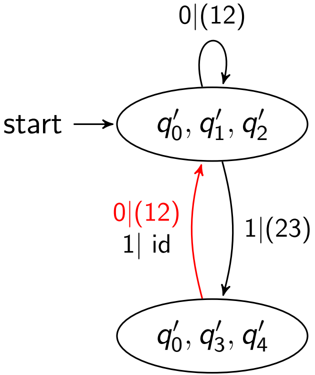

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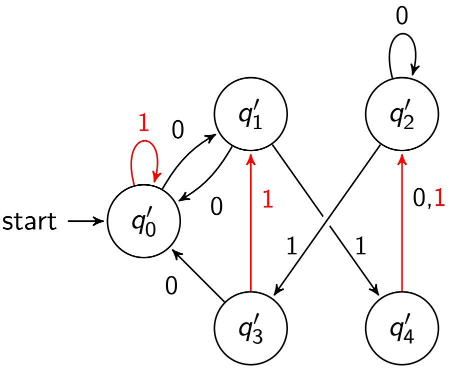

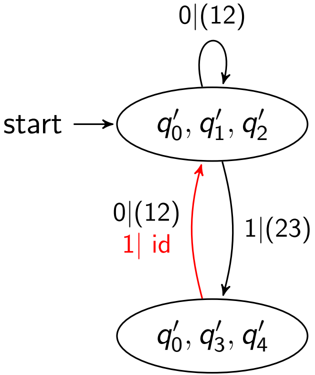

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

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$s_2(n) \bmod 2$ $\qquad$ $T(q_0, n)$

Rewrite the statement in terms of exponential sums:

$$\sum_{\ell<2} \frac{1}{2} \mathrm{e}\left(\frac{\ell(s_2(n) - a)}{2}\right) \qquad \sum_D c_D \cdot D(T(q_0, n))$$

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Theorem (Drmota, M., Spiegelhofer, 2017+)

Let $s_\varphi(n)$ be the Zeckendorf sum-of-digits function and $m(n)$ a bounded multiplicative function. Then we have

$$\sum_{n<N} (-1)^{s_\varphi(n)} m(n) = o(N) \qquad (N \to \infty).$$

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