Analysis of Bidirectional Ballot Sequences and Random Walks Ending in their Maximum

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More-Sums-than-Differences Sets (MSTD)

Definition (Nathanson)

Finite subset $S \subset \mathbb{Z}$ called *More-Sums-than-Differences-Set*:

$$|S + S| > |S - S|.$$

$$S + S := \{s_1 + s_2 \mid s_1, s_2 \in S\}$$
$$S - S := \{s_1 - s_2 \mid s_1, s_2 \in S\}.$$

Subtraction non-commutative: expect $|S - S| > |S + S|$.

Example (Conway, 1960s)

$$\begin{aligned} S &= \{0, 2, 3, 4, 7, 11, 12, 14\}, \\ S + S &= \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \\ &\quad 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 28\}, \\ S - S &= \{-14, -12, -11, -10, -9, -8, -7, -5, -4, -3, -2, -1, 0, \\ &\quad 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 14\}. \end{aligned}$$

Construct MSTD sets

Aim: construct many MSTD-subsets of $\{0, \ldots, n-1\}$.

Zhao's Construction of MSTD Sets

Theorem (Zhao 2010)

Let

$$L = \{0, 2, 3, 7, 8, 9, 10\},$$
$$M \subseteq \{11, \ldots, n-12\},$$
$$R = \{n-11, n-10, n-9, n-8, n-6, n-3, n-2, n-1\}$$

such that every prefix and every suffix of $\{11, \ldots, n-12\}$ *has more than half of its elements in* M.
$\implies S := L \cup M \cup R$ *is MSTD-set.*

$$\pm(n-7) \notin S - S \qquad \implies \qquad |S - S| \leq 2n - 3,$$
$$L + L = \{0, \ldots, 20\} \setminus \{1\}, \qquad R + R = \{2n-22, \ldots, 2n-2\}$$

If $M + M = \{22, \ldots, 2n-24\}$:

$$S + S = \{0, \ldots, 2n-2\} \setminus \{1\} \implies |S + S| = 2n - 2.$$

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Zhao's Construction of MSTD Sets (2)

Lemma (Zhao 2010)

Let $M \subseteq \{1, \ldots, m\}$ such that every prefix and every suffix of $\{1, \ldots, m\}$ has more than half of its elements in M. Then $M + M = \{2, \ldots, 2m\}$.

Proof. WLOG let $2 \leq x \leq m$. Pigeonhole principle: y, $x - y \in M$ for some $y \in \{1, \ldots, x - 1\}$. Thus $x = y + (x - y) \in M + M$. □

Definition

Bidirectional Ballot Sequence: word over $\{0, 1\}$ such that every prefix and every suffix contains strictly more ones than zeros.

Bijection between sets M as above and bidirectional ballot sequences of length m.

$\#$ MSTD-subsets of $\{0, \ldots, n - 1\} \geq B_{n-22}$, the number of bidirectional ballot sequences of length $n - 22$.

Number of Bidirectional Ballot Sequences

B_n: number of bidirectional ballot sequences of length n.

Theorem (Zhao 2010)

$$\frac{B_n}{2^n} = \Theta\left(\frac{1}{n}\right).$$

Theorem (Bousquet-Mélou–Ponty 2008)

$$\frac{B_n}{2^n} \sim \frac{1}{4n}.$$

Conjecture (Zhao 2010)

$$\frac{B_n}{2^n} = \frac{1}{4n} + \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right).$$

Theorem (Hackl–H–Prodinger–Wagner 2015)

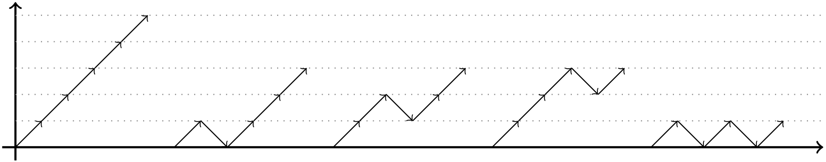
$$\frac{B_n}{2^n} = \frac{1}{4n} + \frac{1}{6n^2} + \frac{7}{45n^3} + \frac{10}{63n^4} + O\left(\frac{\log n}{n^5}\right).$$

Admissible Random Walks

Bidirectional Ballot Sequence: word over $\{0,1\}$ such that every prefix and every suffix contains strictly more ones than zeros.

Definition

A random walk on $\mathbb{Z}$ is called *admissible of height $h \geq 0$*, if $\in [0, h]$ at all times and ends in h.
It is called *admissible*, if it is admissible of some height h.



Bijection: Bidirectional Ballot Sequences of length $n+2 \longleftrightarrow$ admissible walks of length n.

Analysis of Admissible Random Walks

$q_n^{(h)}$: probability that random path of length n is admissible of height h

q_n: probability that random path of length n is admissible

- Generating function for $(q_n^{(h)})_{n\geq 0}$?
- Explicit expression for $q_n^{(h)}$?
- Asymptotic expression for q_n for $n \to \infty$?
- Height of an admissible random path of length n:
  - moments,
  - local limit theorem.

Generating Function for $(q_n^{(h)})_{n\geq 0}$

$q_n^{(h)}$: probability that random path of length n is admissible of height h

Chebyshev polynomials of the second kind

$$U_{h+1}(x) = 2xU_h(x) - U_{h-1}(x) \text{ for } h \geq 1, U_0(x) = 1, U_1(x) = 2x.$$

Proposition

$$q_n^{(h)} = 2[z^{n+1}]\frac{1}{U_{h+1}(1/z)} \qquad (h \geq 0, n \geq 0).$$

$(h+1) \times (h+1)$ transfer matrix

$$M_h = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & \cdots & 0 \\ 0 & 1/2 & 0 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 1/2 \\ 0 & 0 & 0 & \ldots & 1/2 & 0 \end{pmatrix}.$$

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Explicit Expression for $q_n^{(h)}$

$q_n^{(h)}$: probability that random path of length n is admissible of height h

Theorem

$$q_{n-2}^{(h)} = \frac{4}{2^n} \sum_{k\geq 0} \frac{2\upsilon_{h,k}^2 - \frac{n}{2}}{(n-1)\frac{n}{2}} \binom{n}{\frac{n}{2} - \upsilon_{h,k}} \cdot [\![h \equiv n \bmod 2]\!]$$

for $\upsilon_{h,k} := (h+2)(2k+1)/2$, $h \geq 0$ *and* $n \geq 2$.

Prune tails, approximate binomial coefficient in central region ($|\alpha| \leq n^{2/3}$)

$$\binom{n}{\frac{n}{2} - \alpha} \sim \frac{2^{n+1/2}}{\sqrt{n\pi}} \exp\left(-\frac{2\alpha^2}{n}\right),$$

complete tails ...

Harmonic Sum

$$q_{n-2} \sim \frac{8\sqrt{2}}{\sqrt{n\pi}} \frac{1}{n-1} \sum_{\substack{h,k\geq 0 \\ h\equiv n \bmod 2}} \frac{2\upsilon_{h,k}^2 - \frac{n}{2}}{n} \exp\Big(-\frac{2\upsilon_{h,k}^2}{n}\Big).$$

Use Mellin transform to obtain asymptotics.

Asymptotics

Theorem (Hackl–H–Prodinger–Wagner 2015)

$$q_n = \frac{1}{n} - \frac{4}{3n^2} + \frac{88}{45n^3} - \frac{976}{315n^4} + O\Big(\frac{\log n}{n^5}\Big).$$

Expectation, variance and rth moment of the height of admissible random walks on $\mathbb{Z}$:

$$\mathbb{E}H_n = \frac{\sqrt{2\pi^3}}{4}\sqrt{n} - 2 + \frac{3\sqrt{2\pi^3}}{16\sqrt{n}} - \frac{539\sqrt{2\pi^3}}{5760\sqrt{n^3}} + O\Big(\frac{1}{\sqrt{n^5}}\Big),$$

$$\mathbb{V}H_n = \frac{28\zeta(3) - \pi^3}{8}n + \frac{224\zeta(3) - 9\pi^3}{48} + O\Big(\frac{1}{n}\Big),$$

$$\mathbb{E}H_n^r \sim \frac{r}{\sqrt{\pi}}\Gamma\Big(\frac{r+1}{2}\Big)(2^{r+1} - 1)2^{-r/2}\zeta(r+1)n^{r/2}.$$

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Local Limit Thoerem

Theorem (Hackl–H–Prodinger–Wagner 2015)

If

$$\frac{6}{\sqrt{\log n}} < \eta := \frac{h}{\sqrt{n}} < \frac{\sqrt{\log n}}{2},$$

then

$$\begin{aligned}\mathbb{P}(H_n = h) &= \frac{q_n^{(h)}}{q_n} \\ &\sim \frac{4\sqrt{2}}{\sqrt{\pi n}} \sum_{k\geq 0} ((2k+1)^2\eta^2 - 1) \exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big) \\ &= \frac{4\pi^2}{\eta^3\sqrt{n}} \sum_{k\geq 1} (-1)^{k-1} k^2 \exp\Big(-\frac{\pi^2 k^2}{2\eta^2}\Big).\end{aligned}$$

Chebyshev Polynomials of the First Kind and Random Walks on $\mathbb{Z}_{\geq 0}$

Chebyshev polynomials of the first and second kind:

$$T_{h+1}(x) = 2xT_h(x) - T_{h-1}(x) \quad \text{for } h \geq 1, \quad T_0(x) = 1, \quad T_1(x) = x,$$
$$U_{h+1}(x) = 2xU_h(x) - U_{h-1}(x) \quad \text{for } h \geq 1, \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

$p_n^{(h)}$: probability that a random walk on $\mathbb{Z}_{\geq 0}$ (transition probability 1 from 0 to 1) is admissible of height h.

$$p_n^{(h)} = 2[z^{n+1}]\frac{1}{T_{h+1}(1/z)}.$$

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