

Coprimality of consecutive elements in a Piatetski-Shapiro sequence

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To the memory of Eduard Wirsing, an inspiring mathematician

Abstract

We show that in any Piatetski-Shapiro sequence $(\lfloor n^c \rfloor)_n$ with c in $(1, +\infty) \setminus \mathbb{N}$, there exist long subsequences of consecutive elements no pair of which are coprime, whereas for any c in $(1, 2)$, there exist infinitely many n such that all the elements in $\{\lfloor n^c \rfloor, \lfloor (n+1)^c \rfloor, \dots, \lfloor (n+H)^c \rfloor\}$ are pairwise coprime for H almost as large as $\min(c-1, 1-c/2) \log n$.

1 Introduction

We pursue the study of the arithmetical properties of Piatetski-Shapiro sequences of integers, i.e. sequences $(\lfloor n^c \rfloor)_n$, where c is a non integer real number larger than one (see [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16]). The arithmetical property we are looking at here is the coprimality of consecutive elements. The two extreme cases we are looking at are, first, whether no pair of elements $(\lfloor (n+h)^c \rfloor, \lfloor (n+k)^c \rfloor)$, with $0 \leq h < k \leq H$, are coprime for some n and H as large as possible, or whether all such pairs are coprime.

In [4], expanded in [5], with L. Spiegelhofer and A. Shubin we studied the occurrence of blocks of consecutive elements in the sequence of the residues of $\lfloor n^c \rfloor$ modulo an integer m . Here, we build on our previous approach to show that there are very long sets of consecutive values of n such that the respective values of $\lfloor n^c \rfloor$ are even: such even numbers are obviously not coprime. More precisely, we have

Theorem 1. *Let $c \in (1, \infty) \setminus \mathbb{N}$. There exist a positive κ and infinitely many integers n such that for any integer h in $[0, n^\kappa]$, the numbers $\lfloor (n+h)^c \rfloor$ are even.*

On the other hand, we expect that for any c in $(1, \infty) \setminus \mathbb{N}$, there are arbitrarily long chains of consecutive elements in the Piatetski-Shapiro sequence $(\lfloor n^c \rfloor)_n$ the elements of which are pairwise coprime. However, here the classical harmonic

analysis approach to the question does not seem powerful enough. But, when c is in $(1, 2)$, an *ad hoc* combinatorial approach permits to prove the following

Theorem 2. *Let c be in $(1, 2)$ and $0 < \alpha < \min(c - 1, 1 - c/2)$. There exist infinitely many n such that, for any integer $H \leq \alpha \log n$, all the elements in the sequence $\{[n^c], [(n+1)^c], \dots, [(n+H)^c]\}$ are pairwise coprime.*

2 Proof of Theorem 1

2.1 A little lemma

Lemma 1. *Let x, u and v be real numbers with $0 \leq u \leq v < 1$ and q and a be integers with $0 \leq a < q$. The two following properties are equivalent*

$$\{x\} \in [u, v] \quad \text{and} \quad [x] \equiv a \pmod{q} \quad (1)$$

and

$$\left\{ \frac{x}{q} \right\} \in \left[\frac{a+u}{q}, \frac{a+v}{q} \right]. \quad (2)$$

Proof. The first part of (1) is equivalent to

$$\frac{x}{q} \in \left[\frac{1}{q}[x] + \frac{u}{q}, \frac{1}{q}[x] + \frac{v}{q} \right]$$

and thus (1) is equivalent to

$$\exists K \in \mathbb{Z}: \frac{x}{q} \in \left[K + \frac{a+u}{q}, K + \frac{a+v}{q} \right],$$

which is equivalent to (2). \square

2.2 Reduction of Theorem 1

By Taylor expansion, we have

$$(n+h)^c = \sum_{t=0}^{\lfloor c \rfloor} h^t \left(\left\lfloor \binom{c}{t} n^{c-t} \right\rfloor + \left\{ \binom{c}{t} n^{c-t} \right\} \right) + h^{\lfloor c+1 \rfloor} (n+\theta h)^{\{c\}-1}, \quad (3)$$

for some θ in $(0, 1)$ which depends on c, n, h . We used the binomial notation $\binom{c}{t}$ for c a real number and t a non-negative integer, namely $\binom{c}{t} = c(c-1)\cdots(c-t+1)/t!$.

Let us assume that we can find a positive τ such that for infinitely many n we have

$$\forall t \in [0, \lfloor c \rfloor] \cap \mathbb{Z}: \left\lfloor \binom{c}{t} n^{c-t} \right\rfloor \text{ is even and } \left\{ \binom{c}{t} n^{c-t} \right\} \leq n^{-\tau}. \quad (4)$$

If we take $\kappa < \min(\tau/\lfloor c \rfloor, (1 - \{c\})/\lfloor c + 1 \rfloor)$, then, when n is large enough, $(n + h)^c$ is the sum of an even integer and a positive real number less than 1 for any $h \leq n^\kappa$; thus $\lfloor (n + h)^c \rfloor$ is an even integer for $h \leq n^\kappa$ which is precisely the statement of Theorem 1. Hence, we just have to show that (4) holds.

2.3 Proof of Theorem 1

We shall use the multidimensional version of the Erdős and Turán inequality, as given in [7] (Theorem 1.21, page 15).

Proposition 1. *Let $X = (x_1, x_2, \dots, x_N)$ be a finite sequence of elements of \mathbb{R}^s and let $D_N(X)$ denote its discrepancy defined by*

$$\sup_{\substack{0 \leq a_i < b_i \leq 1 \\ 1 \leq i \leq s}} \left| \frac{1}{N} \# \left\{ n \in [1, N] : (\{x_n^1\} \cdots \{x_n^s\}) \in \prod_{i=1}^s [a_i, b_i] \right\} - \prod_{i=1}^s (b_i - a_i) \right|.$$

For any positive integer K we have

$$D_N(X) \leq \left(\frac{3}{2} \right)^s \left(\frac{2}{K+1} + \sum_{0 < \|k\|_\infty \leq K} \frac{1}{r(k)} \left| \frac{1}{N} \sum_{n=1}^N e(k \cdot x_n) \right| \right), \quad (5)$$

where $e(\bullet) = \exp(2\pi i \bullet)$, $u \cdot v$ denote the usual scalar product of two elements u and v in \mathbb{R}^s and $r(k) = \prod_{i=1}^s \max\{1, |k_i|\}$ for $k = (k_1, k_2, \dots, k_s) \in \mathbb{Z}^s$.

In order to apply Proposition 1 we need upper bounds for trigonometric sums.

Proposition 2. *Let $c > 1$ be a real number which is not an integer. There exist positive real numbers C, α, η such that for any non-zero $(\lfloor c \rfloor + 1)$ -tuple $(k_0, k_1, \dots, k_{\lfloor c \rfloor})$ of integers in $[-N^\eta, N^\eta]$ one has*

$$\sum_{n \leq N} e \left(\sum_{t=0}^{\lfloor c \rfloor} \binom{c}{t} k_t n^{c-t}/2 \right) \leq CN^{1-\alpha}. \quad (6)$$

Proof. This is a straightforward application of the results stated and proved in Chapter 2 of [8], entitled *The simplest van der Corput estimates*. We just give a hint of the proof, not mentioning that the different cases we consider separately according to the order of growth of the argument of the exponential in (6) can indeed be made uniform to lead to Proposition 2.

In the case when $(k_0, k_1, \dots, k_{\lfloor c \rfloor - 2})$ is non-zero, Proposition 2 follows from Theorem 2.8 of [8]. Assume now that $(k_0, k_1, \dots, k_{\lfloor c \rfloor - 2})$ is zero; if $k_{\lfloor c \rfloor - 1}$ is non zero, Proposition 2 follows from Theorem 2.2 of [8] and when $k_{\lfloor c \rfloor - 1} = 0$, then $k_{\lfloor c \rfloor}$ is non zero by assumption and then Proposition 2 follows from the Kusmin-Landau Theorem 2.1 of [8]. \square

Remark 1. *The number 2 in (6) may be replaced by any real number in $[N^{-\gamma}, N^\gamma]$ for a sufficiently small real γ .*

We now end the proof of Theorem 1. We consider the sequence $(x_n)_{1 \leq n \leq N}$ of elements in \mathbb{R}^s , with $s = \lfloor c \rfloor + 1$ defined by

$$x_n = \left(\binom{c}{0} n^c/2, \binom{c}{1} n^{c-1}/2, \dots, \binom{c}{\lfloor c \rfloor} n^{\lfloor c \rfloor}/2 \right)$$

and we apply Proposition 1 with $K = \lfloor N^\eta \rfloor$, where η satisfies Proposition 2. For τ such that

$$0 < (\lfloor c \rfloor + 1)\tau < \min(\alpha, \eta) \quad (7)$$

we have

$$\begin{aligned} & \left| \text{Card} \left\{ n \leq N : x_n \in [0, N^{-\tau}]^{\lfloor c \rfloor + 1} \right\} - N^{1 - (\lfloor c \rfloor + 1)\tau} \right| \\ & \ll_c \frac{N}{K} + \sum_{0 < \|k\|_\infty \leq K} \frac{1}{r(k)} N^{1-\alpha} \ll_{c,\varepsilon} N^{1 - \min(\alpha, \eta) + \varepsilon}, \end{aligned} \quad (8)$$

for any $\varepsilon > 0$. Thus, there exists an integer n in $[1, N]$ such that

$$\forall t \in [0, \lfloor c \rfloor] \cap \mathbb{Z} : \left\{ \binom{c}{t} n^{c-t}/2 \right\} \leq N^{-\tau}.$$

Using $N^{-\tau} \leq n^{-\tau}$ and applying the considerations of Subsections 2.1 and 2.2, this last inequality implies Theorem 1.

3 Proof of Theorem 2

Let c be in $(1, 2)$ and α be in $(0, \min(c-1, 1-c/2))$. We let N be a (large) integer and

$$H = \lfloor \alpha \log(2N) \rfloor \quad \text{and} \quad \Pi_H = \prod_{p \leq H} p. \quad (9)$$

3.1 Reduction

We first show the following proposition, which will be used for the proof of Theorem 2.

Proposition 3. *Let N be a sufficiently large integer, H and Π_H be defined in (9). If n in $(N, 2N]$ satisfies*

$$\{n^c\} \leq 1/3, \quad \{cn^{c-1}\} \leq 1/(3H) \quad (10)$$

$$\Pi_H \mid \lfloor cn^{c-1} \rfloor \quad (11)$$

and

$$\gcd(\lfloor n^c \rfloor, \lfloor cn^{c-1} \rfloor) = 1, \quad (12)$$

then the elements in the sequence $(\lfloor n^c \rfloor, \lfloor (n+1)^c \rfloor, \dots, \lfloor (n+H)^c \rfloor)$ are pairwise coprime.

Proof. In this part, let h and k denote any two distinct integers in $[0, H]$ and n sufficiently large an integer satisfying the hypotheses of Proposition 3.

By Taylor expansion (3), we have, for $1 < c < 2$

$$\begin{aligned} (n+h)^c &= n^c + hcn^{c-1} + (n+\theta h)^{c-2} \\ &= \lfloor n^c \rfloor + \{n^c\} + h\lfloor cn^{c-1} \rfloor + h\{cn^{c-1}\} + h^2(n+\theta h)^{c-2} \end{aligned}$$

and thus, by (10) we have for n sufficiently large

$$\lfloor (n+h)^c \rfloor = \lfloor n^c \rfloor + h\lfloor cn^{c-1} \rfloor. \quad (13)$$

Let p be a prime which is at most equal to H : by (11) it divides $\lfloor cn^{c-1} \rfloor$ and since by (12) $\lfloor n^c \rfloor$ and $\lfloor cn^{c-1} \rfloor$ are coprime, it divides $\lfloor (n+h)^c \rfloor$ for no h in $[1, H]$. Let p be a prime which divides $\lfloor (n+h)^c \rfloor$ and $\lfloor (n+k)^c \rfloor$; as we have just seen, it is larger than H and it divides $\lfloor (n+h)^c \rfloor - \lfloor (n+k)^c \rfloor = (h-k)\lfloor cn^{c-1} \rfloor$; thus it divides $\lfloor cn^{c-1} \rfloor$; thus it divides also $\lfloor n^c \rfloor$, a contradiction to (12). \square

3.2 Proof of Theorem 2

Let $K_1 = c(c-1)2^{c-2}$ and $K_2 = c(c-1)$. By the mean value theorem, we have for positive real numbers m and $\ell \leq m$

$$K_1 \ell m^{c-2} \leq c(m+\ell)^{c-1} - cm^{c-1} \leq K_2 \ell m^{c-2}. \quad (14)$$

By the prime number theorem (PNT), we have $\log \Pi_H \sim H$; since

$$\log(cN^{c-1}/\Pi_H) \sim (c-1)\log N - \alpha \log(2N) \sim (c-1-\alpha)\log N$$

the quantity (cN^{c-1}/Π_H) tends to infinity as N tends to infinity at least like a (small) power of N ; by the PNT, this implies that when N is large enough, there exists a prime number q such that

$$cN^{c-1} \leq q\Pi_H < c(3N/2)^{c-1} - 1, \quad (15)$$

and moreover, it implies that we have $q > H$. We let m be the smallest integer satisfying

$$\lfloor cm^{c-1} \rfloor = q\Pi_H$$

and notice that m is in $[N, 3N/2]$.

When N is large enough, we have $K_2 m^{c-2} \leq 1/(20H)$ and thus there exists ℓ (indeed less than m^{2-c}/K_2) such that

$$\lfloor c(m+\ell)^{c-1} \rfloor = q\Pi_H \quad \text{and} \quad \{c(m+\ell)^{c-1}\} \in \left[\frac{1}{5H}, \frac{1}{4H} \right]. \quad (16)$$

From now on, we denote $m+\ell$ as n_0 .

For $0 \leq k \leq 10H\Pi_H$, we have

$$(n_0+k)^c = \lfloor n_0^c \rfloor + \{n_0^c\} + k\lfloor cn_0^{c-1} \rfloor + k\{cn_0^{c-1}\} + O(k^2 n_0^{c-2}).$$

Because of (16), we have

$$10H\Pi_H\{cn_0^{c-1}\} \geq 2\Pi_H$$

and so, when N is large enough it is possible to find a k in the prescribed range such that

$$\lfloor n_0^c \rfloor + k \lfloor cn_0^{c-1} \rfloor + \lfloor \{n_0^c\} + k\{cn_0^{c-1}\} + O(k^2 n_0^{c-2}) \rfloor \equiv 1 \pmod{\Pi_H} \quad (17)$$

$$\lfloor n_0^c \rfloor + k \lfloor cn_0^{c-1} \rfloor + \lfloor \{n_0^c\} + k\{cn_0^{c-1}\} + O(k^2 n_0^{c-2}) \rfloor \not\equiv 0 \pmod{q} \quad (18)$$

$$\{(n_0 + k)^c\} = \{\{n_0^c\} + k\{cn_0^{c-1}\} + O(k^2 n_0^{c-2})\} \leq 1/3. \quad (19)$$

By (14) and (16), we have

$$\{c(n_0 + k)^{c-1}\} \leq 1/(4H) + 10H\Pi_H K_2 n_0^{c-2} \leq 1/(3H). \quad (20)$$

Finally, collecting (17), (18), (19) and (20), we can apply Proposition 3 with $n = n_0 + k$, thus ending the proof of Theorem 2.

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