Analytic Combinatorics of Lattice Paths with Forbidden Patterns: Enumerative Aspects

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Abstract. This article presents a powerful method for the enumeration of pattern-avoiding words generated by an automaton or a context-free grammar. It relies on methods of analytic combinatorics, and on a matricial generalization of the kernel method. Due to classical bijections, this also gives the generating functions of many other structures avoiding a pattern (e.g., trees, integer compositions, some permutations, directed lattice paths, and more generally words generated by a push-down automaton). We focus on the important class of languages encoding lattice paths, sometimes called generalized Dyck paths. We extend and refine the study by Banderier and Flajolet in 2002 on lattice paths, and we unify several dozens of articles which investigated patterns like peaks, valleys, humps, etc., in Dyck and Motzkin words. Indeed, we obtain formulas for the generating functions of walks/bridges/meanders/excursions avoiding any fixed word (a *pattern*). We show that the autocorrelation polynomial of this forbidden pattern (as introduced by Guibas and Odlyzko in 1981, in the context of regular expressions) still plays a crucial role for our algebraic functions. We identify a subclass of patterns for which the formulas have a neat form. En passant, our results give the enumeration of some classes of self-avoiding walks, and prove several conjectures from the On-Line Encyclopedia of Integer Sequences. Our approach also opens the door to establish the universal asymptotics and limit laws for the occurrence of patterns in more general algebraic languages.

Keywords: Lattice paths, pattern avoidance, finite automata, autocorrelation, generating function, kernel method, asymptotic analysis

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1 Introduction

Combinatorial structures having a rational or an algebraic generating function play a key role in many fields: computer science (analysis of algorithms involving trees, lists, words), computational geometry (integer points in polytopes, maps, graph decomposition), bioinformatics (RNA structure, pattern matching), number theory (integer compositions, automatic sequences and modular properties, integer solutions of varieties), probability theory (Markov chains, directed random walks), see e.g. [3, 12, 21, 35]. They are often the trace of a structure which has a recursive specification in terms of a system of tree-like structures, or of some functional equation solvable by variants of the kernel method [13].

Since the seminal article by Chomsky and Schützenberger on the link between context-free grammars and algebraic functions [15], which also holds for push-down automata [33], many articles encoded and enumerated combinatorial structures via a formal language approach. See e.g. [19, 25, 30] for such an approach on the so-called *generalized Dyck languages*. These languages are in fact equivalent to directed lattice paths, and in this article, we try to understand how some of these fundamental objects can be enumerated when they have the additional constraint to avoid a given pattern. For sure, such a class of objects can be described as the intersection of a context-free language and a rational language; therefore, classical closure properties imply that they are directly generated by another (but huge and clumsy) context-free language. Unfortunately, despite the fact the algebraic system associated with the corresponding context-free grammar is *in theory* solvable by a resultant computation or by Gröbner bases, this leads *in practice* to equations which are so big that no current computer could handle them in memory, even for generalized Dyck languages with 20 different letters.

In this article, we introduce a generic and efficient way to tackle the question of enumerating words avoiding a given pattern (for languages generated by push-down automata) which bypass these intractable equations. For directed lattice paths, our method allows to handle an arbitrary number of letters (i.e., allowed jumps), up to alphabets of thousands of letters, computationally in a few minutes. It relies on an analytic combinatorics approach, and also on the kernel method, which we used in our investigation of enumerative and asymptotic properties of lattice paths [4–7]. This allows to unify the considerations of many articles which investigated natural patterns like peaks, valleys, humps, etc., in Dyck and Motzkin words, corresponding patterns in trees, compositions..., see e.g. [9, 10, 14, 16–18, 20, 26, 29, 31] and all the examples mentioned in our Section 7.

2 Definitions and Notations

Let $\mathcal{S}$, the *set of steps* (or *jumps*), be some finite subset of $\mathbb{Z}$, that contains at least one negative and at least one positive number. A *lattice path with steps from* $\mathcal{S}$ is a finite word $w = [v_1, v_2, \ldots, v_n]$ in which all letters belong to $\mathcal{S}$, visualized as a directed polygonal line in the plane, which starts in the origin and is formed by successive appending of vectors $(1, v_1), (1, v_2), \ldots, (1, v_n)$. The letters that form the path $w = [v_1, v_2, \ldots, v_n]$ are referred to as its *steps*. The *length* of w, to be denoted by $|w|$, is the number of steps in w. The *final altitude* of w, to be denoted by $h(w)$, is the sum of all steps in w, that is $v_1 + v_2 + \ldots + v_n$; visually, it is the y-coordinate of the point where w terminates.

Under this setting, it is usual to consider two restrictions: being the whole path (weakly) above the x-axis, and having final altitude 0 (equivalently, terminating at the x-axis). Consequently, one considers four classes of lattice paths:

1. A *walk* is any path as described above.
2. A *bridge* is a path that terminates at the x-axis.
3. A *meander* is a path that stays (weakly) above the x-axis.
4. An *excursion* is a path that stays (weakly) above the x-axis and also terminates at the x-axis. In other words, an excursion fulfills both restrictions.

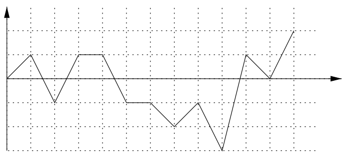
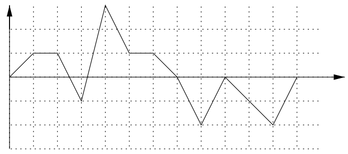
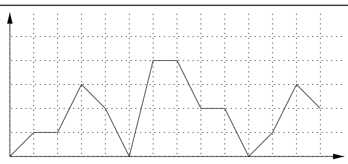
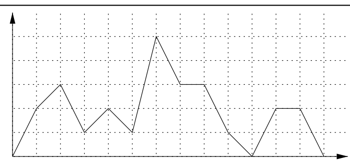
	ending anywhere	ending at 0
on $\mathbb{Z}$	 walks $W(t,u) = \frac{R(t,u)}{K(t,u)}$	 bridges $B(t) = \sum_{i=1}^{e} \frac{u_i'}{u_i} \frac{t}{1 + \frac{R'(t,u_i)}{R(t,u_i)^2} t^{\ell+1} u_i^b - \frac{(\ell-1)t^\ell u_i^b}{R(t,u_i)}}$
on $\mathbb{N}$	 meanders $M(t,u) = \frac{R(t,u)}{K(t,u)} \prod_{i=1}^{c} (u - u_i(t))$	 excursions $E(t) = \frac{(-1)^{c+1}}{t} \prod_{i=1}^{c} u_i(t)$

Table 1. Summary of our results. For the four types of paths and for any set of jumps encoded by $P(u)$, we give the corresponding generating function of such lattice paths avoiding a pattern p (of length ℓ and final altitude b). The formulas involve the autocorrelation polynomial $R(t,u)$ of p, and the small roots u_i of the kernel $K(t,u) := (1 - tP(u))R(t,u) + t^\ell u^b$.

For each of these classes (when no pattern is forbidden), Banderier and Flajolet [4] gave general expressions for the corresponding generating functions and the asymptotics of their coefficients. In the generating functions, the variable t corresponds to the length of a path, and the variable u to its final altitude. $P(u)$ is the *characteristic polynomial* of the set of steps $\mathcal{S}$, defined by $P(u) = \sum_{s \in S} u^s$.

The smallest (negative) number in $\mathcal{S}$ is denoted by $-c$, and the largest (positive) number in $\mathcal{S}$ is denoted by d: that is[1], if one orders the terms of $P(u)$ by the powers of u, one has $P(u) = u^{-c} + \ldots + u^d$.

[1] Some weights (or probabilities, or multiplicities) could be associated with each jump, but we omit them in this article to keep readability. All the proofs would be similar.

3 Lattice Paths with Forbidden Patterns, and Autocorrelation Polynomial

We consider lattice paths with step set $\mathcal{S}$ that avoid a certain *pattern*, that is, an a priori fixed path $p = [a_1, a_2, \ldots, a_\ell]$. To be precise, we define an *occurrence* of p in a lattice path w as a substring of w which coincides with p. If there is no occurrence of p in w, we say that w *avoids* p. For example, the path $[1, 2, 3, 1, 2]$ has two occurrences of $[1, 2]$, but it avoids $[2, 1]$.

Before we state our results, we introduce some notations.

A *presuffix* of p is a non-empty string that occurs in p both as a prefix and as a suffix. In particular, the whole word p is a (trivial) presuffix of itself. If p has one or several non-trivial presuffixes, we say that p exhibits an *autocorrelation* phenomenon. For example, for the pattern $p = [1, 1, 2, 1, 2]$ we have no autocorrelation. In contrast, the pattern $p = [1, 1, 2, 3, 1, 1, 2, 3, 1, 1]$ has three non-trivial presuffixes: $[1]$, $[1, 1]$, and $[1, 1, 2, 3, 1, 1]$, and thus in this case we have autocorrelation.

While analysing the Boyer–Moore string searching algorithm and properties of periodic words, Guibas and Odlyzko introduced in 1981 [22] what turns out to be one of the key characters of our article, the autocorrelation polynomial[2] of the pattern p: For any given word p, let $\mathcal{Q}$ be the set of its presuffixes; the *autocorrelation polynomial* of p is

$$R(t, u) = \sum_{q \in Q} t^{|\bar{q}|} u^{h(\bar{q})}, \tag{1}$$

where $\bar{q}$ denotes the complement of q in p.

For example, consider the pattern $p = [1, 1, 2, 3, 1, 1, 2, 3, 1, 1]$. Its four presuffixes produce four terms of $P(t, u)$ as follows:

q	$\lvert\bar{q}\rvert$	$h(\bar{q})$
$[1]$	9	15
$[1, 1]$	8	14
$[1, 1, 2, 3, 1, 1]$	4	7
$[1, 1, 2, 3, 1, 1, 2, 3, 1, 1]$	0	0

Therefore, for this p we have $R(t, u) = 1 + t^4u^7 + t^8u^{14} + t^9u^{15}$.

Notice that if for some p no autocorrelation occurs, then we have $\mathcal{Q} = \{p\}$ and therefore $R(t, u) = 1$.

Finally, we define the *kernel* as the following Laurent polynomial:

$$K(t, u) := (1 - tP(u))R(t, u) + t^{|p|}u^{h(p)}. \tag{2}$$

Also in our case it can be shown that each root $u = u(t)$ of $K(t, u) = 0$ is either small or large, and that the number of small roots is $e := \max\{c, -h(p)\}$: we shall denote them by $u_1, \ldots, u_e$.

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[2] A similar notion also appears in the work of Schützenberger on synchronizing words [34].

Now we can state our main results. Recall that t is the variable for the length of a path, and u is the variable for its final altitude.

Theorem 1. *Let $\mathcal{S}$ be a set of steps, and let p be a pattern with steps from $\mathcal{S}$. Denote $\ell = |p|$, $b = h(p)$.*

1. *The bivariate generating function for walks avoiding the pattern p, is*
$$W(t,u) = \frac{R(t,u)}{K(t,u)}. \tag{3}$$
If one does not keep track of the final altitude, this yields
$$W(t) = W(t,1) = \frac{1}{1 - tP(1) + t^{\ell}/R(t,1)}. \tag{4}$$
2. *The generating function for bridges avoiding the pattern p is*
$$B(t) = t \sum_{i=1}^{e} \frac{u_i'(t)}{u_i(t)} \frac{R(t,u_i(t))}{R(t,u_i(t)) + \frac{(\partial_t R)(t,u_i(t))}{R(t,u_i(t))} t^{\ell+1} u_i(t)^b - (\ell-1) t^{\ell} u_i(t)^b}, \tag{5}$$
where $u_1, \ldots, u_e$ are the small roots of the kernel $K(t,u)$, as defined in (2).

Definition. For meanders and excursions, the formulas have a noteworthy shape when the pattern p is a *pseudomeander*, i.e. a lattice path which does not cross the x-axis, except, possibly, at the last step. Notice that if the pattern p is a pseudomeander, then $h(p) \geq -c$, and therefore, the number of small roots of $K(t,u)$ is c.

Theorem 2. *Let $\mathcal{S}$ be a set of steps, and let p be a pseudomeander. Denote $\ell = |p|$, $b = h(p)$.*

1. *The bivariate generating function $M(t,u)$ for meanders avoiding the pattern p is*
$$M(t,u) = \frac{R(t,u)}{u^c\, K(t,u)} \prod_{i=1}^{c} \big(u - u_i(t)\big), \tag{6}$$
where $u_1(t), \ldots, u_c(t)$ are the small roots of $K(t,u) = 0$.
If one does not keep track of the final altitude, this yields
$$M(t) = M(t,1) = \frac{R(t,1)}{K(t,1)} \prod_{i=1}^{c} \big(1 - u_i(t)\big). \tag{7}$$
2. *The generating function for excursions avoiding the pattern p is*
$$E(t) = M(t,0) = \frac{(-1)^{c+1}}{t} \prod_{i=1}^{c} u_i(t) \tag{8}$$
if $b > -c$; if $b = c$, then one has $t - t^{\ell}$ rather than t in the denominator.

N.B.: When p is not a pseudomeander, there are still some algebraic expressions, but not as concise.

Remark. Notice that for these four classes of lattice paths, if one forbids a pattern of length 1 or if one uses symbolic weights for each jump, this recovers the formulas from Banderier and Flajolet [4].

4 Automaton and Transfer Matrix A

The following automata, sharing the spirit of the Knuth–Morris–Pratt algorithm, will allow us to tackle pattern avoidance. Let $p = [a_1, \ldots, a_\ell]$ be the "forbidden" pattern. As we construct a lattice path $w = [v_1, v_2, \ldots]$ step by step, it might start "accumulating" p, that is, contain some prefix of p. We introduce ℓ many *states* that indicate how much of p has the path w accumulated at each step. These states, $X_0, X_1, \ldots, X_{\ell-1}$, will be labelled by proper prefixes of p: $X_i = [a_1, a_2, \ldots, a_i]$ (in particular, the first state is labelled by the empty word: $X_0 = \epsilon = [\,]$). We say that $w_{1..m} = [v_1, v_2, \ldots, v_m]$, a prefix of w, is in the state X_i (or, alternatively: w, after its mth step, is in the state X_i), if the label of X_i is the longest proper prefix of p that coincides with a suffix of $w_{1..m}$: $v_{m-i+1} = a_1$, $v_{m-i+2} = a_2$, …, $v_m = a_i$.

If w is in the state X_i after certain step v_m, then its state after the next step v_{m+1} is uniquely determined by i and v_{m+1} (unless $i = \ell - 1$ and $v_{m+1} = a_\ell$ which is impossible since this would yield an occurrence of the forbidden pattern). This gives a finite automaton completely determined by $P(u)$ and p. Its states are labelled by $X_0, \ldots, X_{\ell-1}$, and for $i, j \in \{0, \ldots, \ell-1\}$ we have an arrow labelled λ from X_i to X_j if j is the maximum number such that X_j is a suffix of $X_i\lambda$. Its transition matrix will be denoted by A: it is an $\ell \times \ell$ matrix, and its (i, j) entry is the sum of all terms u^λ such that there is an arrow labelled λ from X_{i-1} to X_{j-1}. See Figure 1 for an example.

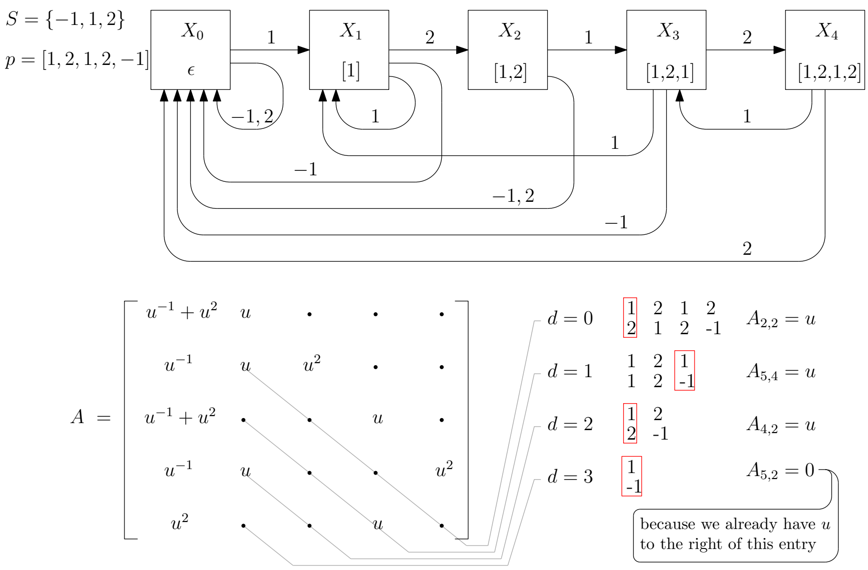

Fig. 1. The automaton and the transfer matrix A for $\mathcal{S} = \{-1, 1, 2\}$ and the pattern $p = [1, 2, 1, 2, -1]$. (The 0 entries of A are replaced by dots.)

The matrix A has several general properties. In particular, for all i, j such that $j > i+1$, we have $A_{i,j} = 0$; for each i, $1 \le i \le \ell - 1$, we have $A_{i,i+1} = a_i$; for each j, $2 \le j \le \ell$, every entry in the jth column is either 0 or a_{j-1}; for each i, $1 \le i \le \ell - 1$, the sum of the entries in the ith row is $P(u)$; the sum of the entries in the ℓ-th row is $P(u) - a_\ell$.

The "essential" entries of A are those with $2 \le j \le i$. They can be determined by the following procedure, which is also illustrated in Figure 1. For $d = 0, 1, \ldots, \ell-2$ we compare $[a_1, a_2, \ldots, a_{\ell-d-1}]$ with $[a_{d+2}, a_{d+3}, \ldots, a_\ell]$. If they coincide ($a_\beta = a_{d+1+\beta}$ for all $\beta = 1, 2, \ldots, \ell - d - 1$), then all the entries $A_{i,j}$ with $i - j = d$, $j \ge 2$, are 0. Otherwise, if β is the smallest number such that $a_\beta \ne a_{d+1+\beta}$, then $A_{d+\beta+1,\beta+1} = u^\beta$, unless a smaller d yielded u^β in the same row, to the right of this position (if this happens, $A_{d+\beta+1,\beta+1} = 0$).

5 The Structure of the Kernel: $\det(I - tA)$

In what follows, an important role will be played by the matrix $I - tA$, specifically by its determinant and by the sum of elements in the first row of its adjoint. In particular the role of $\det(I - tA)$ in our study is the analog of the role played by $1 - tP(u)$ in the study of Banderier and Flajolet [4], but our equation is more involved.

Theorem 3. *Let $\mathcal{S}$ be a set of steps, and let p be a pattern with steps from $\mathcal{S}$. Then for A, the transfer matrix of the automaton, we have*

$$\det(I - tA) = K(t,u) = (1 - tP(u))R(t,u) + t^{|p|}u^{h(p)}, \tag{9}$$

$$(1\ 0\ \cdots\ 0)\,\mathrm{adj}(I - tA)\,(1\ 1\ \cdots\ 1)^\top = R(t,u), \tag{10}$$

where $K(t,u)$ and $R(t,u)$ are the kernel and the autocorrelation polynomial, as defined in Equations (1) and (2). In particular, in the case without autocorrelation we have $\det(I - tA) = 1 - tP(u) + t^{|p|}u^{h(p)}$, and the sum of the entries in the first row of $\mathrm{adj}(I - tA)$ is 1.

Proof (sketch). Consider first that every step s of S has a symbolic weight t_s. Let W be the generating function of walks avoiding p with these weights, R the autocorrelation polynomial of p, $P = \sum_s t_s$ the polynomial encoding the different steps s, and t_p the weight of p. Using the construction of [21, p. 60], we find that: $W = \frac{R(t,u)}{(1-P(u))R(t,u)+t_p}$. By Cramer's rule, W is also equal to the left-hand side of (10) divided by the left-hand side of (9). Since $(1 - P(u))R(t,u) + t_p$ is an irreducible polynomial with degree ℓ, the two rational representations of W are identical. We conclude by specializing t_s to $tu^{h(s)}$ for all s. ☐

6 Proofs of the Generating Functions for Walks, Bridges, Meanders, and Excursions

Proof of Theorem 1 for walks. Let $W(t,u)$ be the bivariate generating function for the number of walks with steps from $\mathcal{S}$ avoiding a fixed pattern p. For $\alpha = 1, 2, \ldots, \ell$, we denote by $W_\alpha = W_\alpha(t,u)$ the corresponding bivariate generating function restricted to those walks that terminate in state α. Then we have the following vectorial functional equation:

$$(W_1\ W_2\ \cdots\ W_\ell) = (1\ 0\ \cdots\ 0) + t\ (W_1\ W_2\ \cdots\ W_\ell)\ A,$$
$$(W_1\ W_2\ \cdots\ W_\ell)\ (I - tA) = (1\ 0\ \cdots\ 0),$$
$$(W_1\ W_2\ \cdots\ W_\ell) = (1\ 0\ \cdots\ 0)\ \frac{\operatorname{adj}(I-tA)}{|I-tA|}.$$

Therefore, the generating function for $W(t,u)$, which is the sum of the generating functions $W_a(t,u)$ over all states, is equal to

$$W(t,u) = (W_1\ W_2\ \cdots\ W_\ell)\ (1\ 1\ \cdots\ 1)^\top =$$
$$= \frac{(1\ 0\ \cdots\ 0)\operatorname{adj}(I-tA)\,(1\ 1\ \cdots\ 1)^\top}{|I-tA|} = \frac{R(t,u)}{(1-tP(u))R(t,u) + t^{|p|}u^{h(p)}},$$

where the last equality follows from Theorem 3. □

Proof of Theorem 1 for bridges. In order to find the univariate generating function $B(t)$ for bridges, we need to extract the coefficient of $[u^0]$ from $W(t,u)$. To this end, we assume that t is a sufficiently small fixed number, extract the coefficient of a (univariate) function by means of Cauchy's integral formula, and apply the residue theorem:

$$B(t) = [u^0]W(t,u) = \frac{1}{2\pi i}\int_{|u|=\varepsilon} \frac{W(t,u)}{u}\,du = \sum_i^e \operatorname{Res}_{u=u_i(t)} \frac{W(t,u)}{u},$$

where $u_1, \ldots, u_e$ are the small roots of $K(t,u)$. By the formula for residues of rational functions, we have

$$\operatorname{Res}_{u=u_i(t)} \frac{W(t,u)}{u} = \operatorname{Res}_{u=u_i(t)} \frac{R(t,u)}{u\left((1-tP(u))R(t,u) + t^\ell u^b\right)} =$$
$$= \left.\frac{R(t,u)}{\frac{d}{du}\left(u\left((1-tP(u))R(t,u) + t^\ell u^b\right)\right)}\right|_{u=u_i(t)}.$$

The denominator of this expression is

$$-tuP'(u)R(t,u) + u(1-tP(u))R_u(t,u) + bt^\ell u^b\big|_{u=u_i(t)}. \tag{11}$$

Next, we differentiate $K(t,u_i) = 0$ with respect to t and obtain an expression for $P'(u_i(t))$. When we substitute it to (11), we obtain (5). □

Proof of Theorem 2 for meanders and excursions. Similarly to our notation above, we denote by $M(t,u)$ the bivariate generating function for meanders, and by $M_\alpha(t,u)$ the bivariate generating function for meanders that terminate in state α. Then we have the following vectorial functional equation:

$$(M_1\ M_2\ \cdots\ M_\ell) = (1\ 0\ \cdots\ 0) + t\,(M_1\ M_2\ \cdots\ M_\ell)\ A - t\,[u^{<0}]\big((M_1\ M_2\ \cdots\ M_\ell)\,A\big),$$

where $[u^{<0}]$ denotes all the terms in which the power of u is negative. The first component of $[u^{<0}]\big((M_1\ M_2\ \cdots\ M_\ell)\,A\big)$ is a sum of several fractions of the form $M_i^j(t,u)/u^\gamma$, where $M_i^j(t,u)$ is the generating function for p-avoiding meanders that terminate in state i at altitude j, and $1 \le \gamma \le c$. We denote this expression by $F(t,u)/u^c$, where $F(t,u)$ is polynomial in t and u. All other components of $[u^{<0}]\big((M_1\ M_2\ \cdots\ M_\ell)\,A\big)$ are 0 because if the path arrives at state X_i with $i > 1$ this means that it accumulated a non-empty prefix of p. And since p is a pseudomeander, w will always remain (weakly) above the x-axis while it accumulates its non-empty prefix. Thus we obtain

$$(M_1\ M_2\ \cdots\ M_\ell)\ (I - tA) = \left(1 - \frac{t}{u^c}F(t,u)\right)(1\ 0\ \cdots\ 0), \tag{12}$$

$$(M_1\ M_2\ \cdots\ M_\ell) = \left(1 - \frac{t}{u^c}F(t,u)\right)(1\ 0\ \cdots\ 0)\frac{\operatorname{adj}(I - tA)}{|I - tA|}. \tag{13}$$

Finally, we multiply it the last identity by $(1\ 1\ \cdots\ 1)$ from the right and obtain, through the use of Theorem 3,

$$M(t,u) = \left(1 - \frac{t}{u^c}F(t,u)\right)\frac{R(t,u)}{K(t,u)}. \tag{14}$$

In order to determine $1 - tF(t,u)/u^c$, we proceed as follows. First, the kernel $K(t,u)$ has precisely c small roots. This follows from the fact that the lowest degree of u in $K(t,u)$ is $-c$: it is contributed by u^{-c} in $P(u)$, while the lowest degree of u in $t^\ell u^b$ is at least $-c$; and this these terms cannot cancel out (the only exception is the trivial case of a one-letter pattern); then it can be shown (for example, by the Newton polygon method) that $K(t,u)$ has c many (distinct) roots whose Puiseux series starts with $const \cdot u^{1/c}$. Further, it can be show in a similar way that all other roots of $K(t,u)$ are large. As usual, we denote the small roots by $u_1, \dots, u_c$.

For each $1 \le i \le c$, we substitute $u = u_i$ into (12). Then the matrix $I - tA$ is singular and therefore it has a (right) eigenvector $\mathbf{v}$ that belongs to the eigenvalue 0. Moreover, the first component of $\mathbf{v}$ is not 0: otherwise the columns of $I - tA$, with the first column excluded, are linearly dependent, which contradicts the structure of the matrix. Then comparing the first components yields the identity $1 - \frac{t}{u_i^c}F(t,u) = 0$, which means that each u_i, $i = 1, \dots, c$, satisfies $u^c - tF(t,u) = 0$, which is a polynomial equation of degree c. This implies $u^c - tF(t,u) = (u - u_1)(u - u_2)\dots(u - u_c)$, We substitute this into (14) and finally obtain the claimed result (6).

Setting $u = 0$ in Formula (6) gives the formula for excursions. □

7 Examples

We end our article with a small set of examples, see Table 2. We can also link some self-avoiding walks and pattern avoiding lattice paths. In fact, the enumeration and the asymptotics of self-avoiding walks in $\mathbb{Z}^2$ is one of the famous open problems of combinatorics and probability theory. As it is classical for intractable problems, many natural subclasses have been introduced, and solved. Our lattice paths avoiding some pattern allow to enumerate many of these subclasses of self-avoiding walks, like partially directed self-avoiding walks with an added constraint of living in a half-plane or a strip [2]. Partially directed walks have three kinds of steps, say n, e and s, and the self-avoiding condition means that factors ns and sn are disallowed. Consider the following three models:

- in the first model, the half-plane is the one over the line $x = 0$; the heights of the steps are $h(\mathsf{n}) = 1$, $h(\mathsf{e}) = 0$ and $h(\mathsf{s}) = -1$;
- in the second model, the half-plane is the one over the line $x = y$; the heights are $h(\mathsf{n}) = 1$, $h(\mathsf{e}) = -1$ and $h(\mathsf{s}) = -1$;
- in the third model, the half-plane is the one over the line $x = -y$; the heights are $h(\mathsf{n}) = 1$, $h(\mathsf{e}) = 1$ and $h(\mathsf{s}) = -1$.

These models are illustrated in Figure 2. Each of them leads to an algebraic generating function, compatible with our formulas.

steps, pattern, model	generating function	OEIS reference[3]
$S = \{-1, 0, 1\}$ $p = [1, 0, \dots, 0, -1]$ bridges	$\frac{1}{\sqrt{1-2t-3t^2+2t^\ell-2t^{\ell+1}+t^{2\ell}}}$	$\ell = 2$: OEIS A051286 (proving a claim therein on Whitney numbers, see also [11])
$S = \{-1, 0, 1\}$ $p = [1, 0, \dots, 0, -1]$ meanders	$\frac{1-3t+t^\ell-\sqrt{1-2t-3t^2+2t^\ell-2t^{\ell+1}+t^{2\ell}}}{2t(1-3t+t^\ell)}$	$\ell = 2$: OEIS A091964 (RNA folding, see [23])
$S = \{-1, 0, 1\}$ $p = [1, 0, \dots, 0, -1]$ excursions	$\frac{1-t+t^\ell-\sqrt{1-2t-3t^2+2t^\ell-2t^{\ell+1}+t^{2\ell}}}{2t^2}$	$\ell = 2$: OEIS A004148 [24] $\ell = 3$: OEIS A114584 [27]
$S = \{1, -1\}$ $p = [1, -1, 1, -1, \dots, 1]$ excursions	$\frac{1-t^{\ell+1}-\sqrt{1-4t^2+2t^{\ell+1}+4t^{\ell+3}-3t^{2\ell+2}}}{2t^2(1-t^{\ell-1})}$	$\ell = 3$: $\approx$ OEIS A001006 (Motzkin numbers [32, 36])
$S = \{1, -1\}$ $p = [1, -1, 1, -1, \dots, -1]$ excursions	$\frac{1-t^{\ell+2}-\sqrt{1-4t^2+6t^{\ell+2}-4t^{2\ell+2}+t^{2\ell+4}}}{2t^2(1-t^\ell)}$	$\ell = 4$: OEIS A078481 (irreducible stack sortable permutations, [8, 28])

Table 2. Our results provide a unified approach for the computation of generating functions for different models. In particular, this solves several conjectures in the On-Line Encyclopedia of Integer Sequences, it also allows to produce many formulas: this recovers several earlier works, often related to Dyck/Motzkin paths.

[3] Such references are links to the web-page dedicated to the corresponding sequence in the On-Line Encyclopedia of Integer Sequences, http://oeis.org.

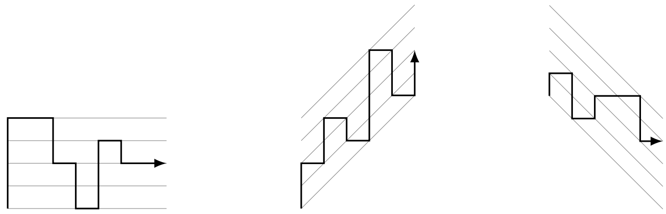

Fig. 2. Some models of self-avoiding walks are encoded by partially directed lattice paths avoiding a pattern (see [2]).

8 Conclusion and Extensions

In this article, we showed how a vectorial generalization of the kernel method allows to enumerate lattice paths avoiding a given pattern. Our approach is flexible, and our future researches include the case of several patterns at once and the general study (enumeration, limit laws) of counting the *number of occurrences* of a given pattern in algebraic structures. The asymptotics are interestingly much more involved than in [4], we analyse these aspects in our companion article [1].

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