Strobl, June 10, 2015

AofA'2015

Symbolic Summation for Combinatorial and Related Problems

Carsten Schneider

SFB F050 Algorithmic and Enumerative Combinatorics
Research Institute for Symbolic Computation
Johannes Kepler University Linz

Der Wissenschaftsfonds

Some of the available summation tools:

Abramov, S.A.: On the summation of rational functions. Zh. vychisl. mat. Fiz. **11**, 1071–1074 (1971)
Abramov, S.A.: The rational component of the solution of a first-order linear recurrence relation with a rational right-hand side. U.S.S.R. Comput. Maths. Math. Phys. **15**, 216–221 (1975). Transl. from Zh. vychisl. mat. mat. fiz. 15, pp. 1035–1039, 1975
Abramov, S.A.: Rational solutions of linear differential and difference equations with polynomial coefficients. U.S.S.R. Comput. Math. Math. Phys. **29**(6), 7–12 (1989)
Abramov, S.A., Petkovšek, M.: D'Alembertian solutions of linear differential and difference equations. In: J. von zur Gathen (ed.) Proc. ISSAC'94, pp. 169–174. ACM Press (1994)
Abramov, S.A., Petkovšek, M.: Rational normal forms and minimal decompositions of hypergeometric terms. J. Symbolic Comput. **33**(5), 521–543 (2002)
Apagodu, M., Zeilberger, D., 2006. Multi-variable Zeilberger and Almkvist–Zeilberger algorithms and the sharpening of Wilf–Zeilberger theory. Advances in Applied Math. 37, 139–152.
Bauer, A., Petkovšek, M.: Multibasic and mixed hypergeometric Gosper-type algorithms. J. Symbolic Comput. **28**(4–5), 711–736 (1999)
Bronstein, M.: On solutions of linear ordinary difference equations in their coefficient field. J. Symbolic Comput. **29**(6), 841–877 (2000)
Chen, S., Jaroschek, M., Kauers, M., Singer, M.F.: Desingularization Explains Order-Degree Curves for Ore Operators. In: M. Kauers (ed.) Proc. of ISSAC'13, pp. 157–164 (2013)
Chen, S., Kauers, M.: Order-Degree Curves for Hypergeometric Creative Telescoping. In: J. van der Hoeven, M. van Hoeij (eds.) Proceedings of ISSAC 2012, pp. 122–129 (2012)
Chyzak, F.: An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math. **217**, 115–134 (2000)
Fasenmyer, M. C., November 1945. Some generalized hypergeometric polynomials. Ph.D. thesis, University of Michigan.
Gosper, R.W.: Decision procedures for indefinite hypergeometric summation. Proc. Nat. Acad. Sci. U.S.A. **75**, 40–42 (1978)
Hendriks, P.A., Singer, M.F.: Solving difference equations in finite terms. J. Symbolic Comput. **27**(3), 239–259 (1999)
Karr, M.: Summation in finite terms. J. ACM **28**, 305–350 (1981)
Karr, M.: Theory of summation in finite terms. J. Symbolic Comput. **1**, 303–315 (1985)
M. Kauers and P. Paule. *The concrete tetrahedron*. Texts and Monographs in Symbolic Computation. SpringerWienNewYork, Vienna, 2011. Symbolic sums, recurrence equations, generating functions, asymptotic estimates.

⋮

Some of the available summation tools:

⋮

Koornwinder, T.H.: On Zeilberger's algorithm and its q-analogue. J. Comp. Appl. Math. **48**, 91–111 (1993)

Koutschan, C.: Creative telescoping for holonomic functions. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 171–194. Springer (2013). ArXiv:1307.4554 [cs.SC]

Paule, P.: Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)

Paule, P.: Contiguous relations and creative telescoping. unpublished manuscript p. 33 pages (2001)

Paule, P., Riese, A.: A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated aproach to q-hypergeometric telescoping. In: M. Ismail, M. Rahman (eds.) Special Functions, q-Series and Related Topics, vol. 14, pp. 179–210. AMS (1997)

Paule, P., Schorn, M.: A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. J. Symbolic Comput. **20**(5-6), 673–698 (1995)

Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. **14**(2-3), 243–264 (1992)

Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$. A. K. Peters, Wellesley, MA (1996)

Petkovšek, M., Zakrajšek, H.: Solving linear recurrence equations with polynomial coefficients. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 259–284. Springer (2013)

Pirastu, R., Strehl, V.: Rational summation and Gosper-Petkovšek representation. J. Symbolic Comput. **20**(5-6), 617–635 (1995)

Wegschaider, K., May 1997. Computer generated proofs of binomial multi-sum identities. Master's thesis, RISC, Johannes Kepler University.

Wilf, H. S., Zeilberger, D., 1992. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. 108 (3), 575–633.

Zeilberger, D., 1990. A holonomic systems approach to special functions identities. J. Comput. Appl. Math. 32, 321–368.

Zeilberger, D.: The method of creative telescoping. J. Symbolic Comput. **11**, 195–204 (1991)

Some of the available summation tools:

⋮

Koornwinder, T.H.: On Zeilberger's algorithm and its q-analogue. J. Comp. Appl. Math. **48**, 91–111 (1993)

Koutschan, C.: Creative telescoping for holonomic functions. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 171–194. Springer (2013). ArXiv:1307.4554 [cs.SC]

Paule, P.: Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)

Paule, P.: Contiguous relations and creative telescoping. unpublished manuscript p. 33 pages (2001)

Paule, P., Riese, A.: A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated aproach to q-hypergeometric telescoping. In: M. Ismail, M. Rahman (eds.) Special Functions, q-Series and Related Topics, vol. 14, pp. 179–210. AMS (1997)

Paule, P., Schorn, M.: A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. J. Symbolic Comput. **20**(5-6), 673–698 (1995)

Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. **14**(2-3), 243–264 (1992)

Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$. A. K. Peters, Wellesley, MA (1996)

Petkovšek, M., Zakrajšek, H.: Solving linear recurrence equations with polynomial coefficients. In: C. Schneider, J. Blümlein (eds.) Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation, pp. 259–284. Springer (2013)

Pirastu, R., Strehl, V.: Rational summation and Gosper-Petkovšek representation. J. Symbolic Comput. **20**(5-6), 617–635 (1995)

Wegschaider, K., May 1997. Computer generated proofs of binomial multi-sum identities. Master's thesis, RISC, Johannes Kepler University.

Wilf, H. S., Zeilberger, D., 1992. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. 108 (3), 575–633.

Zeilberger, D., 1990. A holonomic systems approach to special functions identities. J. Comput. Appl. Math. 32, 321–368.

Zeilberger, D.: The method of creative telescoping. J. Symbolic Comput. **11**, 195–204 (1991)

Here I will restrict to the setting of difference rings/fields.

You've Got Mail (7/2004)

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package
for handling sums with harmonic numbers (among others)
can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously
like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{n,k=1}^{\infty} \frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}; \qquad H_k := \sum_{i=1}^{k} \frac{1}{i}$$

[Arose in the analysis of the simplex algorithm on the Klee-Minty cube (J. Balogh, R. Pemantle)]

$$S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n+1)} \boxed{\sum_{k=1}^{\infty} \frac{H_k}{k(k+n)}}$$

where $$H_k = \sum_{i=1}^{k} \frac{1}{i}.$$

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

Telescoping

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n, k \geq 1$.

Telescoping

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n, k \geq 1$.

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} f(n,k)}$$

Telescoping

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $n, k \geq 1$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $n, k \geq 1$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

solution ☺

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Sigma *computes*: $c_0(n) = n^2$, $c_1(n) = -(n+1)(2n+1)$, $c_2(n) = (n+1)(n+2)$

and

$$g(n,k) := -\frac{kH_k + n + k}{(n+k)(n+k+1)},$$

$$g(n,k+1) := -\frac{(1+n)H_k + n + k + 2}{(n+k+1)(n+k+2)}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \left[c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k)\right]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n) f(n,k) + \sum_{k=1}^{a} c_1(n) f(n+1,k) + \sum_{k=1}^{a} c_2(n) f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$A'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n) \sum_{k=1}^{a} f(n,k) + c_1(n) \sum_{k=1}^{a} f(n+1,k) + c_2(n) \sum_{k=1}^{a} f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\text{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\text{A'}(n) + c_1(n)\text{A'}(n+1) + c_2(n)\text{A'}(n+2)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\mathsf{A'}(n) + c_1(n)\mathsf{A'}(n+1) + c_2(n)\mathsf{A'}(n+2)}$$

$$||\qquad\qquad\qquad\qquad ||$$

$$\frac{a}{(n+1)(a+n+1)} - \frac{(a+1)H_a}{(a+n+1)(a+n+2)} \qquad n^2\mathsf{A'}(n) - (n+1)(2n+1)\mathsf{A'}(n+1) + (n+1)(n+2)\mathsf{A'}(n+2)$$

Summation principles (in difference field/ring setting)

$$n^2\mathbf{A}(n)-(n+1)(2n+1)\mathbf{A}(n+1)+(n+1)(n+2)\mathbf{A}(n+2)=\frac{1}{n+1}$$

Recurrence finder

$$\mathbf{A}(n)=\sum_{k=1}^{\infty}\frac{H_k}{k(k+n)}$$

Summation principles (in difference field/ring setting)

$$n^2\mathbf{A}(n)-(n+1)(2n+1)\mathbf{A}(n+1)+(n+1)(n+2)\mathbf{A}(n+2)=\frac{1}{n+1}$$

Recurrence solver

$$\left\{c_1\frac{nH_n-1}{n^2}+c_2\frac{1}{n}+\frac{nH_n^2-2H_n+nH_n^{(2)}}{2n^2}\;\middle|\;c_1,c_2\in\mathbb{R}\right\}$$

$$\mathbf{A}(n)=\sum_{k=1}^{\infty}\frac{H_k}{k(k+n)}$$

where

$$H_n^{(2)}=\sum_{i=1}^{n}\frac{1}{i^2}$$

Summation principles (in difference field/ring setting)

$$n^2\mathbf{A}(n) - (n+1)(2n+1)\mathbf{A}(n+1) + (n+1)(n+2)\mathbf{A}(n+2) = \frac{1}{n+1}$$

Recurrence solver

$$\mathbf{A}(n) = \sum_{k=1}^{\infty} \frac{H_k}{k(k+n)} \in \boxed{\left\{ c_1 \frac{nH_n - 1}{n^2} + c_2 \frac{1}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}}$$

where

$$H_n^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2}$$

Summation principles (in difference field/ring setting)

$$n^2\mathbf{A}(n) - (n+1)(2n+1)\mathbf{A}(n+1) + (n+1)(n+2)\mathbf{A}(n+2) = \frac{1}{n+1}$$

Summation package Sigma

(based on difference field algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001 –)

$$\mathbf{A}(n) = \sum_{k=1}^{\infty} \frac{H_k}{k(k+n)} = \boxed{0\frac{nH_n - 1}{n^2} + \zeta_2\frac{1}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}}$$

where

$$H_n^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2} \qquad \zeta_z = \sum_{i=1}^{\infty} \frac{1}{i^z}\ (= \zeta(z))$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n^2\mathsf{SUM}[n] - (n+1)(2n+1)\mathsf{SUM}[n+1] + (n+1)(n+2)\mathsf{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[4]= $n^2\mathsf{SUM}[n] - (n+1)(2n+1)\mathsf{SUM}[n+1] + (n+1)(n+2)\mathsf{SUM}[n+2] = \frac{1}{n+1}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= **rec = LimitRec[rec, SUM[n], {n}, a]**

Out[4]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$

In[5]:= **recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → True]**

Out[5]= $\{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n} \frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} - \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[4]= $n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$

In[5]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n], IndefiniteSummation \to True]}$

Out[5]= $\{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n}\frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{2n} - \frac{\sum_{i=1}^{n}\frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n}\frac{1}{i^2}}{2n}\}\}$

In[6]:= $\mathbf{FindLinearCombination[recSol, \{1, \{\zeta_2, 1/2 + \zeta_2/2\}\}, n, 2]}$

Out[6]= $-\frac{\sum_{i=1}^{n}\frac{1}{i}}{n^2} + \frac{\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2}{2n} + \frac{\sum_{i=1}^{n}\frac{1}{i^2}}{2n} + \frac{\zeta_2}{n}$

$$S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n+1)} \underbrace{\boxed{\sum_{k=1}^{\infty} \frac{H_k}{k(k+n)}}}$$

$$= \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}$$

$$S = \sum_{n=1}^{\infty} \frac{H_{n+1}-1}{n(n+1)} \underbrace{\boxed{\sum_{k=1}^{\infty} \frac{H_k}{k(k+n)}}}$$

$$= \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}$$

$$= -4\zeta_2 + (\zeta_2 - 1)\sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3}$$

$$+ \frac{1}{2}\sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2}$$

$$S = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n(n+1)} \underbrace{\boxed{\sum_{k=1}^{\infty} \frac{H_k}{k(k+n)}}}$$

$$= \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}$$

$$= -4\zeta_2 + (\zeta_2 - 1) \sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3}$$

$$+ \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2}$$

$$= -4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5 = 0.999222...$$

J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electron. J. Combin.*, 3:1–27, 1996.
P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Experim. Math.*, 7(1):15–35, 1998.

$$S = \sum_{n=1}^{\infty} \frac{H_{n+1}-1}{n(n+1)} \underbrace{\boxed{\sum_{k=1}^{\infty} \frac{H_k}{k(k+n)}}}$$

$$= \frac{\zeta_2}{n} + \frac{nH_n^2 - 2H_n + nH_n^{(2)}}{2n^2}$$

$$= -4\zeta_2 + (\zeta_2 - 1)\sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^3}$$

$$+ \frac{1}{2}\sum_{i=1}^{\infty} \frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2}$$

$$= -4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5 = 0.999222...$$

J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electron. J. Combin.*, 3:1–27, 1996.
P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Experim. Math.*, 7(1):15–35, 1998.
J. Blümlein and D. J. Broadhurst and J. A. M. Vermaseren, The Multiple Zeta Value Data Mine, *Comput. Phys. Commun.*, 181:582–625, 2010.

Toolbox 1: Indefinite summation

Toolbox 2: Definite summation

Toolbox 3: Special function algorithms

Toolbox 1: Indefinite summation

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = H_k$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (H_k - 1)k.$$

Telescoping

GIVEN $f(k) = H_k$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} H_k} = \boxed{g(n+1) - g(1)}$$

$$= (H_{n+1} - 1)(n+1).$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\,k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\,H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)k \in \mathbb{F}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

Toolbox 1: Indefinite summation – the basic tactic

(a simplified version of Karr's algorithm, 1981)

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad\qquad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1, \dots, t_{e-1})^*, && f_e \in \mathbb{K}(t_1, \dots, t_{e-1})
\end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field (containing $\mathbb{Q}$)

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;
FIND, in case of existence, a $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A $\Pi\Sigma^*$-field for the summand

$\boxed{\mathrm{const}_\sigma \mathbb{F} = \mathbb{Q}}$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, \\ \sigma(k) &= k+1, & \mathcal{S}\, k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$: $\boxed{d = 1}$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$: $d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$: $b = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$: $d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\, d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$: $b = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

Polynomial Solution: FIND

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h].$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2+\sigma(g_1h+g_0)\Big] -\Big[g_2\,h^2+g_1h+g_0\Big]=h$$

coeff. comp.

$$\sigma(g_2)-g_2=0$$

$g_2 = c \in \mathbb{Q}$

$$\Big[\sigma(c)\big(h+\tfrac{1}{k+1}\big)^2+\sigma(g_1h+g_0)\Big] -\Big[c\,h^2+g_1h+g_0\Big]=h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\Big[c\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[c\, h^2 + g_1 h + g_0\Big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\big[\tfrac{2h(k+1)+1}{(k+1)^2}\big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\frac{2}{k+1}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

g=hk-k

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

$\leftarrow$ $g_0 = -k$, $d = 0$

Toolbox 1: Improved indefinite summation – symbolic simplification

For algorithmic details see:

- CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
- CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
- CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
- CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008. [arXiv:0808.2543v1].
- S.A. Abramov, M. Petkovšek. Polynomial ring automorphisms, rational (w, σ)-canonical forms, and the assignment problem. *J. Symbolic Comput.*, 45(6): 684–708, 2010.
- CS, A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In: A. Carey, D. Ellwood, S. Paycha, S. Rosenberg (eds.) Motives, Quantum Field Theory, and Pseudodifferential Operators, Clay Mathematics Proceedings, vol. 12, pp. 285–308. Amer. Math. Soc (2010). ArXiv:0808.2543
- CS, Parameterized Telescoping Proves Algebraic Independence of Sums. Ann. Comb. **14**(4), 533–552 (2010). [arXiv:0808.2596]
- CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
- CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. To appear in *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014. arXiv:1307.7887 [cs.SC].

For special cases see:

- S.A. Abramov. On the summation of rational functions. Zh. vychisl. mat. Fiz., 11: 1071-1074, 1971.
- P. Paule. Greatest factorial factorization and symbolic summation, J. Symbolic Comput., 20(3): 235-268, 1995.

A difference field approach (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{F}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{a}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)H_k}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)H_k^{(3)}}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$=?$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{F}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{a}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)H_k}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)H_k^{(3)}}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$=\frac{a^2+4a+5}{10(a^2+2a+2)}H_a-\frac{(a-1)(a+1)}{5(a^2+2a+2)}H_a^{(3)}-\frac{2}{5}\sum_{k=1}^{a}\frac{1}{k^2}$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = ?$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n} \frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)\left(\sum_{i=1}^{n} \frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i^3}$$

depth 3 depth 1

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a} (-1)^k H_k{}^2 \binom{n}{k} =?$$

A symbolic summation approach

1. FIND an appropriate $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{F}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a} (-1)^k {H_k}^2 \binom{n}{k} = -\frac{1}{n} \sum_{i_1=1}^{a} \frac{(-1)^{i_1}}{i_1} \binom{n}{i_1} - (a-n)\big(n^2 {H_a}^2 + 2nH_a + 2\big) \frac{(-1)^a \binom{n}{a}}{n^3} - \frac{2}{n^2}$$

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(k) = B(k)$$

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(k) = B(k)$$

- such that all the sums in $B(k)$ are **simplified** as above

Simplification of nested product-sum expressions

$A(k)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,k]`

$B(k)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(k) = B(k)$$

- such that all the sums in $B(k)$ are **simplified** as above
- and such that the arising sums in $B(k)$ are **algebraically independent** (i.e., they do not satisfy any polynomial relation)

Toolbox 2: Definite summation

Summation principles (in difference field/ring setting)

$$n^2\mathbf{A}(n)-(n+1)(2n+1)\mathbf{A}(n+1)+(n+1)(n+2)\mathbf{A}(n+2)=\frac{1}{n+1}$$

Summation package Sigma

(based on difference field algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001 –)

$$\mathbf{A}(n)=\sum_{k=1}^{\infty}\frac{H_k}{k(k+n)} = \boxed{0\frac{nH_n-1}{n^2}+\zeta_2\frac{1}{n}+\frac{nH_n^2-2H_n+nH_n^{(2)}}{2n^2}}$$

where

$$H_n^{(2)}=\sum_{i=1}^{n}\frac{1}{i^2}\qquad \zeta_z=\sum_{i=1}^{\infty}\frac{1}{i^z}\ (=\zeta(z))$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\frac{H_k}{k(k+n)} + c_1(n)\,f(n+1,k) + c_2(n)\,f(n+2,k)}$$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\frac{H_k}{k(k+n)} + c_1(n)\,\frac{H_k}{k(k+n+1)} + c_2(n)\,f(n+2,k)}$$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\frac{H_k}{k(k+n)} + c_1(n)\,\frac{H_k}{k(k+n+1)} + c_2(n)\,\frac{H_k}{k(k+n+2)}}$$

Back to creative telescoping

Given
$$f(n,k)=\frac{H_k}{k(k+n)};$$
Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,\frac{H_k}{k(k+n)}+c_1(n)\,\frac{H_k}{k(k+n+1)}+c_2(n)\,\frac{H_k}{k(k+n+2)}}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F}$$

and a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$ defined by

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\frac{H_k}{k(k+n)} + c_1(n)\,\frac{H_k}{k(k+n+1)} + c_2(n)\,\frac{H_k}{k(k+n+2)}}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(\mathbf{n})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(\mathbf{c}) = \mathbf{c} \quad \forall \mathbf{c} \in \mathbb{Q}(\mathbf{n}),$$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\frac{H_k}{k(k+n)} + c_1(n)\,\frac{H_k}{k(k+n+1)} + c_2(n)\,\frac{H_k}{k(k+n+2)}}$$

A difference field for the **summand:**

Construct a rational function field

$$\mathbb{F} := \mathbb{Q}(n)(\mathbf{k})$$

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(\mathbf{k}) = \mathbf{k} + \mathbf{1}, \qquad\qquad \mathcal{S}\,k = k + 1,$$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\tfrac{H_k}{k(k+n)} + c_1(n)\,\tfrac{H_k}{k(k+n+1)} + c_2(n)\,\tfrac{H_k}{k(k+n+2)}}$$

A difference field for the **summand:**

Construct a rational function field

$(\mathbb{F}, \sigma)$ is a $\Pi\Sigma$-field

$$\mathbb{F} := \mathbb{Q}(n)(k)(\mathbf{h})$$

Karr 1981

and a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k+1, \qquad \mathcal{S}\,k = k+1,$$

$$\sigma(h) = \mathbf{h} + \frac{\mathbf{1}}{\mathbf{k+1}}, \qquad \mathcal{S}\,H_k = H_k + \frac{1}{k+1},$$

Back to creative telescoping

Given

$$f(n,k) = \frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\,\frac{H_k}{k(k+n)} + c_1(n)\,\frac{H_k}{k(k+n+1)} + c_2(n)\,\frac{H_k}{k(k+n+2)}}$$

FIND $g \in \mathbb{F}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g} = \boxed{c_0\,\frac{h}{k(k+n)} + c_1\,\frac{h}{k(k+n+1)} + c_2\,\frac{h}{k(k+n+2)}}$$

Back to creative telescoping

Given

$$f(n,k)=\frac{H_k}{k(k+n)};$$

Find $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1)-g(n,k)}=\boxed{c_0(n)\,\frac{H_k}{k(k+n)}+c_1(n)\,\frac{H_k}{k(k+n+1)}+c_2(n)\,\frac{H_k}{k(k+n+2)}}$$

FIND $g\in\mathbb{F}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g)-g}=\boxed{c_0\,\frac{h}{k(k+n)}+c_1\,\frac{h}{k(k+n+1)}+c_2\,\frac{h}{k(k+n+2)}}$$

$$\downarrow$$

$$c_0=n^2,\ c_1=-(n+1)(2n+1),\ c_2=(n+1)(n+2)$$

$$g=-\frac{kh+n+k}{(n+k)(n+k+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Sigma *computes*: $c_0(n) = n^2$, $c_1(n) = -(n+1)(2n+1)$, $c_2(n) = (n+1)(n+2)$

and

$$g(n,k) := -\frac{kH_k + n + k}{(n+k)(n+k+1)},$$

$$g(n,k+1) := -\frac{(1+n)H_k + n + k + 2}{(n+k+1)(n+k+2)}.$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n) f(n,k) + c_1(n) f(n+1,k) + c_2(n) f(n+2,k) \Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n) f(n,k) + \sum_{k=1}^{a} c_1(n) f(n+1,k) + \sum_{k=1}^{a} c_2(n) f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\sum_{k=1}^{a} f(n,k) + c_1(n)\sum_{k=1}^{a} f(n+1,k) + c_2(n)\sum_{k=1}^{a} f(n+2,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}'(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\mathsf{A}'(n) + c_1(n)\mathsf{A}'(n+1) + c_2(n)\mathsf{A}'(n+2)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A'}(n) := \sum_{k=1}^{a} \underbrace{\frac{H_k}{k(k+n)}}_{=:\, f(n,k)} .$$

FIND $g(n,k)$ and $c_0(n), c_1(n), c_2(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $n, k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\mathsf{A'}(n) + c_1(n)\mathsf{A'}(n+1) + c_2(n)\mathsf{A'}(n+2)}$$

$$||\qquad\qquad\qquad ||$$

$$\frac{a}{(n+1)(a+n+1)} - \frac{(a+1)H_a}{(a+n+1)(a+n+2)} \qquad n^2\mathsf{A'}(n) - (n+1)(2n+1)\mathsf{A'}(n+1) + (n+1)(n+2)\mathsf{A'}(n+2)$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \ldots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n)=\sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n)+\dots+a_d(n)A(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

Note: the sum solutions are highly nested
(possibly with denominators of high degrees)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \ldots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Simplify the solutions (using difference field theory) s.t.
   - the sums are algebraically independent;
   - the sums are flattened;
   - the sums can be given in terms of special functions.

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=1}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions in n.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums in n.
(d'Alembertian solutions)
(Abramov/Bronstein/Petkovšek/CS, in preparation)

4. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums in n.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{H_k}}{\mathbf{k(k+n)}}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[3]= $n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[4]= $n^2\mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $n^2\mathsf{SUM}[n] - (n+1)(2n+1)\mathsf{SUM}[n+1] + (n+1)(n+2)\mathsf{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= **rec = LimitRec[rec, SUM[n], {n}, a]**

Out[4]= $n^2\mathsf{SUM}[n] - (n+1)(2n+1)\mathsf{SUM}[n+1] + (n+1)(n+2)\mathsf{SUM}[n+2] = \frac{1}{n+1}$

In[5]:= **recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → False]**

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= **rec = LimitRec[rec, SUM[n], {n}, a]**

Out[4]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$

In[5]:= **recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → False]**

Out[5]= $\{\{0, \frac{1}{n}\}, \{0, -\frac{1}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i}}{n}\}, \{1, -\frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{k=1}^{n} \frac{\sum_{i=1}^{k} \frac{1}{i}}{k}}{n}\}\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{n}][[1]]$

Out[3]= $n^2\mathsf{SUM}[n] - (n+1)(2n+1)\mathsf{SUM}[n+1] + (n+1)(n+2)\mathsf{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= $\mathbf{rec} = \mathbf{LimitRec}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}], \{\mathbf{n}\}, \mathbf{a}]$

Out[4]= $n^2\mathsf{SUM}[n] - (n+1)(2n+1)\mathsf{SUM}[n+1] + (n+1)(n+2)\mathsf{SUM}[n+2] = \frac{1}{n+1}$

In[5]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}], \mathbf{IndefiniteSummation} \to \mathbf{True}]$

Out[5]= $\{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n} \frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} - \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{H_k}{k(k+n)}$

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] == \frac{(-a-1)H_a}{(a+n+1)(a+n+2)} + \frac{a}{(n+1)(a+n+1)}$

In[4]:= **rec = LimitRec[rec, SUM[n], {n}, a]**

Out[4]= $n^2 \mathrm{SUM}[n] - (n+1)(2n+1)\mathrm{SUM}[n+1] + (n+1)(n+2)\mathrm{SUM}[n+2] = \frac{1}{n+1}$

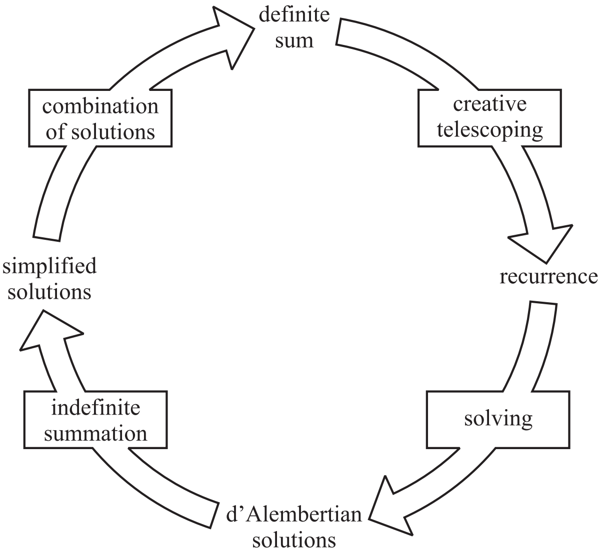
In[5]:= **recSol = SolveRecurrence[rec, SUM[n], IndefiniteSummation → True]**

Out[5]= $\{\{0, \frac{1}{n}\}, \{0, \frac{\sum_{i=1}^{n} \frac{1}{i}}{n} - \frac{1}{n^2}\}, \{1, \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} - \frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n}\}\}$

In[6]:= **FindLinearCombination[recSol, {1, {ζ₂, 1/2 + ζ₂/2}}, n, 2]**

Out[6]= $-\frac{\sum_{i=1}^{n} \frac{1}{i}}{n^2} + \frac{(\sum_{i=1}^{n} \frac{1}{i})^2}{2n} + \frac{\sum_{i=1}^{n} \frac{1}{i^2}}{2n} + \frac{\zeta_2}{n}$

Sigma's summation spiral



Toolbox 3: Special function algorithms

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,...)

$$\boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1-x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta_2 \right) dx, \qquad \zeta_z := \sum_{i=1}^{\infty} 1/i^z$$

Computer algebra and special functions:

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,...)

$$\boxed{\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}}$$

Integral representation:

$$=\int_0^1 \frac{x^n-1}{1-x}\left(\int_0^x \frac{\int_0^y \frac{1}{1-z}dz}{y}dy-\zeta_2\right)dx, \qquad \zeta_z:=\sum_{i=1}^{\infty}1/i^z$$

Asymptotic expansion:

$$=\left(\frac{1}{30n^5}-\frac{1}{6n^3}+\frac{1}{2n^2}-\frac{1}{n}\right)\ln(n)$$
$$-\frac{1}{100n^5}-\frac{1}{6n^4}+\frac{13}{36n^3}-\frac{1}{4n^2}-\frac{1}{n}+2\zeta_3+O\left(\frac{\ln(n)}{n^6}\right).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{N}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{H_j}{j}}{i}}{k}=-\frac{21\zeta_2^2}{20}\frac{1}{N}+\frac{1}{8N^2}+\frac{295}{216N^3}-\frac{1115}{96N^4}+O(N^{-5})$$
$$+\Big(\frac{1}{2N}-\frac{3}{4N^2}+\frac{19}{12N^3}-\frac{5}{N^4}+O(N^{-5})\Big)\zeta_2$$
$$+2^N\Big(\frac{3}{2N}+\frac{3}{2N^2}+\frac{9}{2N^3}+\frac{39}{2N^4}+O(N^{-5})\Big)\zeta_3$$
$$+\Big(\frac{1}{N}+\frac{3}{4N^2}-\frac{157}{36N^3}+\frac{19}{N^4}+O(N^{-5})\Big)(\log(N)+\gamma)$$
$$+\Big(\frac{1}{2N}-\frac{3}{4N^2}+\frac{19}{12N^3}-\frac{5}{N^4}+O(N^{-5})\Big)(\log(N)+\gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$\sum_{k=1}^{N}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{1+2i}}{j^2}}{(1+2k)^2} = \Big(-3+\frac{35\zeta_3}{16}\Big)\zeta_2-\frac{31\zeta_5}{8}$$
$$+\frac{1}{N}-\frac{33}{32N^2}+\frac{17}{16N^3}-\frac{4795}{4608N^4}+O(N^{-5})$$
$$+\log(2)\Big(6\zeta_2-\frac{1}{N}+\frac{9}{8N^2}-\frac{7}{6N^3}+\frac{209}{192N^4}+O(N^{-5})\Big)$$
$$+\Big(-\frac{7}{4}-\frac{7}{16N}+\frac{7}{16N^2}-\frac{77}{192N^3}+\frac{21}{64N^4}+O(N^{-5})\Big)\zeta_3$$
$$+\Big(\frac{1}{16N^2}-\frac{1}{8N^3}+\frac{65}{384N^4}+O(N^{-5})\Big)(\log(N)+\gamma)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{N} \frac{4^j H_{j-1}}{\binom{2j}{j} j^2} = 7\zeta_3 + \sqrt{\pi}\sqrt{N}\Bigg\{\Bigg[-\frac{2}{N} + \frac{5}{12N^2} - \frac{21}{320N^3} - \frac{223}{10752N^4} + \frac{671}{49152N^5} + \frac{11635}{1441792N^6} - \frac{1196757}{136314880N^7} - \frac{376193}{50331648N^8} + \frac{201980317}{18253611008N^9} + O(N^{-10})\Bigg]\ln(\bar{N}) - \frac{4}{N} + \frac{5}{18N^2} - \frac{263}{2400N^3} + \frac{579}{12544N^4} + \frac{10123}{1105920N^5} - \frac{1705445}{71368704N^6} - \frac{27135463}{11164188672N^7} + \frac{197432563}{7927234560N^8} + \frac{405757489}{775778467840N^9} + O(N^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]

Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Discovery of algebraic relations

mutliple Zeta-values

$$\sum_{i=1}^{\infty} \frac{1}{i^3} \sum_{j=1}^{i} \frac{(-1)^j}{j^2} \sum_{k=1}^{j} \frac{1}{k}$$

(Comprehensive literature: M.E. Hoffman, D. Zagier, P. Cartier, M. Petitot/H.N. Minh/C. Costermans, D.J. Broadhurst, D. Kreimer, M. Waldschmidt, D.M. Bradley, J. Vermaseren, J. Bümlein, etc.)

combining known relations of the sum and integral representations

Discovery of algebraic relations(J. Ablinger, J. Blümlein, CS)

mutliple Zeta-values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^j}{j^2}\sum_{k=1}^{j}\frac{1}{k}$$

(Comprehensive literature: M.E. Hoffman, D. Zagier,
P. Cartier, M. Petitot/H.N. Minh/C. Costermans,
D.J. Broadhurst, D. Kreimer, M. Waldschmidt,
D.M. Bradley, J. Vermaseren, J. Bümlein, etc.)

combining known relations of the sum and integral representations

cyclotomic Zeta-values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^j}{(2j+1)^2}\sum_{k=1}^{j}\frac{1}{k}$$

Discovery of algebraic relations(J. Ablinger, J. Blümlein, CS)

mutliple Zeta-values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^j}{j^2}\sum_{k=1}^{j}\frac{1}{k}$$

(Comprehensive literature: M.E. Hoffman, D. Zagier,
P. Cartier, M. Petitot/H.N. Minh/C. Costermans,
D.J. Broadhurst, D. Kreimer, M. Waldschmidt,
D.M. Bradley, J. Vermaseren, J. Bümlein, etc.)

combining known relations of the sum and integral representations

cyclotomic Zeta-values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{(-1)^j}{(2j+1)^2}\sum_{k=1}^{j}\frac{1}{k}$$

generalized multipe Zeta-values

$$\sum_{i=1}^{\infty}\frac{1}{i^3}\sum_{j=1}^{i}\frac{1}{2^j j^2}\sum_{k=1}^{j}\frac{1}{k}$$

The full machinery:

Toolbox 1 + Toolbox 2 + Toolbox 3

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously
like it comes out to exactly 1.
Is there a way I can automatically decide this?
The sum may be written in many ways, but one is:
```

$$\sum_{n,k=1}^{\infty} \frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}; \qquad H_k := \sum_{i=1}^{k} \frac{1}{i}$$

[Arose in the analysis of the simplex algorithm on the Klee-Minty cube (J. Balogh, R. Pemantle)]

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $\textbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]=
$$3\sum_{i=1}^{\infty}\frac{\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k}}{j}}{i^2}-2\sum_{j=1}^{\infty}\frac{\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k}}{j^3}+\frac{1}{3}\Big(3\sum_{j=1}^{\infty}\frac{\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k^2}}{j^2}-3\sum_{k=1}^{\infty}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k^4}\Big)-$$
$$2\sum_{k=1}^{\infty}\frac{\sum_{l=1}^{k}\frac{1}{l^3}}{k^2}+\Big(\sum_{l=1}^{\infty}\frac{1}{l^2}\Big)\Big(-\sum_{k=1}^{\infty}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k^2}+\sum_{l=1}^{\infty}\frac{1}{l^3}-1\Big)+z_2\Big(\sum_{k=1}^{\infty}\frac{\sum_{l=1}^{k}\frac{1}{l}}{k^2}-1\Big)+\sum_{l=1}^{\infty}\frac{1}{l^5}$$

The full machinery:

In[1]:= << **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k^{2}(H_{n+1}-1)^{2}}{k\,(k+n)\,n}]$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k^{2}(H_{n+1}-1)^{2}}{k\,(k+n)\,n}]$

Out[5]= $-10\zeta_3 + \zeta_2^2\left(\frac{58\zeta_3}{5} - \frac{29}{5}\right) - 10\zeta_5 + \zeta_2(-\zeta_3 + 13\zeta_5 - 4) + \frac{457\zeta_7}{8}$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{H_k(H_{n+1}-1)}{k\,(k+n)^2 n^2}]$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2-2\zeta_3+4\zeta_2\zeta_3+2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{H_k(H_{n+1}-1)}{k\,(k+n)^2 n^2}]$

Out[5]= $2\zeta_3+\zeta_2^2(\frac{17\zeta_3}{10}+\frac{17}{10})+\zeta_2(2\zeta_3-3\zeta_5-4)-\frac{9\zeta_5}{2}+\frac{3\zeta_7}{16}$

The full machinery:

In[1]:= << **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{H_k H_n H_{n+l+k}}{k\,(k+n)\,(k+n+l+1)^2}]$

The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[3]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{H_k(H_{n+1}-1)}{kn(n+1)(k+n)}]$

Out[4]= $-4\zeta_2 - 2\zeta_3 + 4\zeta_2\zeta_3 + 2\zeta_5$

In[5]:= $\mathbf{EvaluateMultiSum}[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{H_k H_n H_{n+l+k}}{k\,(k+n)\,(k+n+l+1)^2}]$

Out[5]= $3\zeta_3^2 - \frac{15\zeta_5}{2} + \zeta_2(9\zeta_5 - 6\zeta_3) + \frac{149\zeta_7}{16} + \frac{114}{35}\zeta_2^3$

Example 1: Unfair permutations

joint work with H. Prodinger, S. Wagner

- We are given n players.

- We are given n players.
- Player i: chooses randomly a number (all different)

- We are given n players.
- Player i: chooses randomly a number (all different)
- The player with the highest number gets n dices

- We are given n players.
- Player i: chooses randomly a number (all different)
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

- We are given n players.
- Player i: chooses randomly a number (all different)
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

  $$\vdots$$

  The player with the lowest number (looser) gets 1 dice.

- We are given n players.
- Player i: chooses randomly a number (all different)
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

  $$\vdots$$

  The player with the lowest number (looser) gets 1 dice.
- We get a random permutation

$$\begin{array}{c} \text{player} \\ \text{dices} \end{array} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

- We are given n players.
- Player i: chooses randomly i numbers and takes the largest (best) one
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

  $$\vdots$$

  The player with the lowest number (looser) gets 1 dice.
- We get an unfair permutation

$$\begin{array}{c} \text{player} \\ \text{dices} \end{array} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

- We are given n players.
- Player i: chooses randomly i numbers and takes the largest (best) one
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

  $$\vdots$$

  The player with the lowest number (looser) gets 1 dice.
- We get an unfair permutation

$$\begin{matrix}\text{player} \\ \text{dices}\end{matrix} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

anti-inversion:

$$i < j \text{ and } a_i < a_j$$
$$\Updownarrow$$
$$i < j \text{ and j beats i}$$

- We are given n players.
- Player i: chooses randomly i numbers and takes the largest (best) one
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

  $$\vdots$$

  The player with the lowest number (looser) gets 1 dice.
- We get an unfair permutation

$$\begin{matrix} \text{player} \\ \text{dices} \end{matrix} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

anti-inversion:

$$i < j \text{ and } a_i < a_j$$
$$\Updownarrow$$
$$i < j \text{ and j beats i}$$

probability:

$$\frac{j}{i+j}$$

- We are given n players.
- Player i: chooses randomly i numbers and takes the largest (best) one
- The player with the highest number gets n dices
  The player with the second highest number gets $n-1$ dices.

  $$\vdots$$

  The player with the lowest number (looser) gets 1 dice.
- We get an unfair permutation

$$\begin{array}{c} \text{player} \\ \text{dices} \end{array} \qquad \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \in S_n$$

anti-inversion:

$$i < j \text{ and } a_i < a_j$$
$$\Updownarrow$$
$$i < j \text{ and j beats i}$$

probability:

$$\frac{j}{i+j}$$

expected number of anti-inversions:

$$\boxed{\sum_{1 \le i < j \le n} \frac{j}{i+j}}$$

Theorem (Prodinger, Wagner).

$A_n =$ no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\leq i<j\leq n} \frac{j}{i+j}$$

Theorem (Prodinger, Wagner).

A_n = no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\leq i<j\leq n}\frac{j}{i+j}=\frac{1}{16}\big(-8n^2-8n-1\big)H_n+\frac{1}{8}(2n+1)^2H_{2n}-\frac{5n}{8}$$

Theorem (Prodinger, Wagner).

$A_n =$ no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\le i<j\le n} \frac{j}{i+j} = \frac{1}{16}\big(-8n^2-8n-1\big)H_n + \frac{1}{8}(2n+1)^2 H_{2n} - \frac{5n}{8}$$

$$= 0.3465735903n^2 - 0.4034264097n + O(\log n)$$

fair case $= 0.25n^2 - 0.25n$

Theorem (Prodinger, Wagner).

$A_n =$ no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\leq i<j\leq n} \frac{j}{i+j} = \frac{1}{16}\big(-8n^2-8n-1\big)H_n + \frac{1}{8}(2n+1)^2 H_{2n} - \frac{5n}{8}$$

The variance of A_n is

$$\begin{aligned} &2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\leq i<j<k\leq n} \frac{k}{i+j+k} \\ &+2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+k)(i+j+k)} \\ &-2\sum_{1\leq i<j<k\leq n} \frac{j}{i+j}\cdot\frac{k}{j+k} - 2\sum_{1\leq i<j<k\leq n} \frac{k}{i+k}\cdot\frac{k}{j+k} \\ &-2\sum_{1\leq i<j<k\leq n} \frac{j}{i+j}\cdot\frac{k}{i+k} - \sum_{1\leq i<j\leq n} \frac{j^2}{(i+j)^2} + \sum_{1\leq i<j\leq n} \frac{j}{i+j} \end{aligned}$$

Theorem (Prodinger, Wagner).

A_n = no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\leq i<j\leq n} \frac{j}{i+j} = \frac{1}{16}\big(-8n^2-8n-1\big)H_n + \frac{1}{8}(2n+1)^2 H_{2n} - \frac{5n}{8}$$

The variance of A_n is

$$2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\leq i<j<k\leq n} \frac{k}{i+j+k}$$
$$+2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+k)(i+j+k)}$$
$$-2\sum_{1\leq i<j<k\leq n} \frac{j}{i+j}\cdot\frac{k}{j+k} - 2\sum_{1\leq i<j<k\leq n} \frac{k}{i+k}\cdot\frac{k}{j+k}$$
$$-2\sum_{1\leq i<j<k\leq n} \frac{j}{i+j}\cdot\frac{k}{i+k} - \sum_{1\leq i<j\leq n} \frac{j^2}{(i+j)^2} + \sum_{1\leq i<j\leq n} \frac{j}{i+j}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\frac{jk}{(i+j)(j+k)}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\boxed{\sum_{i=1}^{j-1}\frac{jk}{(i+j)(j+k)}}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\boxed{\sum_{i=1}^{j-1}\frac{jk}{(i+j)(j+k)}}$$

$$||$$

summation spiral

$$\boxed{\frac{1}{j+k}jk\sum_{r=1}^{j}\frac{1}{-1+2r}-\frac{jkH_j}{2(j+k)}-\frac{k}{2(j+k)}}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\Big[\frac{1}{j+k}jk\sum_{r=1}^{j}\frac{1}{-1+2r}-\frac{jkH_j}{2(j+k)}-\frac{k}{2(j+k)}\Big]$$

$$\sum_{k=3}^{n} \boxed{\sum_{j=2}^{k-1} \Big[\frac{1}{j+k} jk \sum_{r=1}^{j} \frac{1}{-1+2r} - \frac{jkH_j}{2(j+k)} - \frac{k}{2(j+k)}\Big]}$$

$$\sum_{k=3}^{n} \boxed{\sum_{j=2}^{k-1} \Big[\frac{1}{j+k} jk \sum_{r=1}^{j} \frac{1}{-1+2r} - \frac{jkH_j}{2(j+k)} - \frac{k}{2(j+k)}\Big]}$$

$$||$$

summation spiral

$$\boxed{\begin{aligned} & -k^2 \sum_{s=1}^{k} \frac{\sum_{r=1}^{s} \frac{1}{-1+2r}}{s} + \big((k-1)k + k^2 H_k\big) \sum_{r=1}^{k} \frac{1}{-1+2r} \\ & -\frac{1}{4}k^2 H_k^2 - \frac{1}{4}k^2 H_k^{(2)} - \frac{1}{4}k(2k-3)H_k + \frac{1}{4} \end{aligned}}$$

$$\sum_{k=3}^{n}\Big[\quad -k^2\sum_{s=1}^{k}\frac{\sum_{r=1}^{s}\frac{1}{-1+2r}}{s}+\big((k-1)k+k^2H_k\big)\sum_{r=1}^{k}\frac{1}{-1+2r}$$

$$-\frac{1}{4}k^2H_k^2-\frac{1}{4}k^2H_k^{(2)}-\frac{1}{4}k(2k-3)H_k+\frac{1}{4}\Big]$$

$$\sum_{k=3}^{n}\Big[\quad -k^2\sum_{s=1}^{k}\frac{\sum_{r=1}^{s}\frac{1}{-1+2r}}{s}+\big((k-1)k+k^2H_k\big)\sum_{r=1}^{k}\frac{1}{-1+2r}$$
$$-\frac{1}{4}k^2H_k^2-\frac{1}{4}k^2H_k^{(2)}-\frac{1}{4}k(2k-3)H_k+\frac{1}{4}\Big]$$

$$||$$

summation spiral

$$n(n+1)(2n+1)\Big[-\frac{1}{6}\sum_{s=1}^{n}\frac{\sum_{r=1}^{s}\frac{1}{-1+2r}}{s}-\frac{1}{12}\big)H_n-\frac{1}{24}H_n^2$$
$$+\big(\frac{1}{6}\sum_{r=1}^{n}\frac{1}{-1+2r}-\frac{1}{24}H_n^{(2)}+\frac{1}{6}\sum_{r=1}^{n}\frac{1}{-1+2r}\Big]$$
$$-\frac{1}{8}(2n+1)^2\sum_{r=1}^{n}\frac{1}{-1+2r}+\frac{1}{12}(n+1)(4n+1)H_n+\frac{7n}{24}$$

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\frac{jk}{(i+j)(j+k)}$$

$$||$$

$$n(n+1)(2n+1)\Big[-\frac{1}{6}\sum_{s=1}^{n}\frac{\sum_{r=1}^{s}\frac{1}{-1+2r}}{s}-\frac{1}{12})H_n-\frac{1}{24}H_n^2$$

$$+(\frac{1}{6}\sum_{r=1}^{n}\frac{1}{-1+2r}-\frac{1}{24}H_n^{(2)}+\frac{1}{6}\sum_{r=1}^{n}\frac{1}{-1+2r}\Big]$$

$$-\frac{1}{8}(2n+1)^2\sum_{r=1}^{n}\frac{1}{-1+2r}+\frac{1}{12}(n+1)(4n+1)H_n+\frac{7n}{24}$$

Theorem (Prodinger, Wagner).

A_n = no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\leq i<j\leq n} \frac{j}{i+j} = \frac{1}{16}\big(-8n^2-8n-1\big)H_n + \frac{1}{8}(2n+1)^2 H_{2n} - \frac{5n}{8}$$

The variance of A_n is

$$2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\leq i<j<k\leq n} \frac{k}{i+j+k}$$

$$+2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+j)(i+j+k)} + 2\sum_{1\leq i<j<k\leq n} \frac{kj}{(i+k)(i+j+k)}$$

$$-2\sum_{1\leq i<j<k\leq n} \frac{j}{i+j}\cdot\frac{k}{j+k} - 2\sum_{1\leq i<j<k\leq n} \frac{k}{i+k}\cdot\frac{k}{j+k}$$

$$-2\sum_{1\leq i<j<k\leq n} \frac{j}{i+j}\cdot\frac{k}{i+k} - \sum_{1\leq i<j\leq n} \frac{j^2}{(i+j)^2} + \sum_{1\leq i<j\leq n} \frac{j}{i+j}$$

Theorem (Prodinger, Wagner, CS).

$A_n =$ no. of anti-inversions of a random unfair permutation of length n.

Then the mean of A_n is

$$\sum_{1\leq i<j\leq n} \frac{j}{i+j} = \frac{1}{16}\big(-8n^2-8n-1\big)H_n + \frac{1}{8}(2n+1)^2 H_{2n} - \frac{5n}{8}$$

The variance of A_n is

$$\frac{n(29+126n+72n^2)}{216} + \frac{35+108n+81n^2-27n^3}{162}H_n$$
$$+ \frac{-3-16n-10n^2+8n^3}{12}H_{2n} + \frac{-16+27n-54n^3}{108}H_{3n}$$
$$+ \frac{n(1+3n+2n^2)}{6}\left(3H_{2n}^{(2)} - 2H_n^{(2)} + 4\sum_{1\leq i\leq 2n}\frac{(-1)^i H_i}{i}\right)$$
$$+ \frac{8}{27}\sum_{i=1}^{n}\frac{1}{3i-2} + \frac{(-1)^n n}{4}\left(\sum_{i=1}^{n}\frac{(-1)^i}{i} - \sum_{i=1}^{3n}\frac{(-1)^i}{i}\right),$$

Example 2: Super-congruences

(S. Ahlgren, E. Mortenson, R. Osburn, Sigma)

Sigma's contribution to harmonic number congruences

- S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \mod p$$

Sigma's contribution to harmonic number congruences

- S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \mod p$$

- E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \mod p$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j) \equiv 0 \mod p$$

Sigma's contribution to harmonic number congruences

- S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}) \equiv 0 \mod p$$

- E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} (1 + 3jH_{j+\frac{p-1}{2}} - 3jH_j) \equiv 0 \mod p$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} (1 + 2jH_{j+\frac{p-1}{2}} - 2jH_j) \equiv 0 \mod p$$

- R. Osburn:

$$p^2 E_2(p) + p\, E_1(p) + p^0 E_0(p) \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$p^2 E_2(p)$

$+pE_1(p)$

$+p^0 E_0(p) \quad \equiv \quad (-1)^{\frac{p-1}{2}} \mod p^3$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+pE_1(p)$$

$$+p^0 \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2 E_2(p)$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[\sum_{j=1}^{\frac{p-3}{2}}\Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\Big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[\sum_{j=1}^{\frac{p-3}{2}}\Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\big)\Big)\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

$$\sum_{j=1}^{\frac{p-3}{2}} \Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \Big(1 + 4j\big(H_{j+\frac{p-1}{2}} - H_j\big)$$

$$+ j^2\Big(2\big(H_{j+\frac{p-1}{2}} - H_j\big)^2 + H_j^{(2)} - H^{(2)}_{j+\frac{p-1}{2}}\Big)\Big)$$

$$\sum_{j=1}^{n-1}\Big(\frac{(-1)^j}{\binom{n}{j}\binom{j+n}{j}}$$

$$+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+4j\big(H_{j+n}-H_j\big)$$

$$+j^2\big(2\big(H_{j+n}-H_j\big)^2+H_j^{(2)}-H_{j+n}^{(2)}\big)\big)$$

$$\sum_{j=1}^{n-1}\Big(\frac{(-1)^j}{\binom{n}{j}\binom{j+n}{j}}$$

$$+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+4j\big(H_{j+n}-H_j\big)$$

$$+j^2\big(2\big(H_{j+n}-H_j\big)^2+H_j^{(2)}-H_{j+n}^{(2)}\big)\big)$$

$$||$$

summation spiral

$$(-1)^n((n+1)(2n+1)-\binom{2n}{n})$$

$$\sum_{j=1}^{\frac{p-3}{2}} \Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+ \sum_{j=0}^{\frac{p-1}{2}} (-1)^j \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \Big(1 + 4j\big(H_{j+\frac{p-1}{2}} - H_j\big)$$

$$+ j^2\Big(2\big(H_{j+\frac{p-1}{2}} - H_j\big)^2 + H_j^{(2)} - H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)$$

$$||$$

$$(-1)^{\frac{p-1}{2}} \Big(\big(\frac{p-1}{2} + 1\big)p - \binom{p-1}{\frac{p-1}{2}}\Big)$$

For a prime $p > 2$,

$$p^2\Bigg[\sum_{j=1}^{\frac{p-3}{2}}\Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\big)\Big)\Bigg]$$

$$+p\Bigg[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Bigg]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$(-1)^{\frac{p-1}{2}}\left(\left(\tfrac{p-1}{2}+1\right)p-\binom{p-1}{\frac{p-1}{2}}\right)$$

$$\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\left(1+j\left(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\right)\right)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$-(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}$$

$$\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$-(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}$$

$$\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

$$\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(-2Hj+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}\big)\big)$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+j\big(-2Hj+H_{j+n}+H_{-j+n}\big)\big)$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+n}{j}\big(1+j\big(-2Hj+H_{j+n}+H_{-j+n}\big)\big)$$

$$||$$

summation spiral

$$-\frac{3}{2}(-1)^n n(n+1)\sum_{j=1}^{n}\frac{\binom{2j}{j}}{j}+(-1)^n(2n+1)\binom{2n}{n}$$

$$\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(-2Hj+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}\big)\big)$$

$$||$$

$$-\frac{3}{2}(-1)^{\frac{p-1}{2}}\Big(\frac{p^2}{4}-\frac{1}{4}\Big)\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}+(-1)^{\frac{p-1}{2}}p\binom{p-1}{\frac{p-1}{2}}$$

For a prime $p > 2$,

$$p^2\Big[$$

$$-(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}$$

$$\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$-(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}$$

$$\Big]$$

$$+p\Big[-\frac{3}{2}(-1)^{\frac{p-1}{2}}\Big(\frac{p^2}{4}-\frac{1}{4}\Big)\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}+(-1)^{\frac{p-1}{2}}p\binom{p-1}{\frac{p-1}{2}}\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[\quad 0 \quad \Big]$$

$$+p\Big[-\frac{3}{2}(-1)^{\frac{p-1}{2}}\Big(\frac{p^2}{4}-\frac{1}{4}\Big)\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[$$

$$0$$

$$\Big]$$

$$+p\Big[\frac{3}{8}(-1)^{\frac{p-1}{2}}\sum_{j=1}^{\frac{p-1}{2}}\frac{\binom{2j}{j}}{j}\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

For a prime $p > 2$,

$$p^2\Big[\sum_{j=1}^{\frac{p-3}{2}}\Big(\frac{(-1)^j}{\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}}$$

$$+\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+4j\big(H_{j+\frac{p-1}{2}}-H_j\big)$$

$$+j^2\Big(2\big(H_{j+\frac{p-1}{2}}-H_j\big)^2+H_j^{(2)}-H_{j+\frac{p-1}{2}}^{(2)}\Big)\Big)\Big]$$

$$+p\Big[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{\frac{p-1}{2}}{j}\binom{j+\frac{p-1}{2}}{j}\Big(1+j\big(+H_{j+\frac{p-1}{2}}+H_{-j+\frac{p-1}{2}}-2H_j\big)\Big)\Big]$$

$$+p^0\sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

Sigma's contribution to harmonic number congruences

- S. Ahlgren (2001):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} \left(H_{j+\frac{p-1}{2}} - H_{\frac{p-1}{2}}\right) \equiv 0 \mod p$$

- E. Mortenson (2003):

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{j+\frac{p-1}{2}}{j} \left(1+3jH_{j+\frac{p-1}{2}} - 3jH_j\right) \equiv 0 \mod p$$

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j} \binom{j+\frac{p-1}{2}}{j} \left(1+2jH_{j+\frac{p-1}{2}} - 2jH_j\right) \equiv 0 \mod p$$

- R. Osburn/CS (2008):

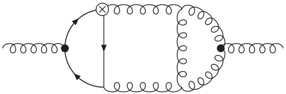
$$p\,\frac{3}{8}(-1)^{\frac{p-1}{2}} \sum_{j=1}^{\frac{p-1}{2}} \frac{\binom{2j}{j}}{j} + \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^2 16^{-j} \equiv (-1)^{\frac{p-1}{2}} \mod p^3$$

Example 3: Feynman integrals

joint work with J. Ablinger, A. Behring, J. Blümlein, A. Hasselhuhn, A. de Freitas, C. Raab, M. Round, F. Wissbrock (RISC–DESY)

Evaluation of Feynman diagrams

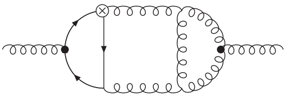
(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

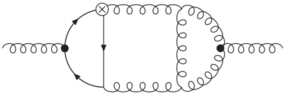
$$\longrightarrow$$

$$\int \Phi(n, \epsilon, x) dx$$

Feynman integrals

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\longrightarrow \quad \int \Phi(n,\epsilon,x)dx$$

Feynman integrals

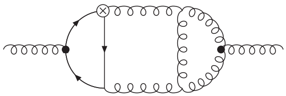
$\Big\downarrow$ **DESY**

$$\sum f(n,\epsilon,k)$$

multi sums

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

DESY

symbolic summation

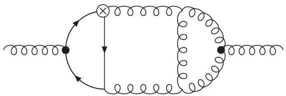
simple sum expressions

$$\sum f(n, \epsilon, k)$$

multi sums

Evaluation of Feynman diagrams

(long term project with J. Blümlein, Deutsches Elektronen–Synchrotron)



Behavior of particles

$$\int \Phi(n, \epsilon, x)dx$$

Feynman integrals

DESY

$$\sum f(n, \epsilon, k)$$

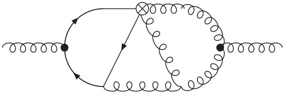
multi sums

symbolic summation

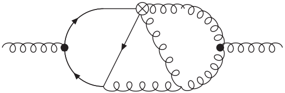
simple sum expressions

processable by physicists

Evaluations required for the
LHC experiment at CERN



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)} + \dots$$

Simplify

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4H_{-j+n-1}-4H_{-j+n-2}-2H_k$$

$$-\left(H_{-l+n-q-2}+H_{-l+n-q-r-s-3}-2H_{r+s}\right)$$

$$+2H_{s-1}-2H_{r+s}\Big] + \textbf{3 further 6–fold sums}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}{H_N}^4+\frac{(17N+5){H_N}^3}{3N(N+1)}+\Big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13H_N^{(2)}}{2}+\frac{5(-1)^N}{2N^2}\Big){H_N}^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2}+\Big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\Big)H_N^{(2)}+\Big(\frac{29}{3}-(-1)^N\Big)H_N^{(3)}$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\Big)H_N+\Big(\frac{3}{4}+(-1)^N\Big)H_N^{(2)\,2}$$
$$-2(-1)^N S_{-2}(N)^2+S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)H_N+\frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\Big)H_N^{(2)}+S_{-2}(N)\Big(10{H_N}^2+\Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)H_N+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)H_N^{(2)}-\frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\Big)H_N^{(3)}+\Big(\frac{19}{2}-2(-1)^N\Big)H_N^{(4)}+\big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\Big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N)+\left(\frac{3}{2}{H_N}^2-\frac{3H_N}{N}+\frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}{H_N}^4 + \frac{(17N+5){H_N}^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13H_N^{(2)}}{2} + \frac{5(-1)^N}{2N^2}\Big){H_N}^2$$

$$+\Big(-\frac{4}{N} \ldots \frac{(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)H_N^{(2)} + \Big(\frac{29}{3} - (-1)^N\Big)H_N^{(3)}$$

$$H_N = \sum_{i=1}^{N}\frac{1}{i}$$

$$+\Big(2+ \ldots - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)H_N + \Big(\frac{3}{4} + (-1)^N\Big)H_N^{(2)\,2}$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)H_N + \frac{4(-1)^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)H_N^{(2)} + S_{-2}(N)\Big(10{H_N}^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)H_N + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)H_N^{(2)} - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)H_N^{(3)} + \Big(\frac{19}{2} - 2(-1)^N\Big)H_N^{(4)} + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}{H_N}^2 - \frac{3H_N}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}{H_N}^4 + \frac{(17N+5){H_N}^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13H_N^{(2)}}{2} + \frac{5(-1)^N}{2N^2}\Big){H_N}^2$$

$$+ \Big(-\frac{4}{N} \text{[illegible]} (-1)^N(2N+1) \text{[illegible]} \frac{}{N(N+1)} - \frac{13}{N}\Big)H_N^{(2)} + \Big(\frac{29}{3} - (-1)^N\Big)H_N^{(3)}$$

$$+ \Big(2 + 2 \text{[illegible]} - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} \text{[illegible]} \Big)H_N^{(2)\,2}$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26 + 4(\text{[illegible]} \Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)H_N^{(2)} + S_{-2}(N)\Big(10{H_N}^2 + \Big(\frac{\text{[illegible]}(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)H_N + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22 + 6(-1)^N\big)H_N^{(2)} - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)H_N^{(3)} + \Big(\frac{19}{2} - 2(-1)^N\Big)H_N^{(4)} + \big(-6 + 5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20 + 2(-1)^N\big)S_{2,-2}(N) + \big(-17 + 13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1) + 4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24 + 4(-1)^N\big)S_{-3,1}(N) + \big(3 - 5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}{H_N}^2 - \frac{3H_N}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$H_N = \sum_{i=1}^{N}\frac{1}{i}$$

$$H_N^{(2)} = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}{H_N}^4 + \frac{(17N+5){H_N}^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13H_N^{(2)}}{2} + \frac{5(-1)^N}{2N^2}\Big){H_N}^2$$

$$+\Big(\dots \frac{(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)H_N^{(2)} + \Big(\frac{29}{3} - (-1)^N\Big)H_N^{(3)}$$

$$+\big(2+\dots - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} \dots \big){H_N^{(2)}}^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(\dots\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(\dots)} - \frac{5}{\dots}\Big)H_N^{(2)} + S_{-2}(N)\Big(10{H_N}^2 + \Big(\frac{\dots}{N(N+1)}$$

$$+\frac{4(3N\dots}{N(N+\dots} \dots)^N\big)H_N^{(2)} - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N\dots}{N(\dots} \dots + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(\dots}{\dots} \dots\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N\dots}{N(N+1)} \dots S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}{H_N}^2 - \frac{3H_N}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$H_N = \sum_{i=1}^{N} \frac{1}{i}$$

$$H_N^{(2)} = \sum_{i=1}^{N} \frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

Summarizing:

If you have

Summarizing:

If you have

unfair permutations/monster sums/. . .

Summarizing:

If you have

unfair permutations/monster sums/...

super congruences/identities/...,

Summarizing:

If you have

unfair permutations/monster sums/...

super congruences/identities/...,

give the presented machinery a try!