Combinatorial Models for Cooperation Networks

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Abstract. We analyze special random network models – so-called *thickened trees* – which are constructed by random trees where the nodes are replaced by local clusters. These objects serve as model for random real world networks. It is shown that under a symmetry condition for the cluster sets a local-global principle for the degree distribution holds: the degrees given locally through the choice of the cluster sets directly affect the global degree distribution of the network. Furthermore, we show a superposition property when using clusters with different properties while building a thickened tree.

1 Introduction

There has been substantial interest in random graph models where vertices are added to the graph successively and are connected to several already existing nodes according to some given law. The so-called Albert-Barabási model (see [1]) joins a new node to an existing one with probability proportional to the degree. This is called preferential attachment and the motivation for introducing such models is to model various *real-word graphs* like the internet or social networks.

It turns out that the preferential attachment rule of the Albert-Barabási model is not in an unambiguous way. One rigorous approach is due to [3]. They introduced a random multigraph which is built of random forests which are then formed into multigraphs by partitioning the vertex set and identifying vertices in the same block of the partition. It was shown in [4] that the degree distribution of these graphs satisfies asymptotically a *power law*, that is, the graphs are *scale free*. Furthermore, the models fulfils the preferential attachment rule given by Albert and Barabási.

2 Thickened trees

The model of thickened trees was introduced in [5]. The starting point was to construct a model of scale free graphs which is locally clustered, but the global structure is tree-like. This fits with observations from real world networks. In particular the design of this model was motivated to describe cooperation networks, where one usually has small groups with a stong interaction and some connections to other groups. Of course, there might be circles in a cooperation network but they are usually rare so that we neglect them.

The idea how to get such a graph is to start with a scale free tree and then "thicken" the tree by substituting the nodes by clusters. Then the initial tree causes the global tree-like structure while the inserted clusters cause the local, highly clustered structure.

The clusters are not produced by an evolution process. Nevertheless we think that our model has several advantages and can be used to explain several properties that are observed in practice:

- There is large flexibility in choosing the structure of local clusters and, thus, can be adapted to the situation.
- The model is feasable for an analytic treatment.
- It can be used to study (analytically) the influence of local changes of the network to the global behaviour.

In the following we present a brief explanation how the model is constructed: We first introduce an evolution process that leads to a labelled plane rooted tree. The process starts with the root that is labeled with 1. Then inductively at step j a new node (with label j) is attached to any previous node of out-degree k with probability proportional to $k+1$. These kinds of trees are also called *plane oriented recursive trees* (PORTs).

Note that if a node v of a plane (rooted) tree has out-degree k, then there are exactly $k+1$ ways of attaching a new node to v, each leading to a different plane tree. Of course, if v is different from the root then its degree d equals $d = k+1$.

PORTs are scale free trees. The degree distribution is given by

$$\lim_{n\to\infty} p_n(d) = \frac{4}{d(d+1)(d+2)} \sim \frac{4}{d^3},$$

where $p_n(d)$ denotes the probability that a random node in random PORT of size n has degree d (see [9] or [8]).

Now, we introduce a substitution process that creates random graphs that have a global tree structure that is governed by plane oriented recursive trees.

For every $k \geq 0$ let $\mathcal{T}_k$ denote a non-empty set of labelled graphs with *half edges* attached to their nodes in such a way that each graph receives in total $k+1$ half edges. The half edges are also ordered from 0 to k. Now consider the following random process. Take a tree T according to the PORT-model. Then we substitute every node v of out-degree k in the following way: cut v and one half of each edge incident with v. Then take a randomly chosen graph G of $\mathcal{T}_k$ and *glue* the $k+1$ half edges of G to those left in T by the cutting of v respecting the given order, that is, the half edge coming from the predecessor of v is glued to the 0th half edge of G and the 1st, 2nd,..., kth successor of v is attached to the 1st, 2nd, ..., kth half edge of G, respectively. Further we relabel all nodes in the new graph $G = G(T)$ in a way that is consistent with the original labelling. We denote the graphs that are obtained by this process *thickened trees* or more precisely *thickened PORTs*.

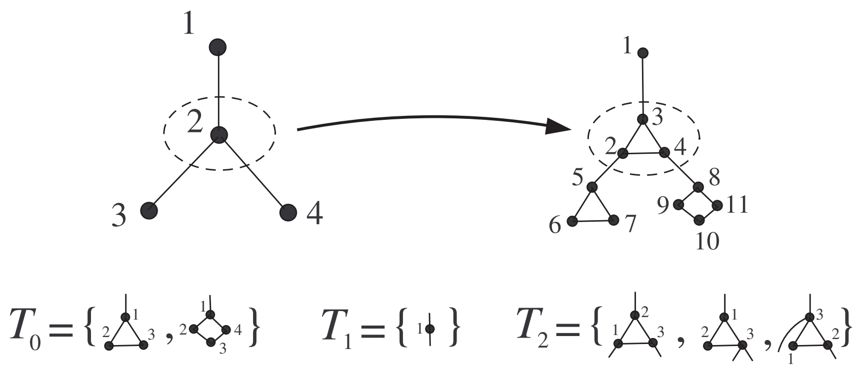

Fig. 1. A simple example of a thickened PORT. The original tree has only nodes of out-degree 0,1, or 2. So the choice of all sets $\mathcal{T}_k$ with $k > 2$ is not relevant for the thickening process. For example, the node with label 2 is cut along the circular dashed line. Since it has out-degree 2, we choose one of the three graphs in $\mathcal{T}_2$ (here the first one was chosen) and glue it into the corresponding space. Applying the same procedure to all nodes and relabelling afterwards yields the graph on the right hand side, a thickened PORT.

3 Generating functions for thickened PORTs and a Local-Global-Principle

Consider the formal solution $y = y(z, x_0, x_1, x_2, \ldots)$ of the differential equation

$$y' = \sum_{k \geq 0} x_k y^k,$$

where $'$ denotes differentiation with respect to z. Then $y = y(z, x_0, x_1, x_2, \ldots)$ can be considered as a power series in $z, x_0, x_1, \ldots$ By construction the coefficient

$$[z^n x_0^{k_0} x_1^{k_1} \ldots]\, y(z, x_0, x_1, x_2, \ldots)$$

is exactly the number of PORTs T of size n and with k_j nodes of out-degree j ($j \geq 0$).

For every $k \geq 0$ let $\mathcal{T}_k$ denote a non-empty set of labelled graphs with $k+1$ additional *half edges* $\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_k$. Furthermore, let

$$t_k(z) = \sum_{G \in \mathcal{T}_k} \frac{z^{|G|}}{|G|!}$$

denote the exponential generating function of these graphs.

The generating function of the numbers g_n of thickened trees with n vertices, $g(z) = \sum_{n \geq 1} g_n \frac{z^n}{n!}$, is then

$$g(z) = y(z, t_0(z)/z, t_1(z)/z, \ldots).$$

We are interested in the number $N_d(G)$ of nodes of degree d in a graph G. Therefore we consider the bivariate generating function

$$t_k^{(d)}(z,u) = \sum_{G \in \mathcal{T}_k} \frac{z^{|G|}}{|G|!} u^{N_d(G)},$$

Here the the half-edges $\tilde{e}_0, \ldots, \tilde{e}_k$ contribute to the node degrees as well. Then the generating function

$$g(z,u) = y(z, t_0^{(d)}(z,u)/z, t_1^{(d)}(z,u)/z, \ldots)$$

encodes the distribution of nodes of degree d of thickened trees.

Set

$$T_d(z,y,u) = \frac{1}{z} \sum_{k \geq 0} t_k^{(d)}(z,u) y^k.$$

Then the following results were shown in [5].

Lemma 1. *Set*

$$G_d(z,y,u) = \int_0^y \frac{dt}{T_d(z,t,u)}.$$

Then $g(z,u)$ satisfies the functional equation

$$G_d(z,g,u) = z.$$

Theorem 1. *Let $\mathcal{T}_k$ be substitution sets (as described above) such that the equation*

$$X = \int_0^1 \frac{dt}{T_d(X,t,1)}$$

has a unique positive solution $X = \rho$ in the region of convergence of $T_d(z,y,u)$ and that $T_d(z,y,u)$ can be represented as

$$T_d(z,y,u) = \frac{C_0(z,y) + C_1(z,y)(1-y)^{r'} y^{d+\alpha}(u-1) + \mathcal{O}\left((1-y)^{r'}(u-1)^2\right)}{(1-y)^r}, \tag{1}$$

where r' and r are real numbers with $0 < r' \leq r$, α is an integer, $C_0(z,y)$ and $C_1(z,y)$ are power series that contain $z = \rho$ and $y = 1$ in their regions of convergence and that satisfy $C_i(\rho, y) \neq 0$ for $i = 0,1$ and $0 \leq y \leq 1$. Moreover, assume that the $\mathcal{O}(\cdot)$-term is uniform in a neighbourhood of $z = \rho$ and $y = 1$.

Let $p_n(d)$ denote the probability that a random node in a thickened PORT of size n has degree d. Then the limits

$$\lim_{n \to \infty} p_n(d) =: p(d)$$

exist and we have, as $d \to \infty$,

$$p(d) \sim \frac{C}{d^{r+r'+1}}.$$

Furthermore, for every $d \geq 0$ let $X_n^{(d)}$ denote the number of nodes of degree d in a random thickened PORT of size n. Then $X_n^{(d)}$ satisfies a central limit theorem

$$\frac{X_n^{(d)} - \mathbb{E}\, X_n^{(d)}}{\mathbb{V}\, X_n^{(d)}} \xrightarrow{d} N(0,1),$$

where $\mathbb{E}\, X_n^{(d)}$ and $\mathbb{V}\, X_n^{(d)}$ are both asymptotically proportional to n.

The above theorem shows that thickened trees, where the generating function $T_d(z, y, u)$ has the form (1), are scale free. Furthermore, the tail of the degree distribution has the order $r + r' + 1$. The parameters r and r' just depend on the structure of the cluster sets. Thus, a local change of the model, that is, a modification of the cluster sets, changes the global degree distribution.

In what follows we will show that the (quite technical) condition (1) is satisfied in very general situations. Furthermore we focus on the question how the structure of the cluster sets influences the paramters r and r'. In fact, we will formulate a proper *local-global-principle* for the order of the degree distribution.

We start with the following

Definition 1. *Let $\mathcal{M}$ be a set of graphs, where some vertices are marked and some or all of the marked vertices have (outgoing) half edges attached to them. Additionally we assume that one vertex is attached to a distinguished (ingoing) half edge. We say that $\mathcal{M}$ satisfies the symmetry condition if the following property holds. Suppose that $G \in \mathcal{M}$ has k' marked vertices and k (outgoing) half edges. Then every graph of that kind, where k (outgoing) half edges are attached to these k' marked vertices in an arbitrary way is also contained in $\mathcal{M}$.*

Note that the above definition assumes marked nodes in the graphs whereas the graphs in the cluster sets $\mathcal{T}_k$ do not have marked nodes. In order to apply Definition 1 we mark nodes in the graphs of $\mathcal{T}_k$ according to the following scheme. First, partition $\mathcal{T}_k$ into isomorphy classes. Now consider one particular isomorphy class and mark in each graph all those nodes, for which another graph of the class exists which has an outgoing half edge attached to the corresponding node.

Example 1. In [5] thickened trees with cluster set as shown in Figure 1 where studied. Here the sets $\mathcal{T}_k$ satisfy the symmetry condition. Since each $\mathcal{T}_k$ with $k \geq 2$ contains a triangular graph where both end vertices of the bottom are incident to a half edge, the two bottom vertices have to be marked in *every* graph of $\mathcal{T}$. The symmetry condition requires now, that all triangular graphs where the half edges are attached to the two bottom vertices (not necessarily involving both of them) are elements of $\mathcal{T}_k$, which is indeed true.

We will first show that the condition (1) is satisfied for a very special cluster set.

Theorem 2. *Consider a family of thickened trees such that all cluster sets $\mathcal{T}_k$, $k \geq k_0$ sufficiently large, contain only isomorphic copies of one graph (the same*

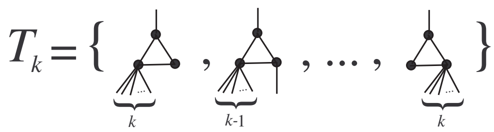

Fig. 2. Cluster sets containing all triangular graphs where the outgoing half edges are separated from the ingoing half edge.

for all $k \geq k_0$) of size m and k' marked vertices and satisfy the symmetry condition. Then we have

$$T_d(z, y, u) = \frac{z^{m-1}}{m!} \frac{1 + C(y)(1-y)(u-1) + \mathcal{O}\left((1-y)(u-1)^2\right)}{(1-y)^{k'}}, \qquad (2)$$

where

$$C(y) = c_{k'-1} + c_{k'-2}(1-y) + \cdots + c_0(1-y)^{k'-1}$$

is a polynomial of degree $k' - 1$, that is, $c_0 \neq 0$.

Corollary 1. *Suppose that the conditions of Theorem 2 are satisfied. Then (by applying Theorem 1) we obtain for the resulting thickened PORT family (with $r = k'$, $r' = 1$)*

$$p(d) = \lim_{n\to\infty} p_n(d) \sim \frac{C}{d^{k'+2}}.$$

This result means that a tail behaviour of the form $C\,d^{-3}$ for usual PORTs is changed into a behaviour of this form. The difference in the exponent equals $k' - 1$ which can be seen as the *additional degree of freedom* we have when we choose the k half edges among the k' marked vertices.

Overall, this means that (under the symmetry condition or by assuming (1)) the local structure of the clusters determine in a relatively simple way the global behaviour of the degree distribution. This can be seen as a *local-global-principle*.

Proof. For $k \geq k_0$ there are k' marked vertices for attaching a half edge to the graph in $\mathcal{T}_k$ and by the symmetry condition every distribution of the half edges among those k' places must lead to a graph in $\mathcal{T}_k$. Hence we obtain $\binom{k+k'-1}{k'-1}$ possible configurations. Since this implies that

$$t_k^{(d)}(z, 1) = \binom{k + k' - 1}{k' - 1} \frac{z^m}{m!}.$$

Thus

$$T_d(z, y, u) = \frac{z^{m-1}}{m!} \frac{1 + \mathcal{O}\left((1-y)(u-1)\right)}{(1-y)^{k'}}.$$

Now let $a_k^{(d)}(u) = m! t_k^{(d)}(z, u)/z^m$. Since the coefficient of u in $a_k^{(d)}(u)$ is the number of configurations where exactly one vertex has degree d is of order $k^{k'-2}$

we surely have $c_{k'-1} \neq 0$. Similar arguments show that all the coefficients of the higher powers of u in $a_k^{(d)}(u)$ are polynomials in k of degree less than k'. This implies the shape of the error term in (2) and completes the proof.

Remark 1. Theorem 2 applies to the triangular cluster set of Example 1, where we have $k' = 2$ and consequently a tail of resulting degree distribution of the form $p(d) \sim C\,k^{-4}$. This is in accordance with the exact result (see [5])

$$p(d) = \frac{12}{(d-1)d(d+1)(d+2)} \qquad (d \geq 4).$$

Another example (which is also discussed in [5]) is again based on triangular graphs, however, we distinguish now between left half edges and right half edges that can be additionally attached to the vertex with the ingoing half edge, see Figure 3. This distinction is necessary by the interpretation of PORTs as plane (or ordered) trees. Formally, such a cluster set can be handled by giving two marks to this vertex, that is, $k' = 2 + 1 + 1 = 4$. Obviously, Theorem 2 applies in such a situation, too. Hence, the tail of the resulting degree distribution is of the form $p(d) \sim C\,d^{-6}$. This is again in accordance with the exact result (see [5])

$$p(d) = \frac{1600}{(d-1)d(d+1)(d+2)(d+3)(d+4)} \qquad (d \geq 4).$$

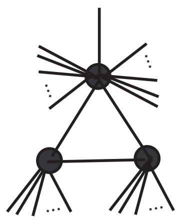

Fig. 3. Triangular cluster where the vertex with the ingoing half edge has two marks and the two other vertices just one.

4 Inserting Clusters of Different Type

Theorem 2 only applies if the cluster set is of a very simple form. We will next investigate what happens if we choose cluster sets containing clusters of different type, that is, of different degrees of freedom for attaching half edges. In order to understand in which way different tail behaviours caused by classes of clusters compete, it suffices to consider the case of two different types. Since $t_k^{(d)}(z,u)$ is the generating function counting all clusters with respect to size and number of nodes of degree d, adding another class of clusters to the cluster sets results in adding the corresponding generating function to the first one. The same holds for the functions $T_d(z,y,u)$. Hence the problem reduces to an analysis of what happens if two behaviours of the forms (1) from Theorem 2 are added.

Theorem 3. *Let $\mathcal{T}_k$ be cluster sets (as described above) such that the equation*

$$X = \int_0^1 \frac{dt}{T_d(X,t,1)}$$

has a unique positive solution $X=\rho$ in the region of convergence of $T_d(z,y,u)$ and that $T_d(z,y,u)$ can be represented as

$$\begin{aligned} T_d(z,y,u) = & \frac{C_0(z,y) + C_1(z,y)(1-y)^{r_1'} y^{d+\alpha}(u-1) + \mathcal{O}\left((1-y)^{r_1'}(u-1)^2\right)}{(1-y)^{r_1}} \\ & + \frac{C_2(z,y) + C_3(z,y)(1-y)^{r_2'} y^{d+\beta}(u-1) + \mathcal{O}\left((1-y)^{r_2'}(u-1)^2\right)}{(1-y)^{r_2}}, \end{aligned}$$

where r_i' and r_i, $i=1,2$, are real numbers with $0 < r_i' \le r_i$ and $r_1 > r_2$; α and β are integers, $C_i(z,y)$, $i=1,2,3,4$, are power series that contain $z=\rho$ and $y=1$ in their regions of convergence and that satisfy $C_i(\rho,y) \neq 0$ for $i=1,3$ and $0 \le y \le 1$ as well as $C_0(\rho,y) + C_2(\rho,y)(1-y)^{r_1-r_2} \neq 0$ for $0 \le y \le 1$. Moreover, assume that the $\mathcal{O}(\cdot)$-terms are uniform in a neighbourhood of $z=\rho$ and $y=1$.

Let $p_n(d)$ denote the probability that a random node in a thickened PORT of size n has degree d. Then the limits

$$\lim_{n\to\infty} p_n(d) =: p(d)$$

exist and we have, as $d \to \infty$,

$$p(d) \sim \frac{C}{d^{\min\{r_1+r_1'+1, 2r_1-r_2+r_2'+1\}}}.$$

Furthermore, for every $d \ge 0$ let $X_n^{(d)}$ denote the number of nodes of degree d in a random thickened PORT of size n. Then $X_n^{(d)}$ satisfies a central limit theorem

$$\frac{X_n^{(d)} - \mathbb{E}\, X_n^{(d)}}{\mathbb{V}\, X_n^{(d)}} \xrightarrow{d} N(0,1),$$

where $\mathbb{E}\, X_n^{(d)}$ and $\mathbb{V}\, X_n^{(d)}$ are both asymptotically proportional to n.

Remark 2. This theorem is – in some sense – a superposition principle for cluster sets. Note that (if $r_2 \le r_1$)

$$\min\{r_1 + r_1' + 1, 2r_1 - r_2 + r_2' + 1\} = 1 + 2\max\{r_1, r_2\} - \max\{r_1 - r_1', r_2 - r_2'\}.$$

Hence, the resulting exponent in the tail of the degree distribution is determined by the bevaviour r_j and the differences $r_j - r_j'$. For those (basic) cluster sets which are covered in Theorem 2 we actually have $r_j' = 1$. Consquently, we obtain $r_1 + 2 = \max\{r_1, r_2\} + 1$ as the resulting exponent. This means that if we interprete r_1 as the degree of freedom to select half edges then the maximum degree of freedom is responsible for the exponent in the degree distribution.

This extends the above formulated *local-global-principle*.

Proof. We start by inspecting the generating function $g(z)$ of all thickened PORTs. For simplicity we assume that the substitution sets $\mathcal{T}_k$ are of a form that $g_n > 0$ for sufficiently large $n \geq n_0$, that is, we exclude, for example, the case that the number of nodes of graphs in $\mathcal{T}_k$ are all congruent to 1 modulo some integer $m > 1$.[1] Then it follows that $|g(z)| < g(|z|)$ if z is not contained in the positive real line.

We first observe that $\rho > 0$ is the only singularity on the circle of convergence $|z| \leq \rho$ and that $g(\rho) = 1$, that is, $g(z)$ is convergent at $z = \rho$. First it is clear that $g(z)$ can be analytically continued starting with $g(0) = 0$ and the functional equation $G_d(z, g(z), 1) = z$. However, if $g(z_0) \neq 1$ for some z_0 contained in the region of convergence of $g(z)$ then we have

$$\left(\frac{\partial}{\partial y} G_d\right)(z_0, g(z_0), 1) = \frac{1}{T_d(z_0, g(z_0), 1)}$$
$$= \frac{(1 - g(z_0))^{r_1}}{C_0(z_0, g(z_0)) + C_2(z_0, g(z_0))(1 - g(z_0))^{r_1 - r_2}} \neq 0.$$

Thus, we can continue analytically with help of the implicit function theorem. Hence, if $g(z)$ has a singularity ρ and if $g(\rho)$ is convergent, then $g(\rho) = 1$. Since $g(z)$ is monotone and analytic it certainly reaches a value with $g(\rho) = 1$ where it has to be singular. Further, ρ is characterized by the equation $G_d(\rho, 1, 1) = \rho$.

Next we characterize the kind of singularity of $g(z)$ at $z = \rho$. By Lemma 1 we have

$$\begin{aligned} z &= \int_0^g \frac{(1-t)^{r_1}}{C_0(z,t) + C_2(z,t)(1-t)^{r_1 - r_2}}\, dt \\ &= \int_0^1 \frac{(1-t)^{r_1}}{C_0(z,t) + C_2(z,t)(1-t)^{r_1 - r_2}}\, dt - \int_g^1 \frac{(1-t)^{r_1}}{C_0(z,t) + C_2(z,t)(1-t)^{r_1 - r_2}}\, dt \\ &=: G(z) - H(z,g) \end{aligned} \tag{3}$$

Hence, by expanding

$$\frac{1}{C_0(z,t) + C_2(z,t)(1-t)^{r_1 - r_2}} = c_0(z) + c_1(z)(1-t) + c_2(z)(1-t)^2 + \dots$$

[1] We call this the aperiodic case. In the periodic case we have to deal with m singularities on the boundary of the circle of convergence of $g(z)$ which are all of the same kind.

we get

$$G(z) - z = c_0(z)(1-g(z))^{r_1+1}\left(1+\mathcal{O}\left(|1-g(z)|\right)\right)$$

which is equivalent to

$$\left(\frac{G(z)-z}{c_0(z)}\right)^{1/(r_1+1)} = (1-g(z))\left(1+\mathcal{O}\left(|1-g(z)|\right)\right). \tag{4}$$

Since $G(\rho) = \rho$ and $C_0(z,y)$ is analytic in z we can represent $(G(z)-z)/c_0(z) = K(z)(1-z/\rho)$. Furthermore, we can invert relation (4) and obtain

$$g(z) = 1 - K(z)^{1/(r_1+1)}\left(1-\frac{z}{\rho}\right)^{1/(r_1+1)} + \mathcal{O}\left(\left|1-\frac{z}{\rho}\right|^{2/(r_1+1)}\right). \tag{5}$$

Since there are no other singularities on the circle $|z| \le \rho$ and $g(z)$ can be analytically continued to a larger range (despite at the point $z=\rho$) it follows from [6] that

$$g_n \sim K(\rho)^{1/(r_1+1)}\frac{\rho^{-n} n^{-\frac{r_1+2}{r_1+1}}}{-\Gamma\left(-\frac{1}{r_1+1}\right)}.$$

Next we determine the asymptotics of the average value $\mathbb{E}\, X_n^{(d)}$. Set $S(z) = \frac{\partial}{\partial u} g(z,1)$. Then it follows from Lemma 1 that

$$\begin{aligned}S(z) = &-\frac{C_0(z,g(z)) + C_2(z,g(z))(1-g(z)^{r_1-r_2}}{(1-g(z))^{r_1}} \\ &\times\left[\int_0^{g(z)} \frac{C_1(z,t)}{(C_0(z,t)+C_2(z,t)(1-t)^{r_1-r_2})^2}(1-t)^{r_1+r_1'}t^{d+\alpha}\,dt\right.\\ &\left.+\int_0^{g(z)} \frac{C_3(z,t)}{(C_0(z,t)+C_2(z,t)(1-t)^{r_1-r_2})^2}(1-t)^{2r_1-r_2+r_2'}t^{d+\beta}\,dt\right]\end{aligned}$$

By (5) and a decomposition of the integral as in (3) we can transform this to

$$\begin{aligned}S(z) = &\frac{1}{K(z)^{\frac{r_1}{r_1+1}}(1-z/\rho)^{\frac{r_1}{r_1+1}}}\left[\int_0^1 \tilde{C}_0(z,t)(1-t)^{r_1+r_1'}t^{d+\alpha}\,dt\right.\\ &\left.+\int_0^1 \tilde{C}_1(z,t)(1-t)^{r_2+r_2'}t^{d+\beta}\,dt\right]\\ &+\frac{1}{K(z)^{\frac{r_2}{r_1+1}}(1-z/\rho)^{\frac{r_2}{r_1+1}}}\left[\int_0^1 \tilde{C}_2(z,t)(1-t)^{r_1+r_1'}t^{d+\alpha}\,dt\right.\\ &\left.+\int_0^1 \tilde{C}_3(z,t)(1-t)^{2r_1-r_2+r_2'}t^{d+\beta}\,dt\right] + \mathcal{O}(1),\end{aligned}$$

where $\tilde{C}_i(z,t)$ are analytic functions. This proves that

$$\mathbb{E}\, X_n^{(d)} = n \cdot \frac{r_1+1}{K(\rho)} \int_0^1 \left[\int_0^1 \tilde{C}_0(z,t)(1-t)^{r_1+r_1'} t^{d+\alpha}\, dt \right. \\ \left. + \int_0^1 \tilde{C}_1(z,t)(1-t)^{2r_1-r_2+r_2'} t^{d+\beta}\, dt \right] + \mathcal{O}\left(n^{\frac{r_2+1}{r_1+1}} \right).$$

Thus, the limit $p(d) = \lim_{n\to\infty} \mathbb{E}\, X_n^{(d)}/n$ exists and is asymptotically given by

$$\begin{aligned} &p(d) \\ &= \frac{r_1+1}{K(\rho)} \left[\int_0^1 \tilde{C}_0(z,t)(1-t)^{r_1+r_1'} t^{d+\alpha}\, dt + \int_0^1 \tilde{C}_1(z,t)(1-t)^{2r_1-r_2+r_2'} t^{d+\beta}\, dt \right] \\ &\sim \frac{C'}{d^{r_1+r_1'+1}} + \frac{C''}{d^{2r_1-r_2+r_2'+1}} \end{aligned}$$

for some constants $C', C'' > 0$.

Finally the proof that the limiting distribution is normal follows from the results and methods in [5, 7] and (5).

Example 2. Let us continue the example where the cluster set $\mathcal{T}_k$ all triangular graphs depicted in Figure 2 and the graph consistung of one single node. That means that – when building the thickened tree – in each substitution step we may substitute a node by a triangular graph or leave the node unchanged (no local clustering in this particular place). Then Theorem 3 says that the exponent -4 in the tail $p(d) \sim C\, d^{-4}$ of the degree distribution remains but the constant C changes.

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