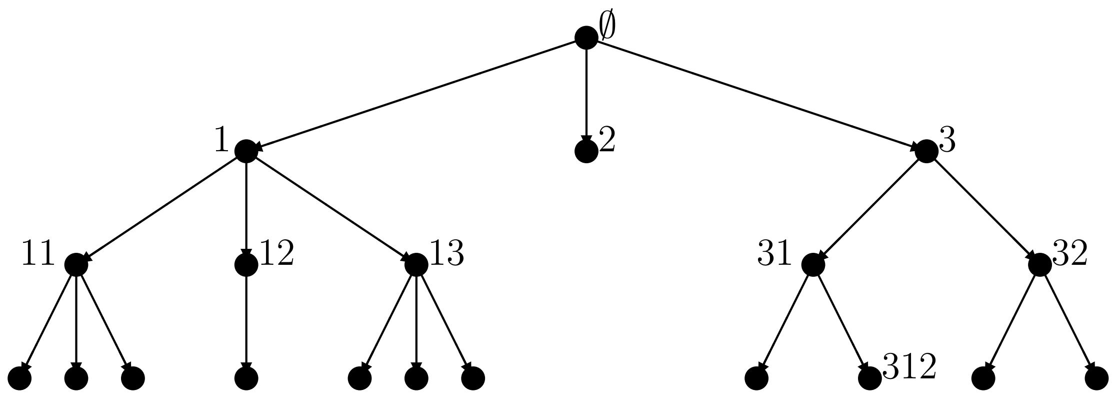
WEIGHTED BRANCHING PROCESS

Uwe Roesler

- **Weighted Branching Process**
- **Stochastic fixed point equations of sum type**
- **Forward and backward view**
- **Stochastic divide-and-conquer algorithms**
- **Quicksort process**

WEIGHTED BRANCHING PROCESS



weighted branching process $(V, L, G, *)$
Ulam-Harris notation $V = \mathbb{N}^* = \cup_{n=0}^{\infty} \mathbb{N}^n$
(genealogical) graph V with directed edges (v, vi)
$(G, *)$ measurable semi group with neutral element and grave, $* : G \times G \to G$
random weight $L_{v,vi}$ on edge (v, vi) with values in G
The object is called Weighted Branching Process (WBP) if

$$(L_{v,vi})_i, \ v \in V \qquad \text{iid rvs}$$

$L:$ paths $\to G$ recursively $L_{v,v}$ neutral element and

$$L_{v,vwi} = L_{v,vw} * L_{vw,vwi}$$

— general dynamical system

$$L_{v,vwx} = L_{v,vw} * L_{vw,vwx}$$

— If only $(L_i)_i$ is given, take iid copies. Markov chain with transitions

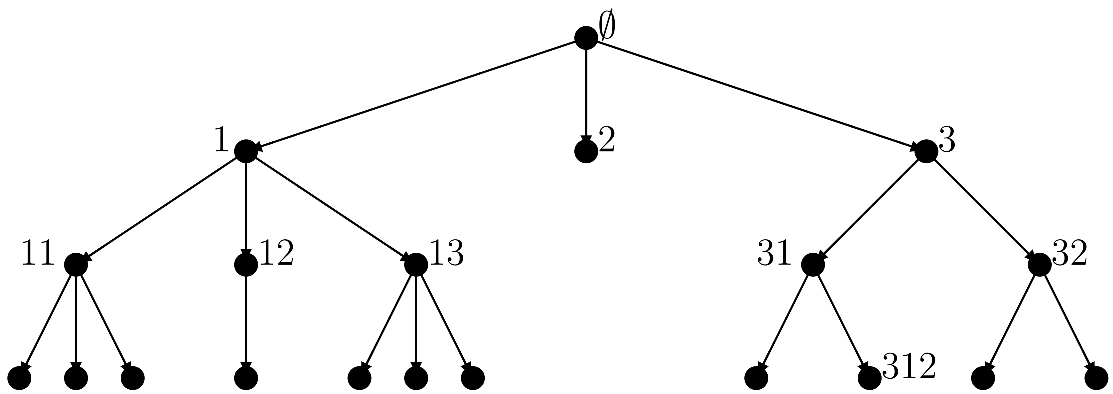
$$k(g,A) = P((g * L_{v,vi})_i \in A)$$

— Given k under weak assumptions exists a random map $(L_i)_i$ with above.
— Our objects random dynamical system.
— $G \subset H^H$ since for $G = H$ identify $T_g = g$

$$T_g : G \to G,\ T_g(h) = g * h \text{ or } T_g(h) = h * g$$

Not: $L_v = L_{\emptyset,v}$
Interpretation: v not observable iff L_v is the grave
Ex: **Free semi group:** $L_i \in G$ generator of G

FORWARD and BACKWARD



Forward Dynamics (Genealogy:) Start with h at root and vertex v gives i-th child its weight via $L_{v,vi}$.

Formal: G acts right on H_l, $*_l : H_l \times G \mapsto H_l$

Ex: Biennaymé-Galton-Watson: $G = \{0, 1\}$ with multiplication. BGW root has weight 1 and $1 *_l L_v = L_v$ values 0 or 1 with interpretation dead or alive.

Backward dynamic: Weight of children determine weight of mother

$$G^{\mathbb{N}} \ni (g_{vi})_i \mapsto \Psi_v((g_{vi})_i)) = g_v$$

Given $g_v \in G,\ |v| = n$ determine $g_v,\ |v| < n$.
Then $n \to \infty$. Weight on δV

STOCHASTIC FIXED POINT EQUATIONS

Stochastic fixed point equation (SFE)

$$X \stackrel{\mathcal{D}}{=} \sum_{i \in I\!N} L_i X_i + C$$

$((L_i)_i, C), X_j,\ j \in I\!N$ independent, $X_j \stackrel{\mathcal{D}}{=} X$.

Ex: **Gauss**

$$X \stackrel{\mathcal{D}}{=} \frac{X_1}{\sqrt{2}} + \frac{X_2}{\sqrt{2}}$$

solved by Gauss distribution $N(0, \sigma^2),\ \sigma^2 > 0$. Unique up to variance

Ex: **Cauchy**

$$X \stackrel{\mathcal{D}}{=} \frac{X_1}{2} + \frac{X_2}{2}$$

solved by symmetric Cauchy(b) distribution, density $x \mapsto \frac{b}{\pi(b^2+x^2)}$ and by constants

Ex: α**-stable-distributions** $\alpha \in (0, 2]$ (=Limit of sums of iid)

X α-stable distribution iff $X \stackrel{\mathcal{D}}{=} aX_1 + bX_2 + c$ for all $a, b \in I\!R$ satisfying $|a|^\alpha + |b|^\alpha = 1$ and exists c.

Replace for all by some specific a, b, c. Unique solution?

Ex: **Bienaymé-Galton-Watson**

$$W \stackrel{\mathcal{D}}{=} \sum_{i \leq N} \frac{1}{m} W_i$$

STOCHASTIC DIVIDE-and-CONQUER ALGORITHMS

Divide problem into smaller ones.
Ex: **Quicksort.**
Running time analysis of algorithms

$$X \stackrel{\mathcal{D}}{=} UX_1 + (1-U)X_2 + C(U)$$

U uniformly distributed and

$$C(x) = 2x\ln x + 2(1-x)\ln(1-x) + 1$$

Solution not unique, Fill-Janson
Many more algorithms now, Neininger,

Quicksort (Roesler '91, contraction method) motivated study of SFE as objects of own interest, Roesler, Alsmeyer,....

MEASURE and PROBABILITY THEORY

• Why in distribution?

$$X \stackrel{\mathcal{D}}{=} \sum_i L_i X_i + C$$

Always exists rvs and common probability space

$$X = \sum_i L_i X_i + C$$

Quick answer: Easier since lower level. More to come.

• Why called fixed point?

Map K from distribution to distributions

$$K(\mu) \stackrel{\mathcal{D}}{=} \sum_i L_i Y_i + C$$

$((L_i)_i, C), Y_j)$ independent, $Y_j \stackrel{\mathcal{D}}{=} \mu$.

Fixed point $K(\mu) = \mu$ solves SFE

Contraction method, overview Rüschendorf-Roesler '01, Neininger

• Where is the dynamics and connection to branching processes?

Iterate the homogeneous SFE

$$X = \sum_i L_i X_i + C = \sum_i \sum_j L_i L_{i,ij} X_{i,j} + \sum_i L_i C_i + C = \ldots$$

Imagine a genealogical tree (possible infinite branches), the mother passes her value randomly transformed to her children. Dependence within a family, but independence for families.

BACKWARD and FORWARD VIEW

For simplicity G the reals with multiplication and grave 0.

Object of interest

$$Z_n := \sum_{|v|=n} L_v$$

Forward and **backward view**, analogous to Markov processes, Letac, Roesler

Forward equation

$$Z_n = \sum_{|v|=n-1} L_v \sum_i L_{v,vi}$$

Forward view is on random variables.
Forward structure is (often) a martingale

$$\frac{Z_n}{m^n} \longrightarrow_n W \qquad m := E \textstyle\sum_i L_i$$

Forward result is a.e. convergence

Backward equation

$$Z_n = \sum_i L_i Z^i_{n-1}$$

Z^i for tree iV.
Backward view is on measures.
Backward structure is an iteration on K, $K(\mu) \stackrel{\mathcal{D}}{=} \sum_i L_i X_i$
Backward result is weak convergence $K^n(\nu) \to_n \mu = K(\mu)$

BACKWARD and FORWARD VIEW 2

Connection:

$$K^n(\delta_1) \stackrel{\mathcal{D}}{=} Z_n$$

For $m = 1$ Z_n is a martingale and converges a.e. to W satisfying

$$W = \sum_i L_i W^i$$

Analogous with

$$R_n = \sum_{|v|<n} L_v *_r C_v$$

$((L_{v,vi})_i, C_v)$ iid, $C_v : \Omega \to H_r$
backward view and convergence $R_n \to R$

$$R_n = \sum_i L_i *_r R^i_{n-1} + C$$

$$R = \sum_i L_i *_r R^i + C$$

ANALOG to KESTEN-STEGUM

Forward dynamic, G the positive reals

$$\mathbb{R}_+ \ni \theta \mapsto m(\theta) = E\sum_i |L_i|^\theta$$

$$m = m(1)$$

$$\frac{Z_n}{m^n}$$

is positive martingale and converges to W

$$W = \sum_i \frac{L_i}{m} W^i$$

Theo Biggins '77, Lyons '97
Suppose not a GWP and $m < \infty$ and $m'(1)$ finite. Then are equivalent
– $P(W = 0) = q$ Extinction probability
– $P(W = 0) < 1$
– $E(W) = 1$
– $E(X \ln^+ X)$ finite for $X = \sum_i L_i$ and $E\sum_i L_i \ln L_i < m \ln m$.
Rem: Biggins Branching random walk is special WBP

Analog $R_n \to R$ by L^p-martingale

$$R = \sum_i L_i *_H R^i + C$$

HOMOGENEOUS SFE

$$X \stackrel{\mathcal{D}}{=} \sum_i L_i X_i$$

Find **all** solutions.

There are **endogenous** solutions (=measurable with respect to all $(L_{v,vi}, C_v)$) and **non endogenous**.

Endogenous by forward dynamics, non endogenous by backward dynamics.

Endogenous are easier.

Endogenous solutions: Biggins '77, Lyons et al. '95
via $\frac{1}{m^n}\sum_{|v|=n} L_v \to X$

Non-endogenous: Kahane-Peyriere 76, Durrett-Liggett '83, Liu 98, Alsmeyer-Roesler 05,

All solutions for real weights Alsmeyer-Biggins-Meiners '12 and Alsmeyer-Meiners 13

Theo All solutions are mixtures of stable ones

$$W^{1/\alpha}Y \qquad E\sum_i L_i^\alpha = 1 \qquad W \stackrel{\mathcal{D}}{=} \sum_i L_i^\alpha W_i$$

INHOMOGENEOUS SFE

$$X \stackrel{\mathcal{D}}{=} \sum_i L_i X_i + C$$

Find **all** solutions.

Endogenous solutions: Roesler '92

Non-endogenous: Rüschendorf principle: general solution =

one inhomogeneous solution + general homogeneous

Final result by Alsmeyer-Meiners '11

Theo All solutions: one special for inhomog + general for homog.

Ex: Quicksort: $\alpha = 1$, W a constant, Y symmetric Cauchy, Fill-Janson

CONVERGENCE in DISTRIBUTION

Contraction method for the operator K

$$K(\mu) \stackrel{\mathcal{D}}{=} \sum_i L_i X_i + C$$

by contraction, Banach Fixed Point Theorem.

Metric on space of distributions with nice properties, like Wasserstein or Zolotarev or....

$$d_p(\mu,\nu) = \inf \|X - Y\|_p$$

$$\xi_p(\mu,\nu) = \inf |Ef(X) - Ef(Y)|$$

$D^{\lfloor p \rfloor} f$ is Hölder $p - \lfloor p \rfloor$ continuous

more .....

Real strength of contraction method shows up for 'dirty' recursions

$$X(n) \stackrel{\mathcal{D}}{=} \sum_i L_i(I(n))X_i(I(n)) + C(I(n))$$

$((L_i(\cdot))_i, C(\cdot), I(n))$ independent, $I(n) < n$

Assume: $I(n) \to_n \infty,\ L_i(n) \to_n L_i,\ C(n) \to_n C$

Hope: $X(n) \to X$ and

$$X \stackrel{\mathcal{D}}{=} \sum_i L_i X_i + C$$

'nice' metric, Roesler 91, Neininger-Rüschendorf 04, 05,

DIVIDE-and-CONQUER ALGORITHMS

Split a problem of some level into problems of smaller level. Continue until solvable.

Quicksort is typical example for stochastic divide-and-conquer algorithm.

Input: n different number

Output: These numbers in natural order

Algorithm **Quicksort:**

– Pick a pivot by random.

– Build list (urn) of strictly smaller, the list containing only the pivot and list of strictly larger numbers than the pivot.

– Store them in this order.

– Recall algorithm (independently) for every list with 2 or more elements.

$Y(n)$ denotes number of comparisons (=running time). Backward view for WBP

$$Y(n) = Y^1(I_n) + Y^2(n+1-I_n) + n - 1$$

Y^1, Y^2, I_n independent, $Y^1 \stackrel{\mathcal{D}}{=} Y^2$, I_n uniformly on $\{0, 1, 2, \ldots, n-1\}$

Then $X(n) := \frac{Y(n)-EY(n)}{n+1}$ satisfies

$$X(n) \stackrel{\mathcal{D}}{=} \frac{I(n)}{n+1} X_1(I(n)) + \frac{n+1-I_n}{n+1} X_2(n+1-I(n)) + C(I(n))$$

$X_1, X_2, I(n)$ independent, $\frac{I(n)}{n+1} \to U$ uniformly distributed, $C(I(n)) \to_n C(U)$

Obtain Quicksort recursion in limit

$$X \stackrel{\mathcal{D}}{=} UX_1 + (1-U)X_2 + C(U)$$

Unique solution in L^p.

BINARY SEARCH TREE

Input $U_n,\ n \in I\!N$ iid uniform distribution on $[0,1]$

Every $U_{|n}$ provides binary search tree $T_n = T(U_1, \ldots, U_n)$

• $(T_n)_n$ is a Markov chain. Transitions choose uniformly an inner leave and add edge.

Régnier '89 found the L^2-martingale

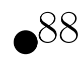
$$X(T_n) = \frac{Y(T_n) - EY(T_n)}{n+1}$$

Provides $(X(T_n)) \to X$ almost surely.

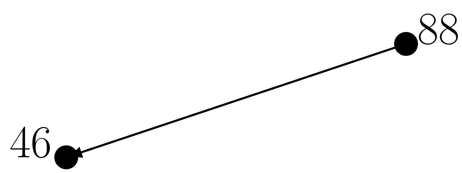
BINARY SEARCH TREE

Input: Sequence $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\neq}$
Here $x = (88, 46, 90, 60, 98, 47, 24, 95, 78)$

$(\mathbf{88}, 46, 90, 60, 98, 47, 24, 95, 78)$



$(\mathbf{88}, \mathbf{46}, 90, 60, 98, 47, 24, 95, 78)$



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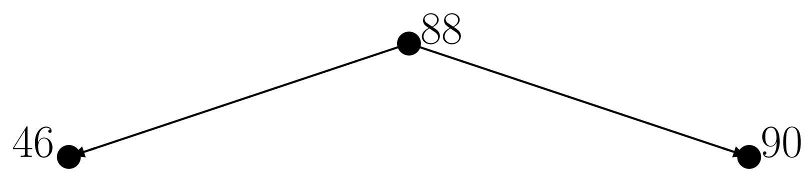

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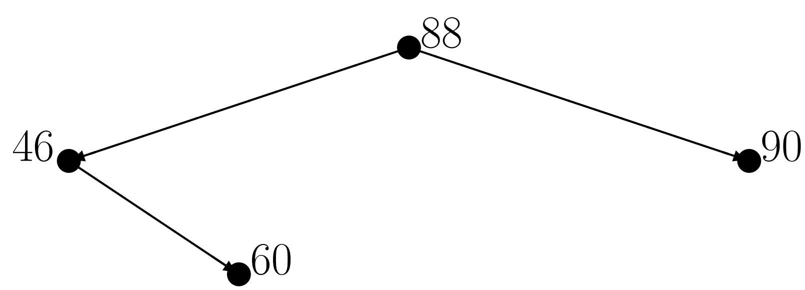

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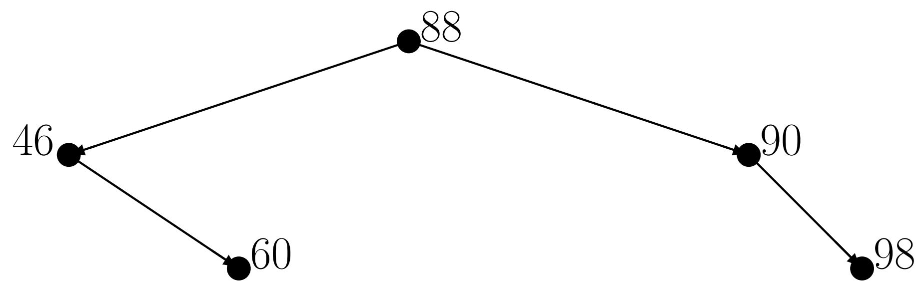

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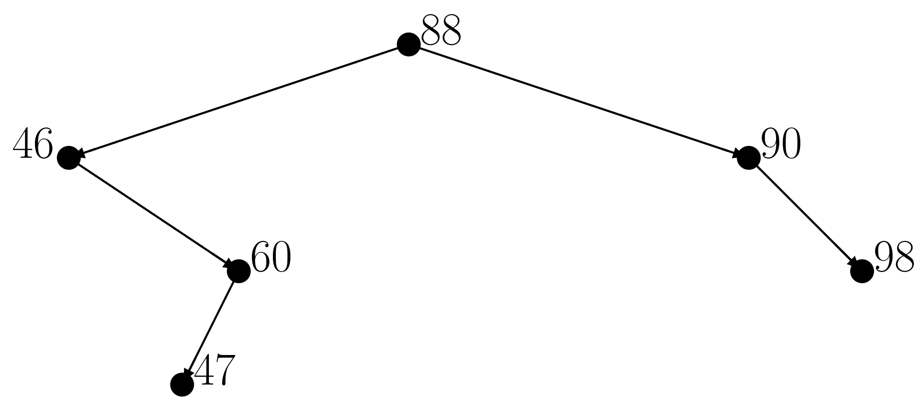

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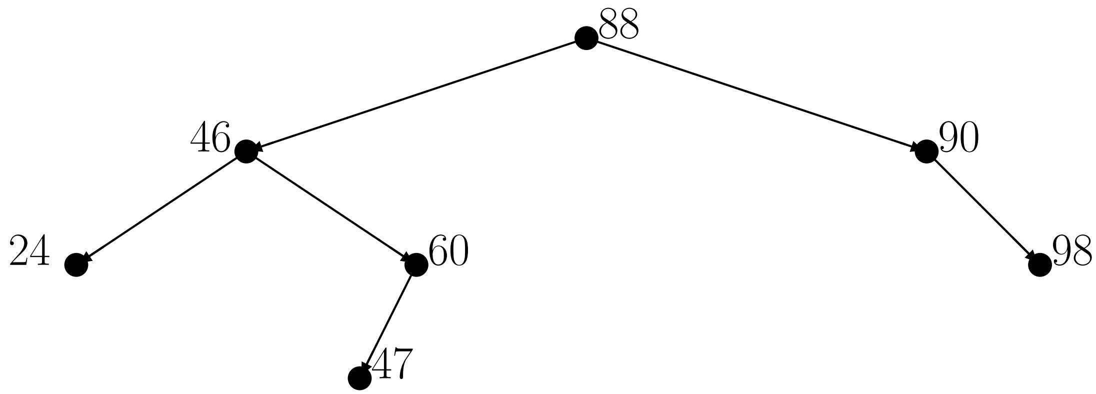

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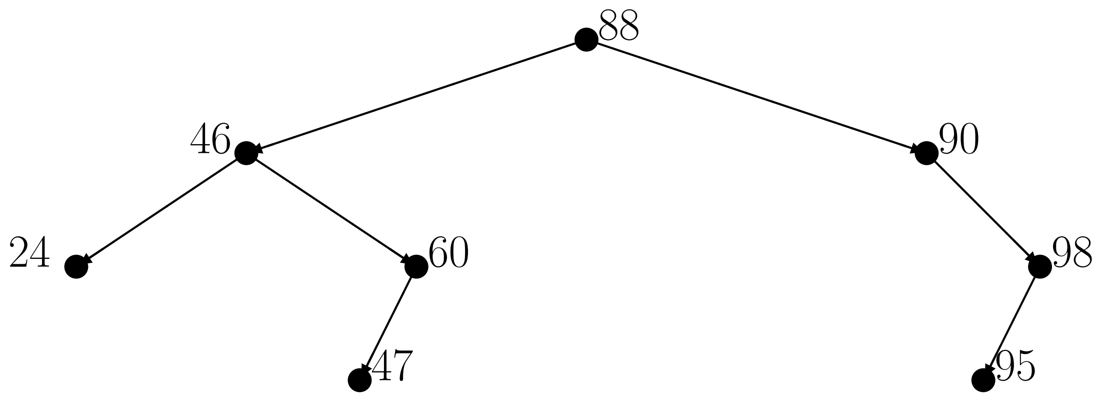
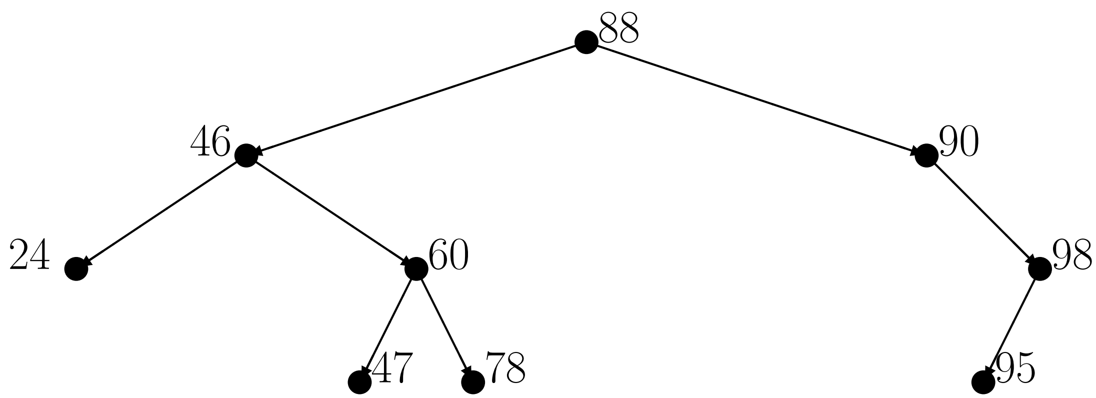


$(\mathbf{88}, \mathbf{46}, \mathbf{90}, \mathbf{60}, \mathbf{98}, \mathbf{47}, \mathbf{24}, \mathbf{95}, \mathbf{78})$



If you traverse from left to right, you obtain the ordered input.

PROCESSES as SOLUTION

Processes $X = X(t)_{t\in[0,1]}$ in D

$$X \stackrel{\mathcal{D}}{=} (\sum_i A_i(t)X_i(B_i(t)) + C(t))_t$$

– $((A_i, B_i)_i, C), X_j,\ j \in \mathbb{N}$ independent,
– $X_j \stackrel{\mathcal{D}}{=} X$
Ex: **Brownian motion**

$$X \stackrel{\mathcal{D}}{=} (1\!\!1_{t<1/2}\frac{1}{\sqrt{2}}X_1(2t) + 1_{t\geq 1/2}\frac{1}{\sqrt{2}}(X_1(1) + X_2(2t-1)))_t$$

Ex: **α-stable processes**
Ex: **Find** Analysis of algorithms,
Grübel-Roesler '96 for $H = D[0,1]$ cadlag functions

$$X \stackrel{\mathcal{D}}{=} (1\!\!1_{t<U}UX_1(\frac{t}{U}) + 1\!\!1_{U\leq t}(1-U)X_2(\frac{t-U}{1-U}) + 1)_t$$

U uniformly distributed on $[0,1]$ Unique solution in $L^p,\ p>1$

Ex: **Quicksort process** on D_-

$$X \stackrel{\mathcal{D}}{=} (UX_1(1 \wedge \frac{t}{U}) + (1-U)X_2(0 \vee \frac{t-U}{1-U}) + C(U,t))_t$$

$$\begin{aligned} X(0) &= 0 \\ C(x,t) &= C(x) + 2\mathbb{1}_{x \geq t}((1-t)\ln(1-t) + (1-x)\ln(1-x) + 1) - (x-t)\ln(x-t) \\ C(x) &= 1 + 2x\ln x + 2(1-x)\ln(1-x) \end{aligned}$$

Unique solution in L^p, $p > 1$ (under $E(X(t)) = 0$)

General: Knof '06 for finite dimensional distribution and Sulzbach, functional contraction method with Zolotarev metric

QUICKSORT on the FLY

Conrado Martinéz: **Partial Quicksort**

Input: sequence of length n

Output: l smallest in order

Procedure: Recall Quicksort always for left most list with 2 or more elements

Publish first smallest then second smallest and so on

Observation: When element published, algorithm had done only necessary comparisons, not more. Take $\frac{l}{n}$ as time.

$Y(n,l)$ number of comparisons

$X(n, \frac{l}{n}) = \frac{Y(n,l) - EY(n,l)}{n}$

Theorem Roesler Let U_i, $i \in I\!N$, be independent uniformly distributed. Then $X(n,\cdot)$ converges almost surely to the Quicksort process X in Skorodhod metric.

Proof: — Formulation on D_- instead of D.
— $X(t) = (1\!\!1_{t\leq U} U X^1(\frac{t}{U}) + 1\!\!1_{t>U}(1-U)X^2(\frac{t-U}{1-U}) + 1\!\!1_{t>U} U Q^1 + C(U,t))_t$
— $R_m = \Sigma_{|v|<n} L_v * C_v \to_m X$ in sup-metric and L^p
— $R_m(U_1, ..., U_n) \to_m X(n,.)$
— Skorodhod J_1-metric not complete.
— τ_n random time change, $X(n,\tau_n) \to X$ in sup-metric and L^p
— $\sup_t |\tau_n(t) - t| \to 0$

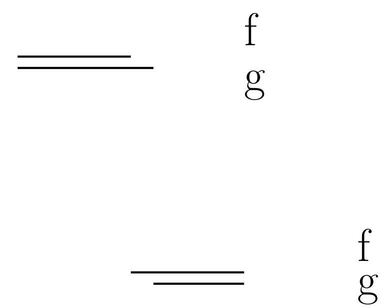
Two stage contraction

SKORODHOD SPACE D

D equipped with Skorodhod J_1-metric d

$$d(f,g) = \inf\{\epsilon > 0 \mid \exists \lambda \in \Lambda : \|f - g \circ \lambda\|_\infty < \epsilon,\ \|\lambda - \mathrm{id}\|_\infty < \epsilon\}$$

where Λ is the set of all bijective increasing functions $\lambda : [0,1] \to [0,1]$.



Big distance in supremum metric, small in Skorodhod metric