Edgeworth expansion for branching random walks and random trees

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Branching random walk (BRW)

Branching random walk models a random cloud of particles on \( \mathbb{Z} \). Random spatial motion of particles is combined with branching.

**Definition of the BRW**

- At time 0: One particle at 0.
- At time \( n \): Every particle located, say, at \( x \in \mathbb{Z} \) is replaced by a random cluster of \( N \) particles located at

\[
x + Z_1, \ldots, x + Z_N.
\]

Here, \( \sum_{k=1}^{N} \delta(Z_k) \) is a point process on \( \mathbb{Z} \).
- All random mechanisms are independent.
Consider a BRW on the lattice \( \mathbb{Z} \). Denote by \( L_n(k) \) the number of particles located at site \( k \in \mathbb{Z} \) at time \( n \in \mathbb{N}_0 \).

**Definition**

The random function \( k \mapsto L_n(k) \) is called the **profile** of the BRW.
Our aim is to obtain an **asymptotic expansion** of the profile as $n \to \infty$. As an application, we obtain a.s. limit theorems with non-degenerate limits for

- the occupation numbers $L_n(k_n)$;
- the mode $u_n := \arg \max_{k \in \mathbb{Z}} L_n(k)$;
- the height $M_n := \max_{k \in \mathbb{Z}} L_n(k) = L_n(u_n)$.

In the setting of random trees these and related quantities were studied by Fuchs, Hwang, Neininger (2006), Chauvin, Drmota, Jabbour-Hattab (2001), Katona (2005), Drmota, Hwang (2005), Devroye, Hwang (2006), Drmota, Janson, Neininger (2008).
**Intensity**

**Definition**

The **intensity** of the BRW at time $n$ is the following measure on $\mathbb{Z}$:

$$\nu_n(\{k\}) := \mathbb{E} L_n(k), \quad k \in \mathbb{Z}.$$ 

**Observation**

$\nu_n$ is the $n$-th convolution power of $\nu_1$.

**Definition and assumption**

Let the **cumulant generating function**

$$\varphi(\beta) := \log \sum_{k \in \mathbb{Z}} e^{\beta k} \nu_1(\{k\})$$

be finite for $|\beta| < \varepsilon$. 
Biggins martingale

Theorem (Uchiyama, 1982, Biggins, 1992)

With probability 1, the martingale

\[ W_n(\beta) := e^{-\varphi(\beta)n} \sum_{k \in \mathbb{Z}} L_n(k)e^{\beta k} \]

converges uniformly on \(|\beta| < \varepsilon\) to some random analytic function \(W_{\infty}(\beta)\).

Remark

The random analytic function \(W_{\infty}\) encodes the “convolution difference” between the distribution of particles in the BRW at time \(n\) and the intensity measure \(\nu_n\).
Theorem (Local form of the “Harris conjecture”)

Let \( \mu = \varphi'(0) \), \( \sigma^2 = \varphi''(0) \) and

\[
\chi_n(k) = \frac{k - \mu n}{\sigma \sqrt{n}}, \quad k \in \mathbb{Z}.
\]

Then, with probability 1,

\[
\frac{L_n(k)}{e^{\varphi(0)n}} = \frac{W_\infty(0)}{\sqrt{2\pi n \sigma}} e^{-\frac{1}{2} \chi_n^2(k)} + o \left( \frac{1}{\sqrt{n}} \right), \quad n \to \infty,
\]

where the \( o \)-term is uniform in \( k \in \mathbb{Z} \).

Remark

The number of particles at time \( n \) is \( \approx W_\infty(0)e^{\varphi(0)n} \).
Edgeworth expansion for the BRW

Theorem (Grübel, Kabluchko, 2015)

Let $\mu = \varphi'(0)$, $\sigma^2 = \varphi''(0)$ and

$$x_n(k) = \frac{k - \mu n}{\sigma \sqrt{n}}, \quad k \in \mathbb{Z}.$$ 

Then, with probability 1 the following asymptotic expansion holds uniformly in $k \in \mathbb{Z}$:

$$\frac{L_n(k)}{e^{\varphi(0)n}} \sim \frac{W_\infty(0)}{\sqrt{2\pi n \sigma}} e^{-\frac{1}{2}x_n^2(k)} \left[ 1 + \frac{F_1(x_n(k))}{\sqrt{n}} + \frac{F_2(x_n(k))}{n} + \ldots \right]$$

where

$$F_1(x) = \left( \frac{\varphi''''(0)}{6\sigma^3} (x^3 - 3x) + \frac{W'(0)}{W_\infty(0)} \frac{x}{\sigma} \right),$$

$$F_2(x) = \ldots.$$
Shift correction
Applications: The mode

Edgeworth expansion can be applied to obtain a.s. limit theorems with non-degenerate limits for

- the occupation numbers $L_n(k_n)$
- the mode $u_n := \arg \max_{k \in \mathbb{Z}} L_n(k)$
- the height $M_n := \max_{k \in \mathbb{Z}} L_n(k) = L_n(u_n)$.

Theorem (Grübel, Kabluchko, 2015)

There is a random variable $N$ such that with probability 1, the mode at time $n > N$ is equal to $\lfloor u^*_n \rfloor$ or $\lceil u^*_n \rceil$, where

$$u^*_n = \varphi'(0)n + \frac{W'(0)}{W(0)} - \frac{\varphi'''(0)}{2\sigma^2}.$$
The mode

Mode $u_n$ as a function of time $n$
Applications: The height

Theorem (Grübel, Kabluchko, 2015)

Let $M_n = \max_{k \in \mathbb{Z}} L_n(k)$ be the height of the BRW at time $n$. The a.s. subsequential limits of the sequence

$$\tilde{M}_n := 2\sigma^2 n \left( 1 - \frac{\sqrt{2\pi n \sigma M_n}}{W_\infty(0)e^{\phi(0)n}} \right)$$

have the form $(\log W_\infty)''(0) + c$, where $c \in I$ and $I \subset \mathbb{R}$ is some compact set. The set $I$

- contains 1 element if $\phi'(0)$ is integer,
- contains finitely many elements if $\phi'(0)$ is rational,
- is an interval of length $1/4$ if $\phi'(0)$ is irrational.
The height

Normalized height $\tilde{M}_n$ as a function of time $n$
Theorem (Grübel, Kabluchko, 2015)

Let $k_n = \lfloor \varphi'(0)n \rfloor + a$, where $a \in \mathbb{Z}$. The a.s. subsequential limits of the sequence

$$\sqrt{2\pi\sigma^3} n^{3/2} e^{-\varphi(0)n} (L_n(k_n) - W_\infty(0) \mathbb{E} L_n(k_n))$$

have the form

$$W'_\infty(0) \left( c + \frac{\varphi'''(0)}{2\sigma^2} \right) - \frac{1}{2} W''\infty(0),$$

where $c \in J$ and $J \subset \mathbb{R}$ is some compact set which can be described explicitly.
Occupation numbers at $k_n = \lfloor \varphi'(0)n \rfloor + a$, $a = -1, 0, 1$