Condensation and symmetry-breaking in the zero-range process with weak disorder

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What is this talk about?

Condensation

= emergence of a macroscopic phenomenon in a stochastic system.

Examples:

- a node of high degree on a random tree/graph
- a site containing a lot of particles in a particles system

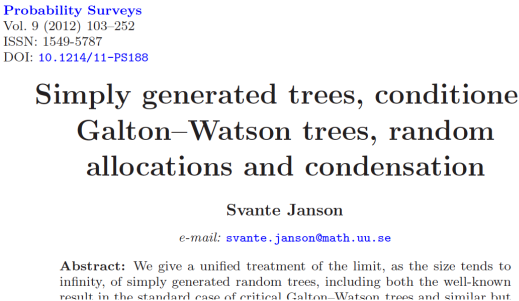
In this talk:

The zero-range process (ZRP) in random environment.

Outline:

1. the homogenenous ZRP - state of the art
2. ZRP in random environment - different interesting condensation phases

The homogeneous ZRP



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Simply generated trees, conditioned Galton–Watson trees, random allocations and condensation

Svante Janson

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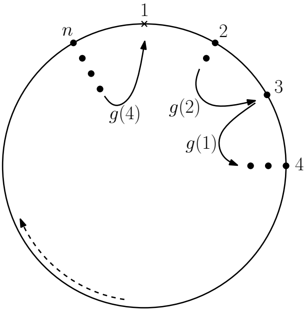
Abstract: We give a unified treatment of the limit, as the size tends to infinity, of simply generated random trees, including both the well-known result in the standard case of critical Galton–Watson trees and similar but

+ [Grosskinsky-Schütz-Spohn ’03]

What is the ZRP?

Continuous time Markov process: m particles moving on n sites.



Function $g : \mathbb{N} \to \mathbb{R}$ encodes the interactions between the particles.

Stationary distribution:

$$\mathbb{P}_{m,n}(q_1, \ldots, q_n) = \frac{1}{Z_{m,n}} \prod_{i=1}^{n} p_{q_i} \, \mathbf{1}_{q_1+\cdots+q_n=m}$$

where $p_k = \prod_{j=1}^{k-1} \frac{1}{g(k)}$

A purely probabilistic description:

Take $(Q_i)_{i\geq 1}$ i.i.d. integer-valued random variables: $\mathbb{P}(Q_i = k) = p_k$.
The law of $(Q_1, \ldots, Q_n \mid \sum_{i=1}^{n} Q_i = m)$ is $\mathbb{P}_{m,n}$.

Aim: understand $\mathbb{P}_{m,n}$ when $m, n \to +\infty$.

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Notations:

- LLN implies $\sum_{i=1}^n Q_i \sim \rho^\star n$ where $\rho^\star = \mathbb{E} Q_1$ "**natural density**"
- We assume that $m = m_n$ depends on n, and that $m/n \to \rho$ "**forced density**"

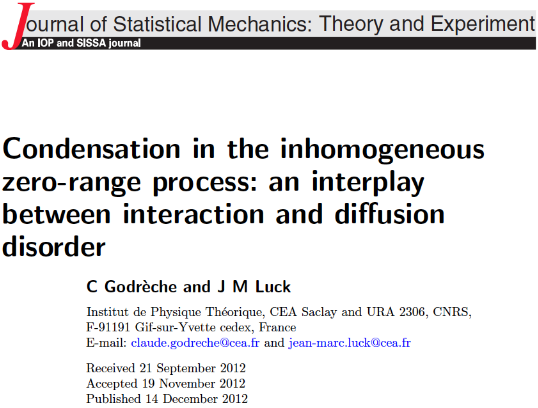
Theorem [Janson (2012)]: condensation

Assume that there exists $\beta > \mathbf{2}$ such that $\boldsymbol{p_k \sim k^{-\beta}}$ $(k \to +\infty)$,
and that $\boldsymbol{\rho > \rho^\star}$, then
conditionally to $S_n := \sum_{i=1}^n Q_i = m$, in probability when $n \to +\infty$,

$$Q_n^{(1)} = (\rho - \rho^\star)n + o(n) \qquad \text{and} \qquad Q_n^{(2)} = o(n).$$

+ further results (fluctuations of $Q_n^{(1)}$)

The ZRP in random environment



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Condensation in the inhomogeneous zero-range process: an interplay between interaction and diffusion disorder

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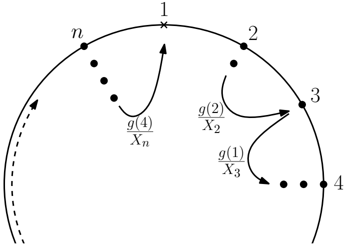

m particles moving on n sites

Random environment: i.i.d. random variables $X_i \in [0,1]$ “**fitnesses**”

Function $g : \mathbb{N} \to \mathbb{R}$ encodes the interactions between the particles

Stationary distribution:

$$\mathbb{P}_{m,n}(q_1,\ldots,q_n) = \frac{1}{Z_{m,n}} \prod_{i=1}^{n} p_{q_i} X_i^{q_i} \, \mathbf{1}_{q_1+\cdots+q_n=m} \qquad \text{where } p_k = \prod_{j=1}^{k-1} \frac{1}{g(k)}$$

A purely probabilistic description:

Take $(X_i)_{i\geq 1}$ i.i.d. random variables on $[0,1]$.
Take $(Q_i)_{i\geq 1}$ independent integer-valued random variables:
$\mathbb{P}(Q_i = k) \propto p_k X_i^k$.
The law of $(Q_1,\ldots,Q_n \mid \sum_{i=1}^n Q_i = m)$ is $\mathbb{P}_{m,n}$.

Precise set-up

Let $(X_i)_{i\geq 1}$ i.i.d. random variables law μ on $[0,1]$.

Let $(Q_i)_{i\geq 1}$ independent random variables

$$\mathbb{P}_{\boldsymbol{X}}(Q_i = k) = \frac{p_k X_i^k}{\Phi(X_i)} \qquad \text{where } \Phi(z) = \sum_{k\geq 0} p_k z^k.$$

We assume:

- $(p_k)_{k\geq 0}$ is a probability distribution
- $\mu([1-h,1]) \sim_{h\to 0} h^{\gamma}$ $(\gamma > 0)$ "**disorder parameter**"
- $p_k \sim_{k\to+\infty} k^{-\beta}$ $(\beta > 1)$ "**interaction parameter**"

Lemma: "Natural density"

In probability when $n \to +\infty$, if $\beta + \gamma > 2$,

$$\frac{1}{n} S_n := \frac{1}{n}\sum_{i=1}^{n} Q_i \to \mathrm{E}\mathbb{E}_{\boldsymbol{X}} Q_1 = \mathrm{E}\left[\frac{X_1 \Phi'(X_1)}{\Phi(X_1)}\right] =: \rho^{\star}.$$

Lemma: “Natural density”

In probability when $n \to +\infty$, if $\beta + \gamma > 2$,

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Proof:

1. if $\beta + \gamma > 2$ then, in $\mathbb{P}_{\boldsymbol{X}}$- probability,
$$\frac{1}{n}\sum_{i=1}^{n} Q_i - \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{\boldsymbol{X}} Q_i \to 0.$$

needs a NSC for the weak LLN for independent RV

2. $(\mathbb{E}_{\boldsymbol{X}} Q_i)_{i\geq 1}$ is a sequence of i.i.d. RV:
$$\mathbb{E}_{\boldsymbol{X}} Q_i = \sum_{k\geq 0} \frac{k p_k X_i^k}{\Phi(X_i)} = \frac{X_i \Phi'(X_i)}{\Phi(X_i)}$$

LLN gives: $\nu_n := \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{\boldsymbol{X}} Q_i \to \mathrm{E}\mathbb{E}_{\boldsymbol{X}} Q_1.$

To conclude:

Prove that
$$\mathrm{E}\mathbb{E}_{\boldsymbol{X}} Q_1 < +\infty$$

Lemma: “Natural density”

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Proof:

To conclude:

Prove that

$$\mathrm{E}\mathbb{E}_{\boldsymbol{X}} Q_1 < +\infty$$

Recall that $\Phi(z) = \sum_{k\geq 0} p_k z^k$ and $p_k \sim k^{-\beta}$:

1. if $\beta > 2$ then, $\mathrm{E}\mathbb{E}_{\boldsymbol{X}} Q_1 \leq \Phi'(1) < +\infty$ qed
2. if $\beta < 2$ then $\phi'(x) \sim_{x\to 1} (1-x)^{\beta-2}$

Fix $u \in \mathbb{R}$ and *think* $u \to +\infty$:

$$\mathrm{P}(\mathbb{E}_{\boldsymbol{X}} Q_1 \geq u) = \mathrm{P}\left(\frac{X_1 \Phi'(X_1)}{\Phi(X_1)} \geq u\right) \approx \mathrm{P}\left((1 - X_1)^{\beta-2} \geq u\right)$$

$$= \mathrm{P}\left(X_1 \geq 1 - u^{\frac{1}{\beta-2}}\right) \sim u^{-\frac{\gamma}{2-\beta}}.$$

Lemma: “Natural density”

In probability when $n \to +\infty$, if $\beta + \gamma > 2$,

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$$= \mathrm{P}\left(X_1 \geq 1 - u^{\frac{1}{\beta-2}}\right) \sim u^{-\frac{\gamma}{2-\beta}} .$$

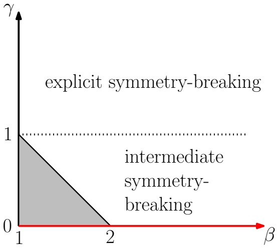
Integrable as soon as $\frac{\gamma}{2-\beta} > 1 \Leftrightarrow \beta + \gamma > 2$ qed

Main results: condensation

Theorem: condensation

Assume that $\beta + \gamma > 2$ and that $m/n \to \rho > \rho^\star$,
then, conditionally to $S_n = m$, in $\mathbb{P}_{m,n}$-probability when $n \to +\infty$,

$$Q_n^{(1)} = (\rho - \rho^\star)n + o(n) \qquad \text{and} \qquad Q_n^{(2)} = o(n).$$



- if $\gamma > 1$ then, the condensate is a.s. located in the largest fitness site:

$$Q_n^{(1)} = Q_{I_n} \qquad \text{where} \qquad X_n^{(1)} = X_{I_n}.$$

- if $\gamma \leq 1$ this is no longer true:
  where is the condensate?

Weak-disorder phase: location of the condensate

Let K_n be the rank (in term of decreasing fitness) of the site containing the condensate.

We already know that $K_n = 1$ a.s. when $\gamma > 1$.

Location of the condensate when $0 < \gamma \leq 1$

Assume $\beta + \gamma > 2$ and $\rho > \rho^\star$,
then in **quenched distribution** when $n \to +\infty$,

$$(n^{\gamma-1} K_n)^{1/\gamma} \to \mathrm{Gamma}(\gamma, \rho - \rho^\star).$$

Definition:

$(Z_n)_{n\geq 1}$ converges in quenched distribution to Z iff:
for all $u \in \mathbb{R}$, for all $\varepsilon > 0$, when $n \to +\infty$,

$$\mathrm{P}\left(|\mathbb{P}_{\boldsymbol{X}}(Z_n \leq u \mid S_n = m) - \mathbb{P}_{\boldsymbol{X}}(Z \leq u)| > \varepsilon\right) \to 0.$$

Weak-disorder phase: fitness of the condensate

Let F_n be the fitness of the site containing the condensate.

We already know that $F_n = \max_{i=1}^n X_i$ a.s. when $\gamma > 1$.

Fitness of the condensate when $0 < \gamma \leq 1$

Assume $\beta + \gamma > 2$ and $\rho > \rho^\star$,
then in **quenched distribution** when $n \to +\infty$,

$$n(1 - F_n) \to \mathrm{Gamma}(\gamma, \rho - \rho^\star).$$

Definition:

$(Z_n)_{n\geq 1}$ converges in quenched distribution to Z iff:
for all $u \in \mathbb{R}$, for all $\varepsilon > 0$, when $n \to +\infty$,

$$\mathrm{P}\left(|\mathbb{P}_{\boldsymbol{X}}(Z_n \leq u \mid S_n = m) - \mathbb{P}_{\boldsymbol{X}}(Z \leq u)| > \varepsilon\right) \to 0.$$

Weak disorder phase: fluctuations

New notations: $\rho_n := \frac{m_n}{n} \to \rho$ and $\nu_n := \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{\boldsymbol{X}} Q_i \to \rho^\star$

Theorem: fluctuations when $\gamma \leq 1$

Assume that $\beta + \gamma > 2$ and $\rho > \rho^\star$. Then,

- if $\beta + \gamma > 3$ then, in P-probability,

$$\frac{Q_n^{(1)} - (\rho_n - \nu_n)n}{\sqrt{n}} \to W,$$

  in distribution, where W is a normal random variable.
- if $2 < \beta + \gamma \leq 3$ then, in P-probability,

$$\frac{Q_n^{(1)} - (\rho_n - \nu_n)n}{n^{\frac{1}{\beta+\gamma-1}}} \to W_{\beta+\gamma-1},$$

  in distribution, where $W_{\beta+\gamma-1}$ is a $(\beta + \gamma - 1)$-stable random variable.

Weak disorder phase: fluctuations

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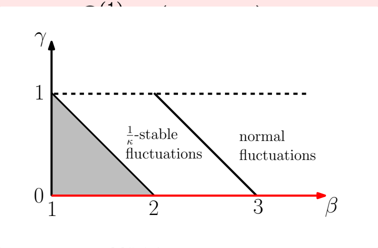

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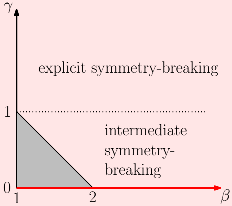
Summary of the results

Condensation



If $\beta + \gamma > 2$ and $\rho > \rho^\star$,

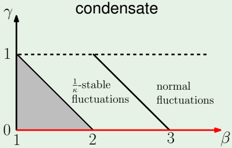
$$Q_n^{(1)} = (\rho - \rho^\star)n + o(n)$$
$$Q_n^{(2)} = o(n)$$

Weak disorder: $\gamma \leq 1$

K_n defined such that $F_n := X_n^{(K_n)}$ is the fitness of the condensate:

$$(n^{\gamma-1}K_n)^{1/\gamma} \to \mathrm{Gamma}(\gamma, \rho - \rho^\star)$$
$$n(1 - F_n) \to \mathrm{Gamma}(\gamma, \rho - \rho^\star)$$

fluctuations of the size of the condensate



A zoom in the proof: where is the condensate?

Assume that we have already proved:

$$\mathbb{P}(S_n = m) = (1 + o(1)) \sum_{i=1}^{n} \mathbb{P}(\text{condensation in site } i)$$

$$= \mathtt{cst}(1 + o(1))n^{-\beta} \sum_{i=1}^{n} X_i^{(\rho-\rho^\star \pm \delta_n)n}$$

What can be said about $\sum_{i=1}^{n} X_i^{\lambda n}$?

Assumption $\mathbb{P}(X_1 \geq 1 - h) \sim h^\gamma$ gives

$$X_n^{(1)} = 1 - \Theta(n^{-1/\gamma})$$

$$X_n^{(2)}/X_n^{(1)} = 1 - \Theta(n^{-1/\gamma})$$

$$\sum_{i=1}^{n} X_i^{\lambda n} \begin{cases} \sim \left(X_n^{(1)}\right)^{\lambda n} & \text{if } \gamma > 1 \\ = \Theta(n^{1-\gamma}) & \text{if } \gamma \leq 1. \end{cases}$$

$\gamma > 1 \Rightarrow$ condensation in the largest fitness site

$\gamma \leq 1 \Rightarrow \mathbb{P}_{\boldsymbol{X}}(S_n = m) = \Theta(n^{1-\beta-\gamma})$

Where is the condensate when $\gamma \leq 1$?

The condensate occurs in the K_n^{th} largest fitness site:

$$\mathbb{P}_{\boldsymbol{X}}((n^{\gamma-1}K_n)^{1/\gamma} \geq u \mid S_n = m)$$

$$\approx \frac{1}{n^{1-\beta-\gamma}} \sum_{i | X_i \geq X_n^{(\lceil u^\gamma n^{1-\gamma} \rceil)}} \mathbb{P}(\text{condensation in site } i)$$

$$\approx \frac{\mathtt{cst}}{n^{1-\beta-\gamma}} \sum_{i \geq u^\gamma n^{1-\gamma}} n^\beta (X_n^{(i)})^{(\rho-\rho^\star)n} \approx \frac{\mathtt{cst}}{n^{1-\gamma}} \int_{u^\gamma n^{1-\gamma}}^{n} (X_n^{(\lfloor x \rfloor)})^{(\rho-\rho^\star)n} dx$$

Let $x = y^\gamma n^{1-\gamma}$:

$$\mathbb{P}_{\boldsymbol{X}}((n^{\gamma-1}K_n)^{1/\gamma} \geq u \mid S_n = m) \approx \mathtt{cst} \int_u^n \left(X_n^{(\lfloor y^\gamma n^{1-\gamma} \rfloor)} \right)^{(\rho-\rho^\star)n} y^{\gamma-1} dy$$

Note that because $\mathbb{P}(X_1 \geq 1-h) \sim h^\gamma$, $\boxed{X_n^{(\lfloor y^\gamma n^{1-\gamma} \rfloor)} \approx 1 - \frac{y}{n}}$

$$\mathbb{P}_{\boldsymbol{X}}((n^{\gamma-1}K_n)^{1/\gamma} \geq u \mid S_n = m) \to \mathtt{cst} \int_u^{+\infty} \mathrm{e}^{(\rho-\rho^\star)y} y^{\gamma-1} dy \quad \text{qed}$$

Outlook and further work

- Actually, we believe:
  - $\rho > \rho^\star \Rightarrow Q_n^{(1)} \sim (\rho - \rho^\star)n$ and $Q_n^{(2)} = O(\ln n)$;
  - $\rho < \rho^\star \Rightarrow Q_n^{(1)} = O(\ln n)$.

  Zoom into the phase transition window: take

$$\rho_n := {}^{m}\!/_{n} = \rho^\star + \lambda n^{-\eta}.$$

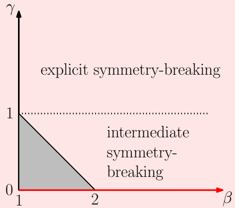
- Our hypothesis ($\beta + \gamma > 2$) induced that $\sum_{i=1}^n Q_i \approx \rho^\star n$. Under weaker hypothesis, we could have

$$\sum_{i=1}^n Q_i \approx \rho^\star n^\kappa \text{ with } \kappa > 1.$$

  Can we observe condensation when ${}^{m_n}\!/_{n^\kappa} \to \rho > \rho^\star$?

Thanks!!!

Condensation



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Weak disorder: $\gamma \leq 1$

K_n defined such that $F_n := X_n^{(K_n)}$ is the fitness of the condensate:

$$n^{\gamma-1} K_n^{1/\gamma} \to \mathrm{Gamma}(\gamma, \rho - \rho^\star)$$
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fluctuations of the size of the condensate