Edgeworth expansion for branching random walks and random trees

Zakhar Kabluchko
Westfälische Wilhelms-Universität Münster

Joint work with
Rudolf Grübel
Leibniz Universität Hannover

AofA 2015, Strobl

June 8, 2015

Branching random walk (BRW)

Branching random walk models a random cloud of particles on $\mathbb{Z}$. Random spatial motion of particles is combined with branching.

Definition of the BRW

- At time 0: One particle at 0.
- At time n: Every particle located, say, at $x \in \mathbb{Z}$ is replaced by a random cluster of N particles located at

$$x + Z_1, \dots, x + Z_N.$$

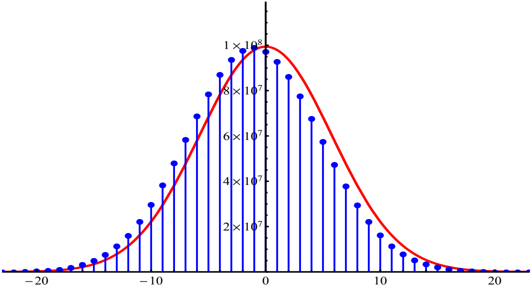
  Here, $\sum_{k=1}^{N} \delta(Z_k)$ is a point process on $\mathbb{Z}$.
- All random mechanisms are independent.

Profile of the branching random walk

Consider a BRW on the lattice $\mathbb{Z}$. Denote by $L_n(k)$ the number of particles located at site $k \in \mathbb{Z}$ at time $n \in \mathbb{N}_0$.

Definition

The random function $k \mapsto L_n(k)$ is called the **profile** of the BRW.



Results

Our aim is to obtain an **asymptotic expansion** of the profile as $n \to \infty$. As an application, we obtain a.s. limit theorems with non-degenerate limits for

- the occupation numbers $L_n(k_n)$;
- the mode $u_n := \arg\max_{k \in \mathbb{Z}} L_n(k)$;
- the height $M_n := \max_{k \in \mathbb{Z}} L_n(k) = L_n(u_n)$.

In the setting of random trees these and related quantities were studied by Fuchs, Hwang, Neininger (2006), Chauvin, Drmota, Jabbour-Hattab (2001), Katona (2005), Drmota, Hwang (2005), Devroye, Hwang (2006), Drmota, Janson, Neininger (2008).

Intensity

Definition

The **intensity** of the BRW at time n is the following measure on $\mathbb{Z}$:

$$\nu_n(\{k\}) := \mathbb{E}L_n(k), \quad k \in \mathbb{Z}.$$

Observation

ν_n is the n-th convolution power of ν_1.

Definition and assumption

Let the **cumulant generating function**

$$\varphi(\beta) := \log \sum_{k\in\mathbb{Z}} \mathrm{e}^{\beta k}\nu_1(\{k\})$$

be finite for $|\beta| < \varepsilon$.

Biggins martingale

Theorem (Uchiyama, 1982, Biggins, 1992)

With probability 1, the martingale

$$W_n(\beta) := \mathrm{e}^{-\varphi(\beta)n} \sum_{k\in\mathbb{Z}} L_n(k)\mathrm{e}^{\beta k}$$

converges uniformly on $|\beta| < \varepsilon$ to some random analytic function $W_\infty(\beta)$.

Remark

The random analytic function W_∞ encodes the "convolution difference" between the distribution of particles in the BRW at time n and the intensity measure ν_n.

Local CLT for the BRW

Theorem (Local form of the "Harris conjecture")

Let $\mu = \varphi'(0)$, $\sigma^2 = \varphi''(0)$ and

$$x_n(k) = \frac{k - \mu n}{\sigma \sqrt{n}}, \quad k \in \mathbb{Z}.$$

Then, with probability 1,

$$\frac{L_n(k)}{\mathrm{e}^{\varphi(0)n}} = \frac{W_\infty(0)}{\sqrt{2\pi n}\,\sigma} \mathrm{e}^{-\frac{1}{2}x_n^2(k)} + o\left(\frac{1}{\sqrt{n}}\right), \; n \to \infty,$$

where the o-term is uniform in $k \in \mathbb{Z}$.

Remark

The number of particles at time n is $\approx W_\infty(0)\mathrm{e}^{\varphi(0)n}$.

Edgeworth expansion for the BRW

Theorem (Grübel, Kabluchko, 2015)

Let $\mu = \varphi'(0)$, $\sigma^2 = \varphi''(0)$ and

$$x_n(k) = \frac{k - \mu n}{\sigma \sqrt{n}}, \quad k \in \mathbb{Z}.$$

Then, with probability 1 the following asymptotic expansion holds uniformly in $k \in \mathbb{Z}$:

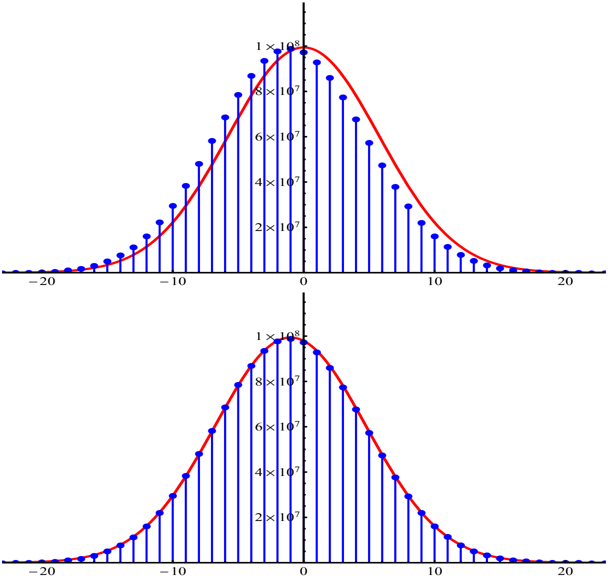
$$\frac{L_n(k)}{\mathrm{e}^{\varphi(0)n}} \sim \frac{W_\infty(0)}{\sqrt{2\pi n}\,\sigma} \mathrm{e}^{-\frac{1}{2}x_n^2(k)} \left[1 + \frac{F_1(x_n(k))}{\sqrt{n}} + \frac{F_2(x_n(k))}{n} + \dots\right]$$

where

$$F_1(x) = \left(\frac{\varphi'''(0)}{6\sigma^3}(x^3 - 3x) + \frac{W'_\infty(0)}{W_\infty(0)}\frac{x}{\sigma}\right),$$

$$F_2(x) = \dots .$$

Shift correction



Applications: The mode

Edgeworth expansion can be applied to obtain a.s. limit theorems with non-degenerate limits for

- the occupation numbers $L_n(k_n)$
- the mode $u_n := \arg\max_{k\in\mathbb{Z}} L_n(k)$
- the height $M_n := \max_{k\in\mathbb{Z}} L_n(k) = L_n(u_n)$.

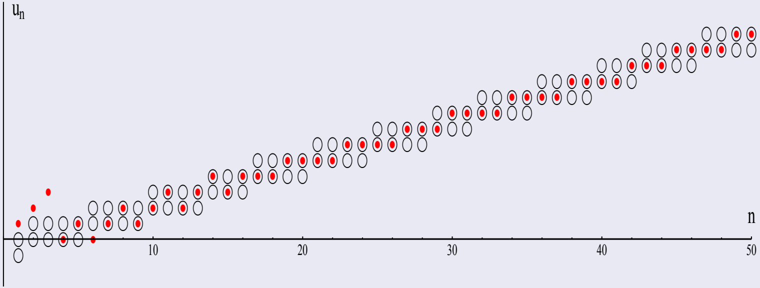
Theorem (Grübel, Kabluchko, 2015)

There is a random variable N such that with probability 1, the mode at time $n > N$ is equal to $\lfloor u_n^* \rfloor$ or $\lceil u_n^* \rceil$, where

$$u_n^* = \varphi'(0)n + \frac{W_\infty'(0)}{W_\infty(0)} - \frac{\varphi'''(0)}{2\sigma^2}.$$

The mode

Mode u_n as a function of time n



Applications: The height

Theorem (Grübel, Kabluchko, 2015)

Let $M_n = \max_{k \in \mathbb{Z}} L_n(k)$ be the height of the BRW at time n. The a.s. subsequential limits of the sequence

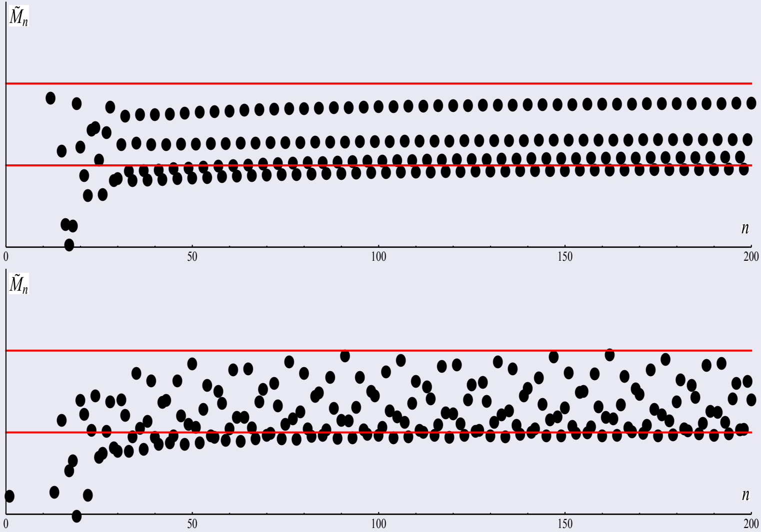
$$\tilde{M}_n := 2\sigma^2 n \left(1 - \frac{\sqrt{2\pi n}\,\sigma M_n}{W_\infty(0)\mathrm{e}^{\varphi(0)n}}\right)$$

have the form $(\log W_\infty)''(0) + c$, where $c \in I$ and $I \subset \mathbb{R}$ is some compact set. The set I

- contains 1 element if $\varphi'(0)$ is integer,
- contains finitely many elements if $\varphi'(0)$ is rational,
- is an interval of length $1/4$ is $\varphi'(0)$ is irrational.

The height

Normalized height $\tilde{M}_n$ as a function of time n



Applications: Occupation numbers

Theorem (Grübel, Kabluchko, 2015)

Let $k_n = \lfloor \varphi'(0)n \rfloor + a$, where $a \in \mathbb{Z}$. The a.s. subsequential limits of the sequence

$$\sqrt{2\pi}\sigma^3 n^{3/2} \mathrm{e}^{-\varphi(0)n}(L_n(k_n) - W_\infty(0)\mathbb{E}L_n(k_n))$$

have the form

$$W'_\infty(0)\left(c + \frac{\varphi'''(0)}{2\sigma^2}\right) - \frac{1}{2}W''_\infty(0),$$

where $c \in J$ and $J \subset \mathbb{R}$ is some compact set which can be described explicitly.

Occupation numbers

Occupation numbers at $k_n = \lfloor \varphi'(0) n \rfloor + a$, $a = -1, 0, 1$