THE DYING FIBONACCI TREE

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1. Introduction

Consider a tree with two types of nodes, say A and B, and the following properties:

1. Let the root be of type A.
2. Each node of type A produces exactly one descendent of each type with probability p and no descendent with probability $1-p$.
3. Nodes of type B produce one descendent of type A with probability p and no descendent with probability $1-p$.

If $p=1$ then the resulting tree is the Fibonacci tree. It can easily be verified that the number of A's in the n-th layer equals the n-th Fibonacci number F_n and the number of B's equals F_{n-1}. Let A_n and B_n denote the number of A's and B's, respectively, in the first n layers of the tree. Then we have

$$\frac{A_n}{A_n+B_n}=\frac{\sum_{i=1}^n F_i}{\sum_{i=2}^{n+1} F_i}=1-\frac{F_{n+1}-F_1}{\sum_{i=2}^{n+1} F_i}$$

Using the well known representation of the Fibonacci numbers $F_n=(\alpha^n-\alpha^{-n})/\sqrt{5}$ where $\alpha=(1+\sqrt{5})/2$ we immediately get

$$\frac{A_n}{A_n+B_n}=1-\frac{\alpha^{n+1}-\alpha^{-n-1}-\alpha+1/\alpha}{(\alpha^2(1-\alpha^n)-\alpha^{-1}(1-\alpha^{-n}))/(1-\alpha)}\sim\frac{1}{\alpha}=\frac{\sqrt{5}-1}{2} \tag{1}$$

We are interested in the distribution of the number of A's and B's conditioned on the total number of nodes for the case $p<1$. In this case there occur trees with a finite number of nodes with positive probability and due to (1) we might conjecture that the ratio $A_n/(A_n+B_n)$ behaves similarly for trees conditioned on the tree size to be n if p is close to 1. This is the topic of the next section. The last section is devoted to the connection between the dying Fibonacci tree and branching processes.

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2. The number of type A nodes in the dying Fibonacci tree

We will consider now the case $p < 1$. Let T denote a dying Fibonacci tree, T_A and T_B the number of A's and B's, respectively, and $|T|$ the total number of nodes. Set $a_{nm} = P\{T_A = n, T_B = m\}$ and $q = 1 - p$. Let $A(u,v) = \sum_{n,m\geq 0} a_{nm} u^n v^m$ be the probability generating function associated to a_{nm}. Furthermore let $B(u,v)$ be the analogous generating function for trees that start with a root of type B. Due to the construction of the dying Fibonacci tree we have the following relations between $A(u,v)$ and $B(u,v)$:

$$
\begin{aligned}
A(u,v) &= u(q + pA(u,v)B(u,v)) \\
B(u,v) &= v(q + pA(u,v))
\end{aligned}
$$

and thus

$$A(u,v) = u(q + vpqA(u,v) + vp^2A^2(u,v)).$$

From this we get

Theorem 2.1. *The probability that a tree with exactly n A's and exactly m B's occurs is given by*

$$P\{T_A = n, T_B = m\} = \frac{1}{n}\binom{n}{m}\binom{m}{n-m-1}p^{n-1}q^{m+1}$$

Proof. The above probability is given by the coefficients a_{nm} which may be determined explicitly by means of Lagrange's inversion formula. We have

$$[u^n]A(u,v) = \frac{1}{n}[z^{n-1}](q + vpqz + vp^2z^2)^n.$$

This implies

$$
\begin{aligned}
[u^n v^m]A(u,v) &= \frac{1}{n}[z^{n-1}v^m](q + vpqz + vp^2z^2)^n \\
&= \frac{1}{n}[z^{n-1}]\binom{n}{m}q^{n-m}p^m(qz + pz^2)^m \\
&= \frac{1}{n}\binom{n}{m}q^{n-m}p^m[z^{n-1-m}](q + pz)^m \\
&= \frac{1}{n}\binom{n}{m}\binom{m}{n-m-1}p^{n-1}q^{m+1}
\end{aligned}
$$

and we are done. □

The distribution of the number of A's in trees of size n is given by

$$\frac{a_{m,n-m}}{\sum_{i+j=n} a_{ij}}.$$

In order to get some information on the behavior of these quantities we modify the generating function $A(u,v)$ to $A(xu,x)$ such that it keeps track on the number of A's and the tree size and use as a lemma the following result of Drmota[2]:

Lemma 2.1. *Let $A(x,u) = \sum_{n,k\geq 0} a_{nk}x^n u^k = \sum_{n\geq 0} \varphi_n(u)x^n$ be a generating function of non-negative numbers $a_{n,k}$ such that there are n_1, n_2, n_3, $k_1 < k_2 < k_3$ with $a_{n_1k_1}a_{n_2k_2}a_{n_3k_3} > 0$ and $\gcd(k_3-k_2, k_2-k_1) = 1$. Set $d = \gcd\{n-l : \varphi_n(u) \neq 0\}$ where $l = \min\{m > 0 : \varphi_m(u) \neq 0\}$. Furthermore let $A(x,u)$ satisfy a functional equation $A = F(A,x,u)$ where the expansion $F(A,x,u) = \sum f_{ijk}A^i x^j u^k$ has non-negative coefficients and suppose that the system of equations*

$$A = F(A,x,u)$$

$$1 = F_A(A,x,u)$$

has positive solutions $A = f_1(u)$, $x = f_2(u)$ for $u \in [a,b]$ such that $(f_1(u), f_2(u), u)$ are regular points of $F(A,x,u)$. In addition suppose that $F_x(f_1(u), f_2(u), u)$ and $F_{AA}(f_1(u), f_2(u), u)$ are positive. Then we have

$$a_{nk} = \frac{d}{2\pi n^2}\frac{g(h(k/n))}{\sigma(h(k/n))}\frac{1}{h(k/n)^k f_2(h(k/n))^n}\left(1+\mathcal{O}\left(n^{-1/2}\right)\right)$$

uniformly for $k/n \in [\mu(a), \mu(b)]$ and $n \equiv l \bmod d$, where

$$g(u) = \left(\left[\frac{xF_x}{F_{AA}}\right](f_1(u), f_2(u), u)\right)^{1/2},$$

$$\mu(u) = \left[\frac{uF_u}{xF_x}\right](f_1(u), f_2(u), u),$$

and $h(u)$ is the inverse function of $\mu(u)$.

If $1 \in (a,b)$ then discrete random variables X_n with $P\{X_n = k\} = a_{nk}/\varphi_n(1)$ are asymptotically normal with mean $EX_n = \mu(1)n + \mathcal{O}(1)$ and variance $\mathcal{O}(n)$. Furthermore we have

$$\varphi_n(1) = \frac{d}{\sqrt{2\pi}} g(1) f_2(1)^{-n} n^{-3/2}(1+\mathcal{O}\left(n^{-1}\right)), \quad \text{as } n \to \infty. \tag{2}$$

As a consequence we get

Theorem 2.2. *Let T be a dying Fibonacci tree and p close to 1. Then the distritbution of the random variable T_A/n conditioned on $|T| = n$ is asymptotically normal with mean value*

$$\mu = \frac{2}{3}\left(1 + \frac{1}{2}q^{1/3} + \mathcal{O}\left(q^{2/3}\right)\right).$$

and variance $\mathcal{O}(1/n)$. Besides, we have

$$P\{|T| = n\} = \frac{g}{\sqrt{2\pi}}\rho^{-n}n^{-3/2}(1 + \mathcal{O}\left(n^{-1}\right))$$

where

$$g = \frac{\sqrt{q}}{\sqrt[3]{2}}\left(1 + \frac{1}{12\sqrt[3]{2}}q^{1/3} + \mathcal{O}\left(q^{2/3}\right)\right), \quad \text{as } q \to 0, \tag{3}$$

and

$$\rho = \frac{1}{\sqrt[3]{4q}}\left(1 - \frac{1}{3\sqrt[3]{2}}q^{1/3} + \mathcal{O}\left(q^{2/3}\right)\right), \quad \text{as } q \to 0.$$

Remark . This means that for p close to 1 large Fibonacci trees contain about twice as many type A nodes as type B nodes and so the conjecture stated in the introduction, namely that the ratio will be close to the golden ratio, is surprisingly false.

Proof. Obviously, $A(xu, x)$ satisfies the functional equation

$$A = F(A, x, u) = xuq + x^2upqA + x^2up^2A^2.$$

Thus we have to show that the system

$$A = xuq + x^2upqA + x^2up^2A^2$$

$$1 = x^2upq + 2x^2up^2A$$

has positive solution $A = f_1(u)$ and $x = f_2(u)$ for $u \in (a, b)$ for some interval (a, b). As the first equation is quadratic in A we can get an explicit expression for A:

$$A = \frac{1 - x^2upqA - \sqrt{x^4p^2q^2u^2 - 4x^3p^2qu^2 - 2x^2pqu + 1}}{2x^2up^2A^2}$$

The second equation means that we have to set the discriminant equal to zero:

$$x^4p^2q^2u^2 - 4x^3p^2qu^2 - 2x^2pqu + 1 = 0. \tag{4}$$

Note that the left hand side is positive if $x = 0$ and negative if $u = 1$ and $x = 1/\sqrt{pq}$. Thus there exists a positive root of the above equation if u lies near 1. Consequently there exists an interval (a, b) containing 1 such that for $u \in (a, b)$ the above system has positive solutions

f_1 and f_2. Furthermore, it is easy to verify that the other assumptions of Lemma 2.1 are also fulfilled. Thus the number of A's in trees of size n is asymptotically normally distributed with mean $\mu(1)$ and variance $\mathcal{O}\left(n^{-1}\right)$. Now let us study the mean in detail, especially for p tending to 1. We have already seen that $x = \mathcal{O}\left(1/\sqrt{pq}\right)$. If q tends to zero then $p^2q/(\sqrt{pq})^3 \to \infty$ while the other terms of (4) remain bounded. Thus $x = o(1/\sqrt{pq})$. This implies that the third order term is the dominant one and we get

$$x = \frac{y}{\sqrt[3]{4p^2q}}, \quad \text{as } q \to 0,$$

where $y = 1 + w$ with $w = o(1)$. Using this and keeping in mind that $y^k = 1 + kw + \mathcal{O}\left(w^2\right)$ and that $p = 1 + \mathcal{O}\left(q\right)$ we get

$$\begin{aligned}
&\frac{1}{4}\left(y^2\sqrt[3]{\frac{q}{2}}\right)^2 - y^3 - y^2\sqrt[3]{\frac{q}{2}} + 1 = 0 \\
\Longrightarrow\ &\frac{1}{4}\sqrt[3]{\frac{q^2}{4}} - 3w - \sqrt[3]{\frac{q}{2}} + o(w) = 0 \\
\Longrightarrow\ &w \sim -\frac{1}{3}\sqrt[3]{\frac{q}{2}}
\end{aligned}$$

Set $s = \sqrt[3]{q/2}$. We will now use this information to get a better asymptotic result via bootstrapping as demonstrated by de Bruijn[1]. We have

$$\begin{aligned}
&\frac{1}{4}(1 + 4w + \mathcal{O}\left(w^2\right))s^2 - 3w - 3w^2 + \mathcal{O}\left(w^3\right) - (1 + 2w + \mathcal{O}\left(w^2\right))s = 0 \\
\Longrightarrow\ &\frac{s^2}{4} - 3w - 3w^2 - s - 2sw + \mathcal{O}\left(w^3\right) = 0 \\
\Longrightarrow\ &w^2 + w\left(1 + \frac{2s}{3}\right) + \frac{s}{3} - \frac{s^2}{12} + \mathcal{O}\left(s^3\right) = 0
\end{aligned}$$

Solving the quadratic equation yields

$$w = -\frac{s}{3} + \frac{7}{36}s^2 + \mathcal{O}\left(s^3\right)$$

and consequently

$$\begin{aligned}
x &= \frac{1}{\sqrt[3]{4q}}\left(1 - \frac{1}{3}\sqrt[3]{\frac{q}{2}} + \frac{7}{36}\sqrt[3]{\frac{q^2}{4}} + \mathcal{O}\left(q\right)\right) \\
&= \frac{1}{\sqrt[3]{4q}}\left(1 - \frac{1}{3}\sqrt[3]{\frac{q}{2}} + \mathcal{O}\left(q^{2/3}\right)\right), \quad \text{as } q \to 0. \qquad (5)
\end{aligned}$$

Since

$$A = \frac{1 - x^2pqA}{2x^2p^2A^2}$$

we get

$$A = \frac{1 - \frac{q^{1/3}}{2\sqrt[3]{2}}\left(1 - \frac{q^{1/3}}{3\sqrt[3]{2}} + \mathcal{O}\left(q^{2/3}\right)\right)}{\frac{q^{-2/3}}{\sqrt[3]{2}}\left(1 - \frac{2q^{1/3}}{3\sqrt[3]{2}} + \mathcal{O}\left(q^{2/3}\right)\right)}$$

$$= \sqrt[3]{2q^2}\left(1 + \frac{1}{6\sqrt[3]{2}}q^{1/3} + \mathcal{O}\left(q^{2/3}\right)\right), \quad \text{as } q \to 0. \tag{6}$$

The mean value $\mu(1)$ we are searching for is given by

$$\mu(1) = \left[\frac{uF_u}{xF_x}\right](A(x(1),1), x(1), 1) = \frac{A}{xq + 2x^2pqA + 2x^2p^2A^2}$$

Using the asymptotic expansions for x and A we get

$$\mu(1) = \frac{2}{3}\left(1 + \frac{1}{2}q^{1/3} + \mathcal{O}\left(q^{2/3}\right)\right).$$

The second statement is an immediate consequence of (2): Note that

$$g(1) = \left(\left[\frac{xF_x}{F_{AA}}\right](f_1(1), f_2(1), 1)\right)^{1/2},$$

$$= \sqrt{\frac{q + xA + xA^2}{2x}}\,(1 + \mathcal{O}(q))$$

and thus inserting (5) and (6) we obtain (3). □

3. The dying Fibonacci tree and branching processes

This section is devoted to the connection between the dying Fibonacci tree and branching processes. We will first present a few basic facts of the theory of branching processes. The reader who is interested in detail may e.g. consult [3].

Consider a particle that produces ξ children after one time unit and assume that ξ is a random variable on the natural numbers. Denote by Z_i the number of particles of the i-th generation (thus $Z_0 = 1$). The stochastic process $(Z_n; n \geq 0)$ is called branching process if the following conditions are fulfilled:

1. The value of Z_{n+1} only depends on Z_n, i.e. $(Z_n; n \geq 0)$ is a Markov chain.
2. The numbers of children of the particles are independent and identically distributed with the distribution of ξ.

Let $\xi_k = P\{\xi = k\} = P\{Z_1 = k\}$. Then the probability generating function associated to the branching process is

$$f(z) = \sum_{k \geq 0} \xi_k z^k$$

and $EZ_1 = f'(1)$. Depending on the value of $f'(1)$ three classes of branching processes can be distinguished: If $f'(1) < 1$ then the process is called subcritical, for $f'(1) > 1$ it is called supercritical and for $f'(1) = 1$ it is called critical. For subcritical processes we have $EZ_n \to 0$, in the supercritical case $EZ_n \to \infty$ holds and in the critical case we have $EZ_n = 1$. The total number of particles that is produced is called the total progeny. It can be shown that $P\{\text{total progeny} = n\}$ tends to zero polynomially if the process is critical and exponentially otherwise.

If a branching process consists of several types of particles then a similar situation occurs. Let a_{ij} be the expectation of the number of particles of type j produced by a particle of type i. Then the indicator for criticality is the largest positive eigenvalue ρ of the matrix (a_{ij}). If $\rho < 1$ the process is subcritical and the expected generation sizes tend to zero. For $\rho > 1$ the process is supercritical and for $\rho = 1$ it is critical. $P\{\text{total progeny} = n\}$ behaves in the same way as for single type branching processes.

The dying Fibonacci tree may obviously be regarded as a branching process with two types of particles. Now let us examine for which p the dying Fibonacci tree is a critical branching process. The matrix of the expectations a_{ij} is given by

$$\begin{pmatrix} p & p \\ p & 0 \end{pmatrix}$$

and the eigenvalues are the solutions of

$$\lambda^2 - p\lambda - p^2 = 0.$$

Thus the largest positive eigenvalue is $p(1+\sqrt{5})/2$. This implies that the dying Fibonacci tree yields a critical branching process if and only if p equals the golden ratio. This fits also with the behaviour of the total progeny: We have by Lemma 2.1

$$P\{\text{total progeny} = n\} = \varphi_n(1) = \frac{1}{\sqrt{2\pi}} x(1)^{-n} n^{-3/2}$$

and $p = (\sqrt{5}-1)/2$ is the only value for which $x(1) = 1$ as can easily be seen by setting $x = 1$ and $u = 1$ in (4).

Let us investigate the expected number of type A particles if p is the golden ratio. It is easy to see that $A(x(1), 1) = 1$ and $xF_x = 1 - p + 2p(1-p) + 2p^2 = 1 + p$ and thus we get

Theorem 3.1. *The dying Fibonacci tree yields a critical branching process if and only if p equals the golden ratio and in this case the ratio of the number of type A nodes and the total number of nodes conditioned on the total progeny tends to the golden ratio.*

References

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[3] B.A. Sevastyanov, *Verzweigungsprozesse,* Akademie–Verlag, Berlin, 1974.