Beyond Cobham's Theorem: Intersections of automatic sets

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TU Wien

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Deterministic Finite Automata

Definition (Automaton - DFA)

$$\mathcal{A} = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

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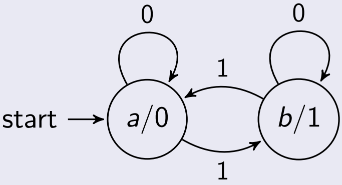

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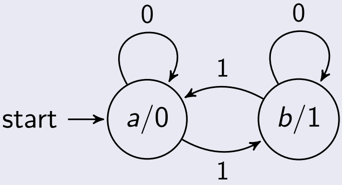

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Automatic sequences/sets

Definition

A sequence is called a k-automatic sequence if it is produced by a k-automaton. A set is called k-automatic if its indicator function is an automatic sequence.

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$$(a(n))_{n\geq 0} = 01101001100101101001011001101001\ldots$$
$$A = \{1, 2, 4, 7, 8, 11, 13, 14, 16, \ldots\}$$

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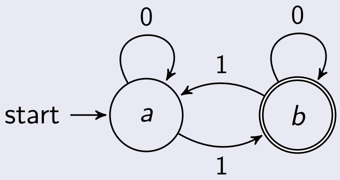

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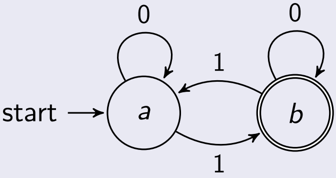

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- Relatively easy to define (structured).
- The subword complexity p_n of an automatic sequence is (at most) linear.
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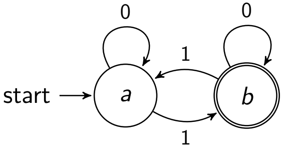
- Thue-Morse: $|A \cap [N]| \sim \frac{N}{2} = \Theta(N)$



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- Missing digits: $|A \cap [N]| = \Theta(N^{\log(3)/\log(4)})$.

- Powers of k: $|A \cap [N]| = \Theta(\log(N))$.

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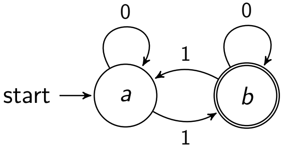

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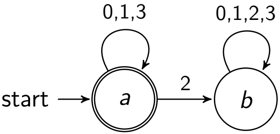

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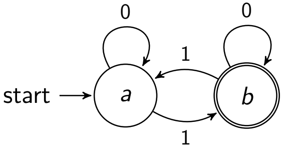

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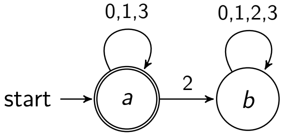

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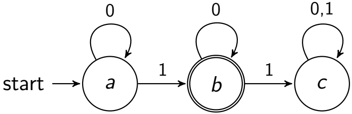

Growth of automatic sets

We distinguish three different growth types:

- Dense automatic sets: $|A \cap [N]| = \Theta(N)$
- Sparse automatic sets: there exist $0 < \alpha < 1, r \in \mathbb{N}$ s.t. $|A \cap [N]| = \Theta(N^\alpha \log^r(N))$
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Being automatic in different bases

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Can a sequence be automatic in multiple bases?

Lemma

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Theorem (Cobham - 1969)

If a sequence $(a(n))_{n \geq 0}$ is both k and l automatic, where $\log(k)/\log(l) \notin \mathbb{Q}$. Then $(a(n))_{n \geq 0}$ is eventually periodic.

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Naive hope for automatic sets

We model an automatic set A by a pseudorandom set where n is chosen with probability $\frac{|A \cap [n]|}{n}$.

If A and B are automatic sets that are "independent", then one could expect:

$$\frac{|(A \cap B) \cap [n]|}{n} \approx \frac{|A \cap [n]|}{n} \cdot \frac{|B \cap [n]|}{n}.$$

Counter example

$A = 3\,\mathbb{N}, B = \{n : s_{10}(n) \equiv 1 \bmod 3\} = 3\,\mathbb{N} + 1.$
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PNT: The number of primes $\leq x$ is asymptotically equal to $\frac{x}{\ln(x)}$.

Cramér's model

One can model the prime numbers as a pseudorandom set $\mathcal{P}'$ where n is chosen with probability $\frac{1}{\ln(n)}$.

Refined Cramér's model

Obviously no prime number (except 2) is even.
We define $\mathcal{P}'_2$ where each odd integer n is chosen with probability $\frac{2}{\ln(n)}$ and each even n with probability 0.
We can do the same for all primes $\leq w$ to obtain $\mathcal{P}'_w$.

The refined Cramér's model also captures periodic biases up to w.

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Unless there is an (obvious) periodic bias, we expect that

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If there is no periodic bias we expect for $|(A \cap B) \cap [N]|$:

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1. Gelfond Problem (Kim; 1999)

It would be interesting to prove that for coprime bases $k, l \geq 2$, and integers m_1, m_2 such that $\gcd(m_1, k-1) = \gcd(m_2, l-1) = 1$ and $r, s \in \mathbb{Z}$ the following holds. There exists some $\lambda > 0$ such that

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The base 3 expansion of every sufficiently large power of 2 contains the digit 2.

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Theorem (Glasscock, Moreira, Richter; 2025)

Let A be a set with missing digits in base k s.t. $|A \cap [N]| = \Theta(N^{\alpha})$ and B be a set with missing digits in base l s.t. $|B \cap [N]| = \Theta(N^{\beta})$, where k, l are multiplicatively independent. Then

$$|A \cap B \cap [N]| \ll N^{\max(0,\alpha+\beta-1)+o(1)}.$$

Expected upper bound for the intersection of sparse automatic sets.

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Arithmetic regularity lemma for (dense) automatic sequences

Theorem (Byszewski, Konieczny, M.; 2023)

Let $a : \mathbb{N} \to \mathbb{C}$ be a primitive k-automatic sequence. Then it has a decomposition as $a = a_{str} + a_{uni}$, where

- a_{str} is a structured part of a, i.e. it can be very well approximated by a periodic sequence.
- a_{uni} is uniform in the sense that for each $d \geq 2$ there exists $\kappa > 0$ such that $||a_{uni}||_{U^d[N]} \ll N^{-\kappa}$.

We expect a_{uni} to only behave like random noise that cancels out!

Remark: We can also assume that a_{str} and a_{uni} satisfy a *carry property* if we allow them to be matrix-valued.

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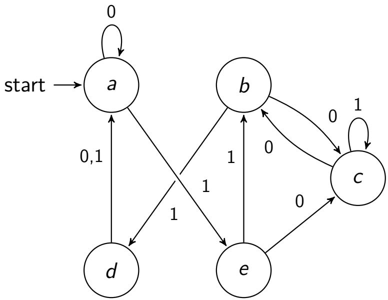

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S(n)	S_0	S_1	S_0	S_0	S_0	S_1	S_0	S_1	S_0	S_1	S_0	S_0	S_0	S_1

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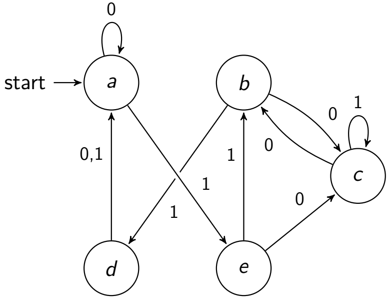

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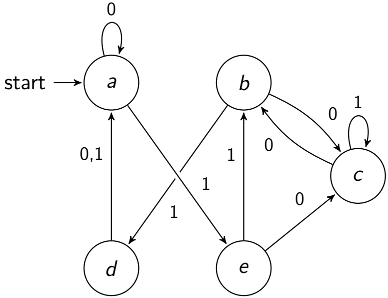

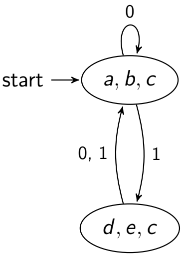

$$S_0 = \{a, b, c\}, \qquad S_1 = \{d, e, c\}$$

$u(n)$	a	e	c	b	b	c	c	d	c	d	b	c	b	c
S(n)	S_0	S_1	S_0	S_0	S_0	S_1	S_0	S_1	S_0	S_1	S_0	S_0	S_0	S_1

Structured part of an automatic sequence

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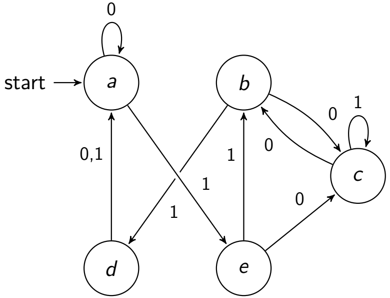

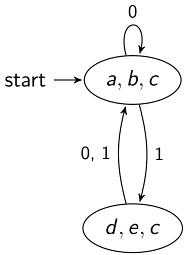

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General philosophy

- The sequence $S(n)$ gives a "coarse picture", which is highly structured, i.e. $S(n)$ is "almost periodic".
- Which element from $S(n)$ is chosen for $u(n)$ behaves "randomly".

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Addendum

Theorem (Shubin, M.; in preparation)

For pairwise coprime $q_1, \ldots, q_m$ and A_i being a q_i-automatic set, we have

$$|\mathbb{P} \cap A_1 \cap \ldots \cap A_m \cap [N]| = \sum_{p \leq N} 1_{A_1, str}(p) \cdots 1_{A_m, str}(p).$$

For A_i being sum of digits modulo m_i, we have $1_{A_i, str}$ is periodic, which recovers the presented theorem.

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Structure of sparse automatic sequences

Theorem (Adamczewski, Konieczny, M; in preparation)

Let A be a sparse automatic set with $|A \cap [N]| = \Theta(N^\alpha \log^r(N))$. Then there exists a decomposition

$$1_A = 1_{A,str} + 1_{A,uni}$$

where we have

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Thue-Morse and "no block 11".

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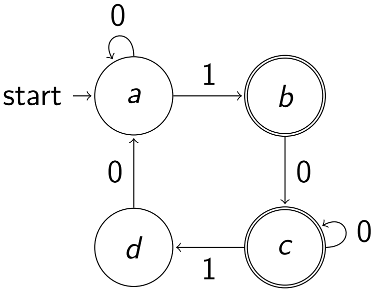

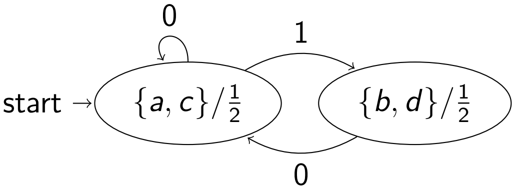

Our results (Overview)

B \ A	dense	sparse ($N^{\alpha+o(1)}$)	arid ($\log(N)^r$)
dense	✓✓	✓	✗
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Let A be a dense k-automatic set and let B be a dense l-automtic set, where k and l are multiplicatively independent.

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The first Gelfond Problem is also true for multiplicatively independent k and l (and not only for coprime k, l).

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The proof relies heavily on the result by Glasscock, Moreira, Richter which in turn utilizes recent progress by Shmerkin and Wu on Furstenberg's conjecture .

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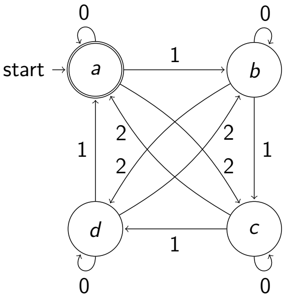
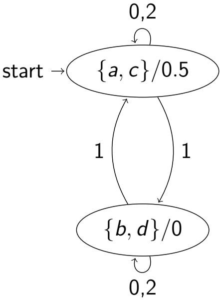
Structured part of $s_k(n)$ mod m

$$a(n) = \begin{cases} 1 & s_3(n) \equiv 0 \bmod 4 \\ 0 & \textit{otherwise} \end{cases}$$

$$a_{str}(n) = \begin{cases} \frac{1}{2} & n \equiv 0 \bmod 2 \\ 0 & \textit{otherwise} \end{cases}$$





Lemma (Adamczewski, Konieczny, M; in preparation)

If $a(n) = 1$ iff $s_k(n) \equiv r \bmod m$.
Then $a_{str}(n) = \frac{\gcd(m,k-1)}{m}$ iff $n \equiv r \bmod \gcd(m, k-1)$.