A NOTE ON "STATE SPACES OF THE SNAKE AND ITS TOUR – CONVERGENCE OF THE DISCRETE SNAKE" BY J.-F. MARCKERT AND A. MOKKADEM

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Abstract. In *State spaces of the snake and its tour – Convergence of the discrete snake* the authors showed a limit theorem for Galton-Watson trees with geometric offspring distribution. In this note it is shown that their result holds for all Galton-Watson trees with finite offspring variance.

In [4] the following process was considered: Let $f(m)$ denote the mth vertex during a depth first search process of a Galton-Watson tree with n vertices. $V_n(m)$ denotes the distance between $f(m)$ and the root. Furthermore, an $\mathbf{R}^d$-valued random variable $y(x)$ is associated to each nonroot vertex x where the $y(x)$ are assumed to be independent. Then to each node x there corresponds a finite random walk $\rho_x = (\rho_x(j))_{j=1,\dots,h(x)}$ where $h(x)$ is the height of x and $\rho_j(x) = \sum_{i=1}^{j} y(\xi_i)$. The nodes $root, \xi_1, \dots, \xi_{h(x)} = x$ comprise the path from the root to x. Now define for integer k and j

$$W_n(k,j) := \rho_{f(k)}(j), \quad R_n(k) := W_n(k, V_n(k))$$

and by linear interpolation for non-integer values of k and j (see [4] for details). Then $(W_n(x,\cdot), V_n(y))_{(x,y)\in[0,2n]^2}$ is called the discrete snake, $(R_n(x), V_n(y))_{(x,y)\in[0,2n]^2}$ the tour of the discrete snake. In [4] it was proved:

Theorem 1. *If there exist a $p > 6$ such that $\mathbf{E}|y(x)|^p < \infty$ and $(R_n(x), V_n(y))_{(x,y)\in[0,2n]^2}$ is the tour of a discrete snake with underlying tree equal to a plane tree (i.e., Galton-Watson with geometric offspring distribution), then*

$$\left(\frac{R_n(2ns)}{n^{1/4}}, \frac{V_n 2nt}{\sqrt{n}}\right) \xrightarrow{w} (r(s), v(t))$$

where $(r(s), v(t)) = (w(s, v(s)), v(t))$ is a Brownian snake scaled by $v(s) = \sqrt{2}e(s)$ with a standard Brownian excursion $e(s)$.

In this note it is shown that, if $\mathbf{E}|y(x)|^p < \infty$ for some $p > 8$, then this theorem is true if the underlying tree is any Galton-Watson process with finite offspring variance.

Let $\check{V}_n(m,l) := \min_{m\le k\le l} V_n(k)$ and $v_n(t) = V_n(2nt)/\sqrt{n}$ and $\check{v}_n(s,t) = \check{V}_n(2ns, 2nt)/\sqrt{n}$. Then in order to generalize the theorem, it suffices to generalize [4, Th. 3.5] to all Galton-Watson trees with finite offspring variance: we must show that for $2sn, 2tn$ integers the inequality

$$\mathbf{P}\left\{|v_n(s) + v_n(t) - 2\check{v}_n(s,t)| \ge \varepsilon\right\} \le \frac{C}{|s-t|} \exp\left(-D\frac{\varepsilon}{\sqrt{|s-t|}}\right) \tag{1}$$

is true for every such Galton-Watson tree, where $C > 0$ and $D > 0$ do not depend on ε, s, and t.

We will estimate this probability by counting the trees for which the depth first search process satisfies the appropriate inequality. Therefore, let $b_{m_1k_1lm_2k_2n}$ be the weighted number of Galton-Watson trees with total progeny n, such that $V(m_1) = k_1$, $V(m_2) = k_2$, $\check{V}(m_1, m_2) = l$. In [3] it is shown that the generating function of these numbers,

$$B_{k_1lk_2}(z, u_1, u_2) = \sum_n \sum_{m_1} \sum_{m_2} b_{m_1k_1lm_2k_2n} z^n u_1^{m_1} u_2^{m_2},$$

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satisfies the relation

$$B_{k_1lk_2}(z,u_1,u_2) = A\left(z(u_1u_2)^2,u_2\right)\phi_1(z,u_1u_2,u_2)^{k_1-l-1}\phi_1(z,u_2,1)^{k_2-l-1} \\ \times \phi_1(z,u_1u_2,1)^{l-1}\phi_2(z,u_1u_2,u_2)A(z,u_2), \qquad (2)$$

where

$$A(z,u) = uz\sum_{i\ge 0}\varphi_i\sum_{j=0}^{i}a(zu^2)^j a(z)^{i-j} = uz\frac{a(zu^2)\varphi(a(zu^2)) - a(z)\varphi(a(z))}{a(zu^2)-a(z)},$$

$$\phi_1(z,u,v) = uvz\frac{\varphi(a(zu^2)) - \varphi(a(zv^2))}{a(zu^2)-a(zv^2)}$$

$$\phi_2(z,u,v,w) = z\sum_{i\ge 2}\varphi_i\sum_{j_1+j_2+j_3=i-2}a(zu^2)^{j_1}a(zv^2)^{j_2}a(zw^2)^{j_3}$$

and $a(z)$ is the generating function for Galton-Watson trees and satisfies a functional equation of the form $a(z) = z\varphi(a(z))$ for some power series $\varphi(t) = \sum_i \varphi_i t^i$ with $\varphi(0) > 0$. It is well known that $a(z)$ (see [2] for a treatment of general functional equations) has a positive singularity on the circle of convergence which we will denote by $z_0 > 0$ in the sequel. Moreover, without loss of generality we may assume that z_0 is the only singularity on the circle of convergence.

Let τ denote the solution of $t\varphi'(t) = \varphi(t)$ and σ^2 the offspring variance. Then with $m_1 = \lfloor \mu_1 n\rfloor$ and $m_2 = \lfloor \mu_2 n\rfloor$ we have

$$\mathbf{P}\left\{\left|v_n\left(\frac{\lfloor\mu_1 n\rfloor}{n}\right) + v_n\left(\frac{\lfloor\mu_2 n\rfloor}{n}\right) - 2\check{v}_n\left(\frac{\lfloor\mu_1 n\rfloor}{n},\frac{\lfloor\mu_2 n\rfloor}{n}\right)\right| \ge \varepsilon\right\}$$
$$= \frac{1}{[z^n]a(z)}[z^n u_1^{m_1}u_2^{m_2}]\sum_{\substack{k,l,m\ge 1\\ |k+m-2l|\ge\lfloor\varepsilon\sqrt{n}\rfloor}} B_{klm}(z,u_1,u_2)$$
$$= \frac{1}{[z^n]a(z)}[z^n u^m v^l]\left(\frac{\phi_1(z,v,1)^{\varepsilon\sqrt{n}} - \phi_1(z,u,v)^{\varepsilon\sqrt{n}}}{(1-\phi_1(z,u,v)/\phi_1(z,v,1))(1-\phi_1(z,v,1))(1-\phi_1(z,u,1))} + \frac{\phi_1(z,u,v)^{\varepsilon\sqrt{n}}}{(1-\phi_1(z,u,v))(1-\phi_1(z,v,1))(1-\phi_1(z,u,1))}\right) \qquad (3)$$

where we used (2) and the substitution $u = u_1$, $v = u_1u_2$ and consequently $m = \lfloor\mu_1 n\rfloor$ and $l = \lfloor(\mu_2-\mu_1)n\rfloor$ in the last step. By Lemma 3.1 in [3] we have the local expansion

$$a(zu^2) \sim \tau - \frac{\tau}{\sigma\sqrt{2}}\sqrt{-\frac{t}{n}-\frac{2s}{m}}$$

for $z = z_0\left(1+\frac{t}{n}\right)$, $u = 1+\frac{s}{m}$ and $n,m\to\infty$ where $m\sim cn, c>0$ and $s,t = o\left(\sqrt{n}\right)$. Consequently, in the same range for z and u and with $v = 1+\frac{r}{l}$ we have (for details cf. [1])

$$\phi_1(z,u,v) \sim 1 - \frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{t}{n}-\frac{2s}{m}} + \sqrt{-\frac{t}{n}-\frac{2}{r}l}\right).$$

Note that there is a representation of the form

$$y(z,u) = \tilde{g}(z,u) - \tilde{h}(z,u)\sqrt{1-\frac{u}{\tilde{f}(z)}}$$

as well, where $g(z,u)$, $h(z,u)$, and $f(z)$ are analytic functions satisfying

$$g(z_0,1) = \tau, h(z_0,1) = \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}} = \frac{\tau\sqrt{2}}{\sigma},\text{ and } f(z_0) = 1.$$

If we call the function on the right-hand side of (3) $F(z,u,v)$. then the desired coefficient can be expressed in terms of the Cauchy integral

$$[z^n u^m v^l]F(z,u,v) = \frac{1}{(2\pi i)^3} \int\limits_{\Gamma_z}\int\limits_{\Gamma_u}\int\limits_{\Gamma_v} \frac{F(z,u,v)}{z^{n+1}u^{m+1}v^{l+1}}\,dv\,du\,dz \tag{4}$$

where the integration contour $\Gamma_z = \Gamma_{z1} \cup \Gamma_{z2} \cup \Gamma_{z3} \cup \Gamma_{z4}$ is chosen as follows:

$$\Gamma_{z1} = \left\{ z = z_0\left(1+\frac{e^{it}}{n}\right) \middle| \alpha \le |t| \le \pi \right\}$$

$$\Gamma_{z2} = \left\{ z = z_0\left(1+\frac{t}{n}e^{i\alpha}\right) \middle| 1 \le |t| \le \log^2 n \right\}$$

$$\Gamma_{z3} = \overline{\Gamma}_{z2}$$

$$\Gamma_{z4} = \left\{ z \middle| |z| = z_0\left|1+\frac{\log^2 n}{n}e^{i\alpha}\right|, \arg\left(1+\frac{\log^2 n}{n}e^{i\alpha}\right) \le |\arg z| \le \pi \right\}$$

The contours Γ_u and Γ_v are identical with Γ_z up to a suitable shift (depending on z) in order to follow the singularity when z is varying (see [3, pp. 452] for a detailed description)

Note that the above expansion of the function ϕ_1 as well as the integrand in (4) are very similar to the ones which occur in the proof of tightness for the contour of Galton-Watson trees (see [3]). Hence we can argue exactly in the same way as in [3, pp.454]: First we estimate the denominator in (3) and show that $Cn^{3/2}$ is an upper bound. Next, applying [3, Lemma 3.5] to the numerator immediately yields the upper bound

$$\frac{C_1}{ml}\exp\left(-C_2\left(\frac{\varepsilon}{\sqrt{\mu_1}}+\frac{\varepsilon}{\sqrt{\mu_2-\mu_1}}\right)\right)\int \frac{|f(z)|^{-l-m}}{|z^{n+1}|}\,dz.$$

Provided that μ_1 and μ_2 stay away from 1, the integral can be shown to be $\mathcal{O}\left(1/z_0^n n\right)$ which implies the exponential bound (1).

If μ_1 and μ_2 are arbitrarily close to 1 (the case where only one of the two values is close to one is trivial, since in this case the distance $|s-t|$ is large), then in [3] we showed the exponential bound

$$\mathbf{P}\left\{v_n\left(\frac{m}{n}\right) \ge \varepsilon\right\} = \frac{C_1}{a_n}[z^{n-1}u^{m-1}]\frac{\phi_1(z,u,1)^k}{1-\phi_1(z,u,1)} \le \frac{C_2}{(n-m)m}\exp\left(-C_3\left(\frac{\varepsilon\sqrt{n}}{\sqrt{m}}+\frac{\varepsilon\sqrt{n}}{\sqrt{n-m}}\right)\right),$$

where C_1, C_2, C_3 are appropriate constants. This in conjunction with the estimate

$$\mathbf{P}\left\{|v_n(\mu_1)+v_n(\mu_2)-2\check{v}_n(\mu_1,\mu_2)| \ge \varepsilon\right\} \le \mathbf{P}\left\{v_n(\mu_1)\ge\varepsilon\right\} + \mathbf{P}\left\{v_n(\mu_2)\ge\varepsilon\right\}$$

yields (1).

References

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