AofA'15, Strobl, Austria, 8-12 June 2015

# A line-breaking construction of the stable trees

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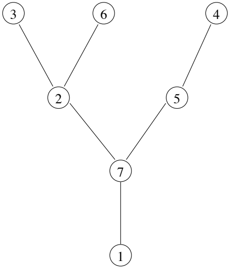
Joint work with Bénédicte Haas (Paris-Dauphine)

# Uniform random trees

Let $\mathbb{T}_n$ be the set of unordered trees on $n$ vertices labelled by $[n] := \{1, 2, \ldots, n\}$.

Write $T_n$ for a tree chosen uniformly from $\mathbb{T}_n$.



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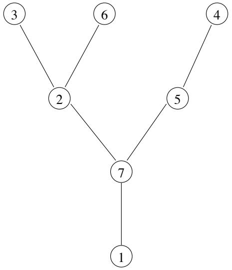



What happens as $n$ grows?

# An algorithm due to Aldous

In order to study $T_n$, it's useful to have a way of building it.

1. Start from the vertex labelled 1.
2. For $2 \leq i \leq n$, connect vertex $i$ to vertex $V_i$ such that

$$V_i = \begin{cases} i - 1 \text{ with probability } 1 - (i-2)/(n-1) \\ \text{uniform on } \{1, 2, \ldots, i-2\} \text{ otherwise.} \end{cases}$$

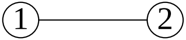
3. Take a uniform random permutation of the labels.

# Aldous' algorithm

Consider $n = 10$.

# Aldous' algorithm

$V_2 = 1$ with probability 1

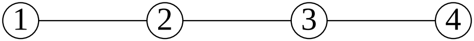


# Aldous' algorithm

$$V_3 = \begin{cases} 1 & \text{with probability } 1/9 \\ 2 & \text{with probability } 8/9 \end{cases}$$

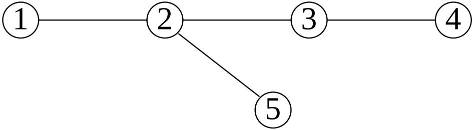
# Aldous' algorithm

$$V_4 = \begin{cases} j & \text{with probability } 1/9,\ 1 \leq j \leq 2 \\ 3 & \text{with probability } 7/9 \end{cases}$$



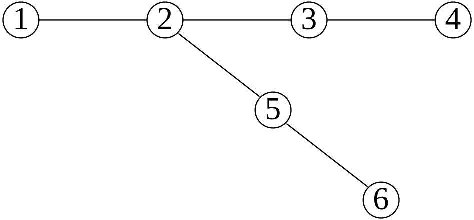
# Aldous' algorithm

$$V_5 = \begin{cases} j & \text{with probability } 1/9,\ 1 \leq j \leq 3 \\ 4 & \text{with probability } 6/9 \end{cases}$$



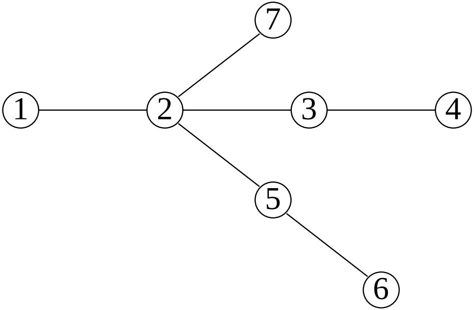
# Aldous' algorithm

$$V_6 = \begin{cases} j & \text{with probability } 1/9,\ 1 \le j \le 4 \\ 5 & \text{with probability } 5/9 \end{cases}$$



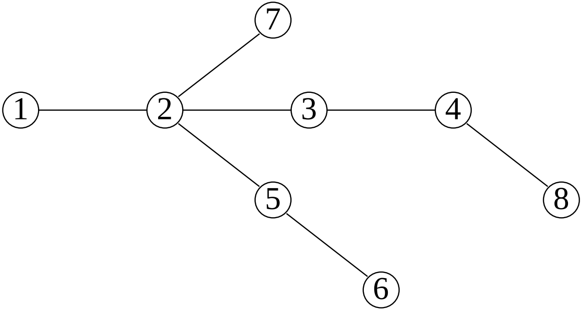
# Aldous' algorithm

$$V_7 = \begin{cases} j & \text{with probability } 1/9,\ 1 \le j \le 5 \\ 6 & \text{with probability } 4/9 \end{cases}$$



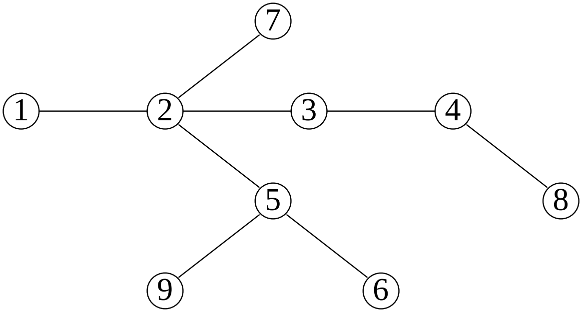
# Aldous' algorithm

$$V_8 = \begin{cases} j & \text{with probability } 1/9,\ 1 \le j \le 6 \\ 7 & \text{with probability } 3/9 \end{cases}$$



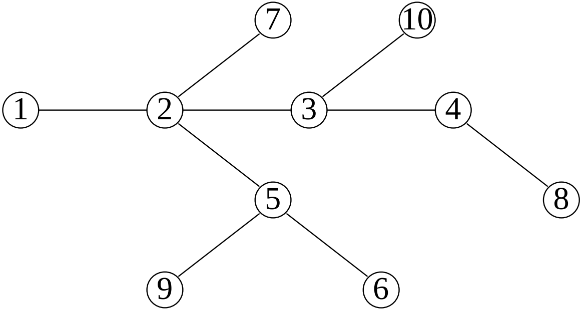
# Aldous' algorithm

$$V_9 = \begin{cases} j & \text{with probability } 1/9,\ 1 \leq j \leq 7 \\ 8 & \text{with probability } 2/9 \end{cases}$$



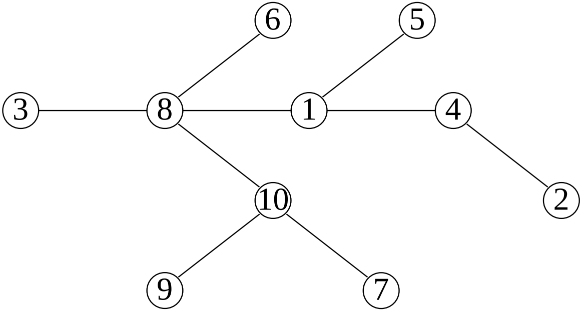
# Aldous' algorithm

$$V_{10} = \begin{cases} j & \text{with probability } 1/9,\ 1 \le j \le 8 \\ 9 & \text{with probability } 1/9 \end{cases}$$



# Aldous' algorithm

Permute.



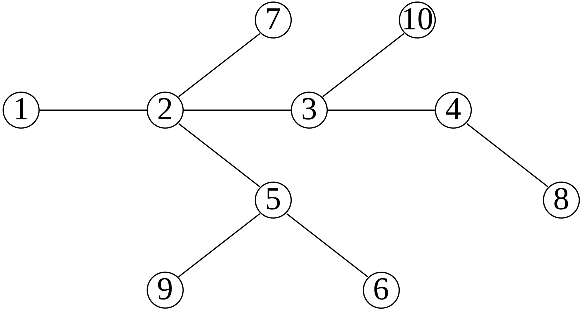
# Typical distances

Consider the tree before we permute. Let

$$L_n = \inf\{i \geq 2 : V_{i+1} \neq i\}.$$

We can use $L_n$ to give us an idea of typical distances in the tree.

In our example, $L_{10} = 4$:



# Typical distances

For $2 \leq i \leq n$, connect vertex $i$ to vertex $V_i$ such that

$$V_i = \begin{cases} i - 1 \text{ with probability } 1 - (i-2)/(n-1) \\ \text{uniform on } \{1, 2, \ldots, i-2\} \text{ otherwise.} \end{cases}$$

$$L_n = \inf\{i \geq 2 : V_{i+1} \neq i\}$$

## Proposition

*As* $n \to \infty$,

$$\mathbb{P}\left(n^{-1/2} L_n > x\right) \to \exp(-x^2/2).$$

# Proof

$$\mathbb{P}\left(n^{-1/2}L_n > x\right) = \mathbb{P}\left(L_n \geq \lfloor xn^{1/2}\rfloor + 1\right)$$
$$= \mathbb{P}\left(2 \to 1, 3 \to 2, \ldots, \lfloor xn^{1/2}\rfloor + 1 \to \lfloor xn^{1/2}\rfloor\right)$$
$$= 1 \cdot \left(1 - \frac{1}{n-1}\right)\left(1 - \frac{2}{n-1}\right)\cdots\left(1 - \frac{\lfloor xn^{1/2}\rfloor - 1}{n-1}\right).$$

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So

$$-\log \mathbb{P}\left(n^{-1/2}L_n > x\right) = -\sum_{i=1}^{\lfloor xn^{1/2}\rfloor - 1} \log\left(1 - \frac{i}{n-1}\right)$$
$$\sim \sum_{i=1}^{\lfloor xn^{1/2}\rfloor - 1} \frac{i}{n} = \frac{\lfloor xn^{1/2}\rfloor(\lfloor xn^{1/2}\rfloor - 1)}{2n} \sim \frac{x^2}{2}.$$

# Typical distances

Once we have built this first stick of consecutive labels, we pick a uniform starting point along that stick and attach a new stick with a random length, and so on.

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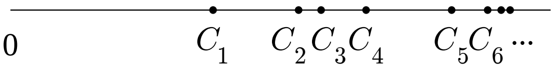
Imagine now that edges in the tree have length 1. The proposition suggests that rescaling edge-lengths by $n^{-1/2}$ will give some sort of limit for the whole tree. The limiting version of the algorithm is as follows.

# Line-breaking construction

Let $E_1, E_2, \ldots$ be independent Exponential(1/2) r.v.'s and set $C_k = \sqrt{\sum_{i=1}^k E_i}$. (Equivalently, let $C_1, C_2, \ldots$ be the points of an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity $t\ dt$.)



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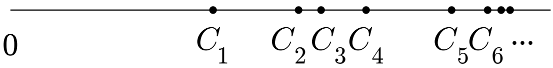


(Note that $\mathbb{P}(C_1 > x) = \mathbb{P}(E_1 > x^2) = \exp(-x^2/2)$.)

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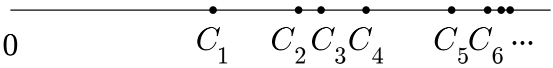
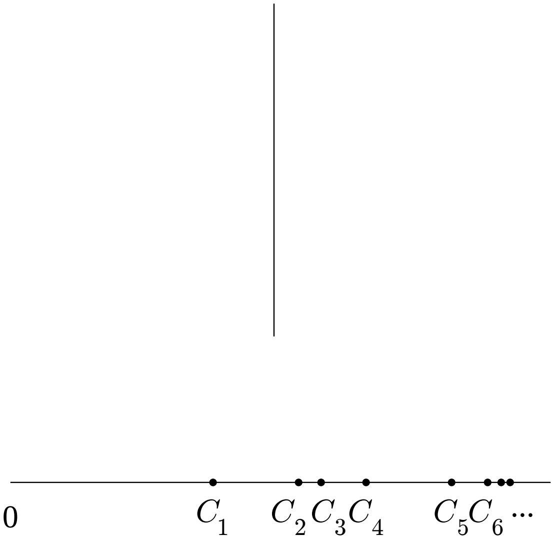



(Note that $\mathbb{P}(C_1 > x) = \mathbb{P}\left(E_1 > x^2\right) = \exp(-x^2/2)$.)

- Consider the line-segments $[0, C_1)$, $[C_1, C_2)$, $\ldots$.
- Start from $[0, C_1)$ and proceed inductively.
- For $i \geq 2$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.

# Line-breaking construction



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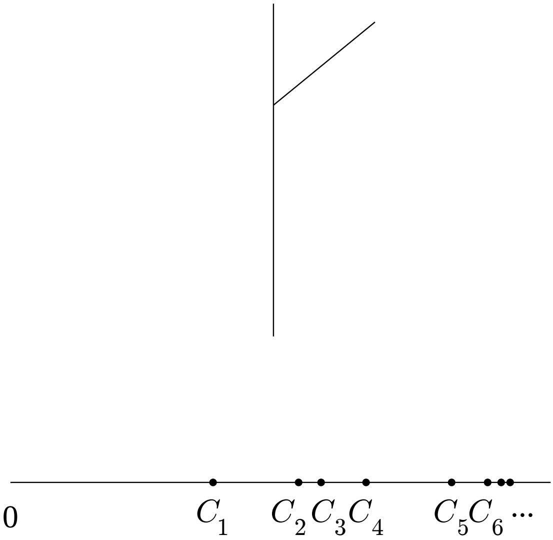

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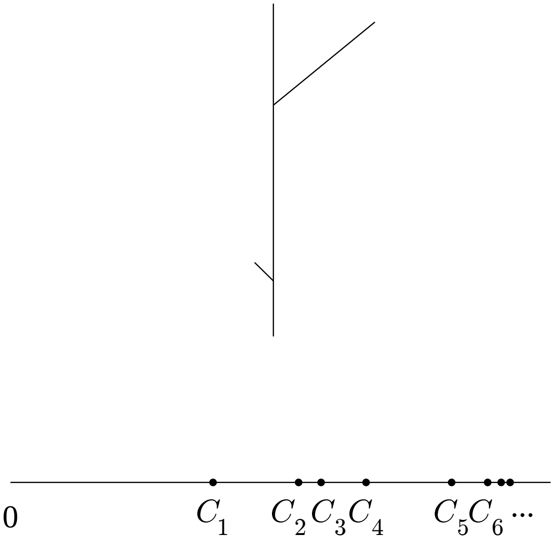

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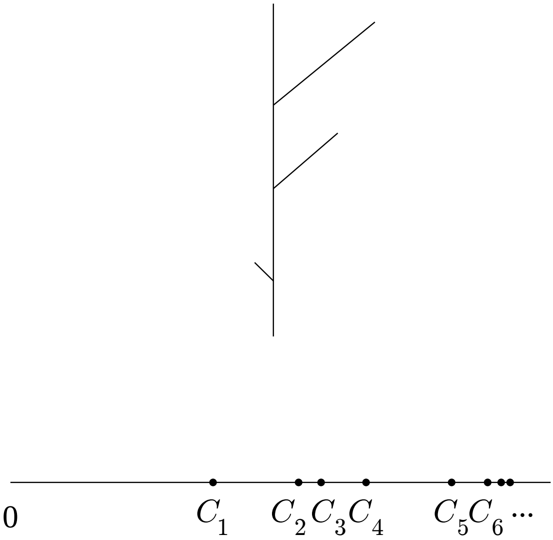

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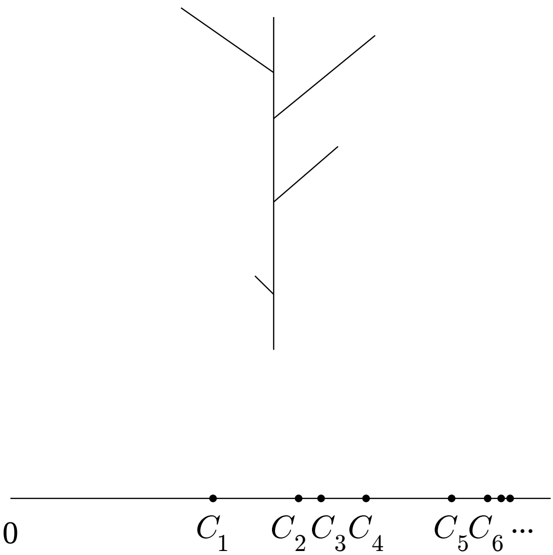

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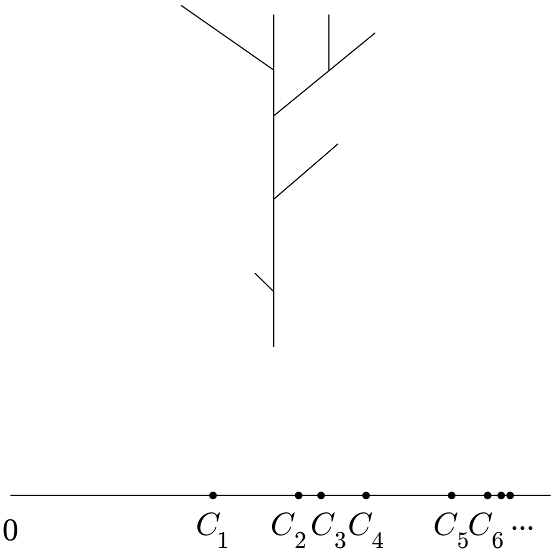
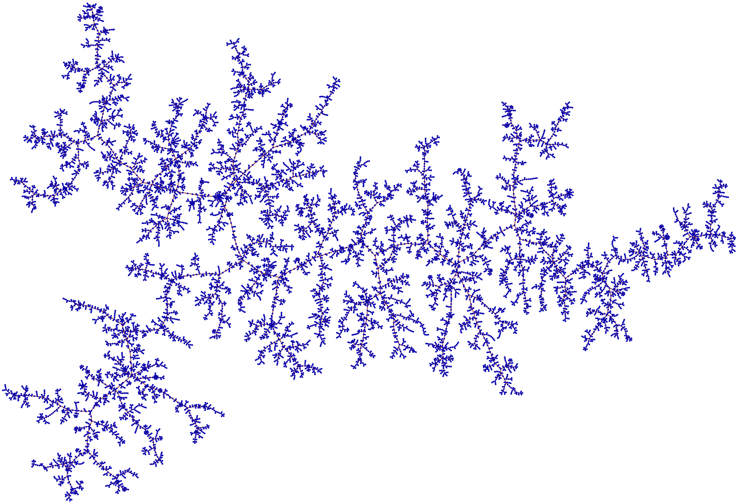


# The Brownian continuum random tree



[Picture by Igor Kortchemski]

# The scaling limit of the uniform random tree

Theorem (Aldous (1991); Le Gall (2005))

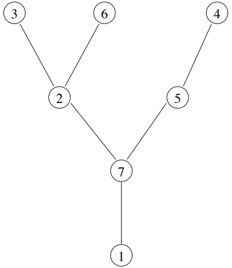
*Then*

$$\frac{1}{\sqrt{n}} T_n \xrightarrow{d} c\mathcal{T}_2 \quad \text{as } n \to \infty$$

*where $\mathcal{T}_2$ is Aldous' Brownian continuum random tree and $c$ is a non-negative constant. (The convergence is in the sense of the Gromov–Hausdorff distance.)*

# Trees as metric spaces

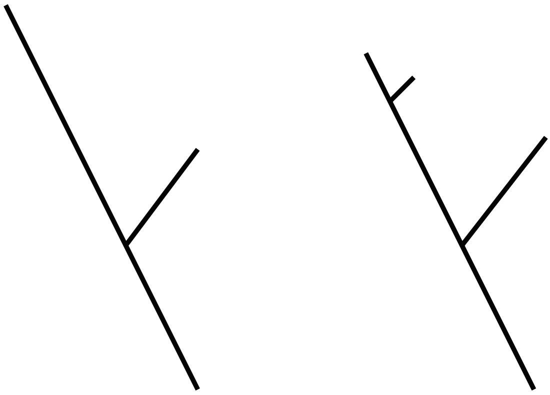
The vertices of $T_n$ come equipped with a natural metric: the graph distance.




We write $\frac{1}{\sqrt{n}} T_n$ for the metric space given by the vertices of $T_n$ with the graph distance divided by $\sqrt{n}$.

# Measuring the distance between metric spaces

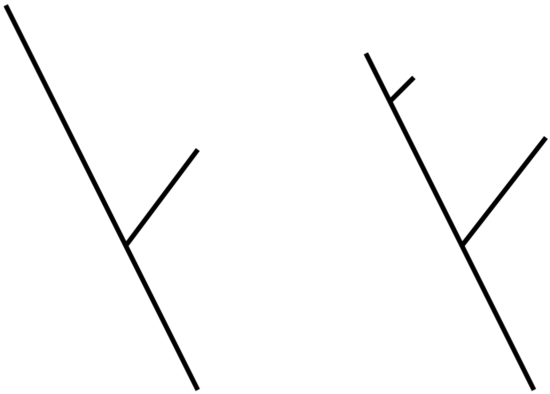
Suppose that $(X, d)$ and $(X', d')$ are compact metric spaces.



# Measuring the distance between metric spaces

Suppose that $(X, d)$ and $(X', d')$ are compact metric spaces.

A correspondence $R$ is a subset of $X \times X'$ such that for every $x \in X$, there exists $x' \in X'$ with $(x, x') \in R$ and vice versa.



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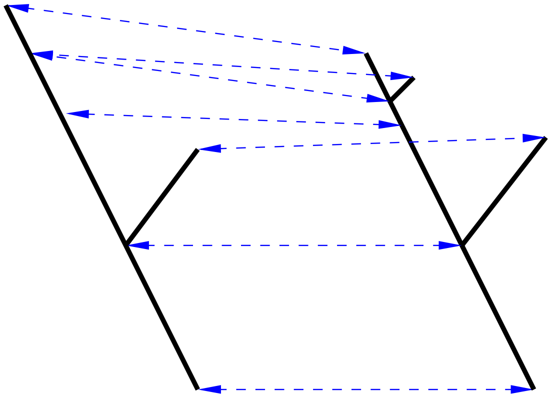
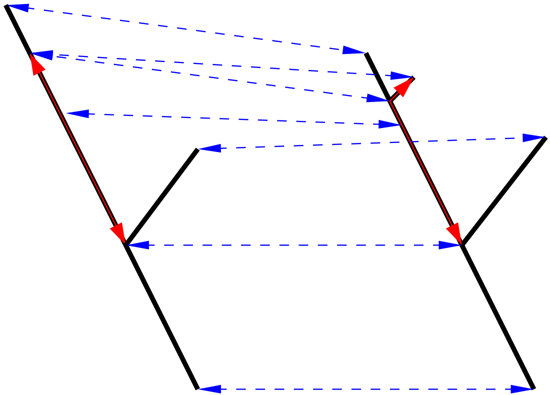


# Measuring the distance between metric spaces

The distortion of $R$ is

$$\text{dis}(R) = \sup\{|d(x,y) - d'(x',y')| : (x,x'),(y,y') \in R\}.$$



# Measuring the distance between metric spaces

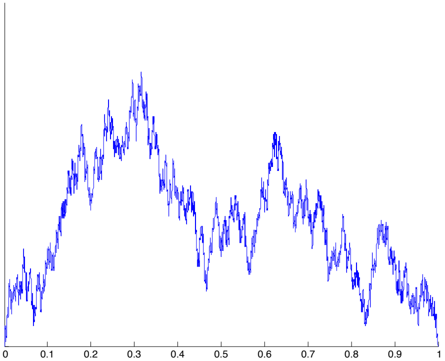
$(X, d)$ and $(X', d')$ are at Gromov-Hausdorff distance less than $\epsilon > 0$ if there exists a correspondence $R$ between $X$ and $X'$ such that $\mathrm{dis}(R) < 2\epsilon$. Write

$$\mathrm{d}_{\mathrm{GH}}((X, d), (X', d')) < \epsilon.$$

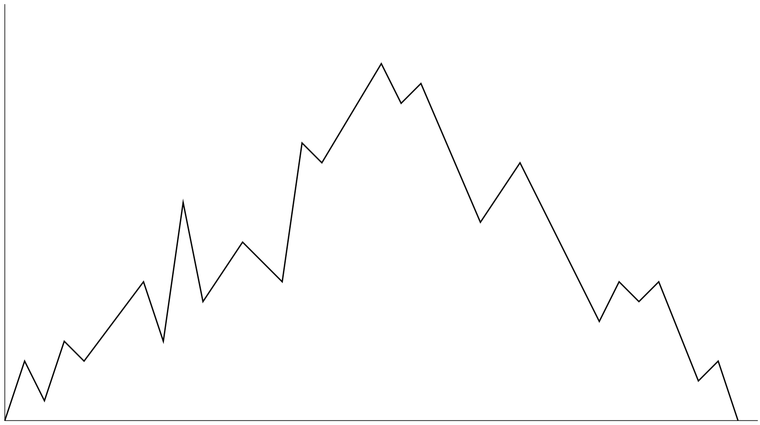
# The Brownian CRT

Why Brownian continuum random tree?

Because $\mathcal{T}_2$ can be obtained by a glueing operation performed on the standard Brownian excursion, $(e(t), 0 \leq t \leq 1)$.



# The Brownian CRT



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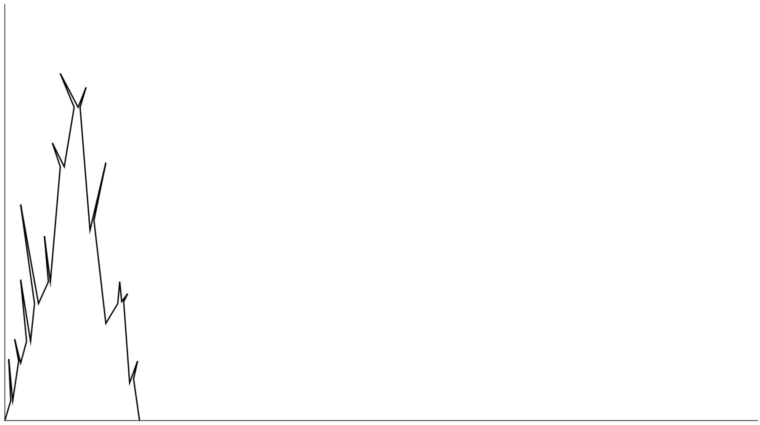

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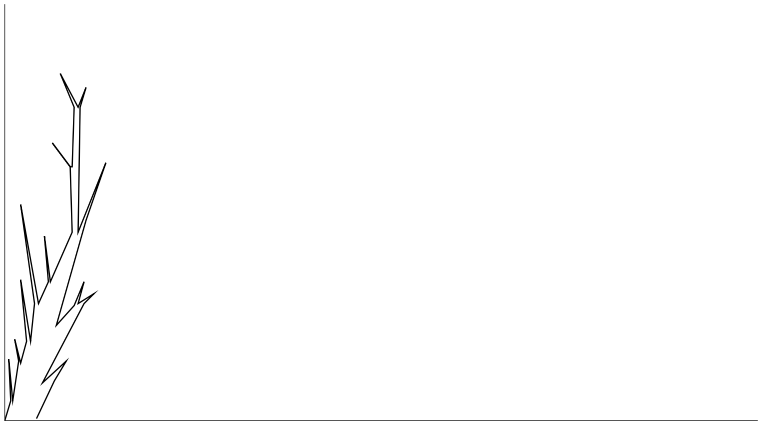

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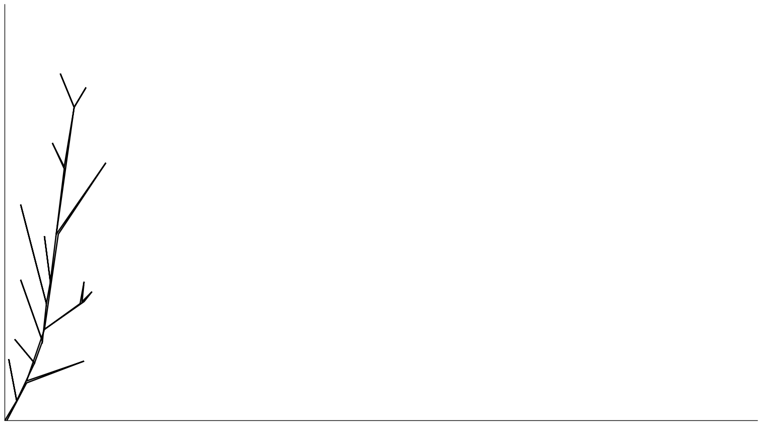

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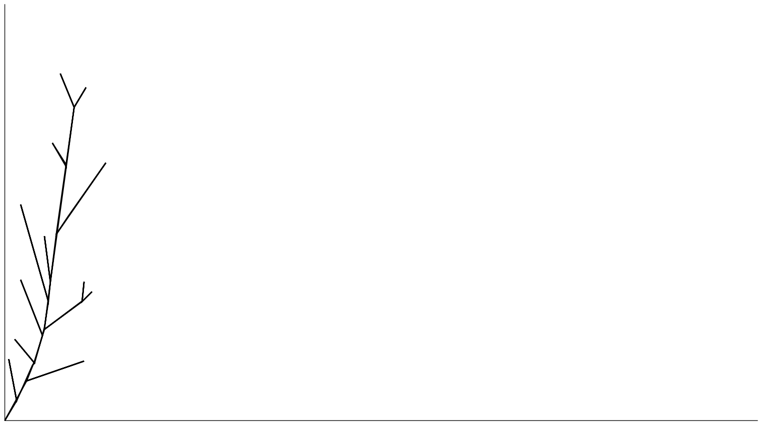

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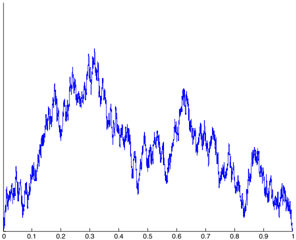
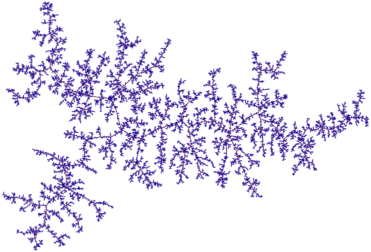



$\longrightarrow$



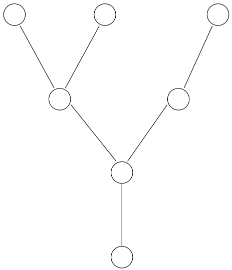
[Pictures by Igor Kortchemski]

# Critical Galton–Watson trees

Consider a Galton–Watson branching process with offspring distribution $(p_k)_{k \geq 0}$.

Suppose that the offspring distribution is critical i.e. $\sum_{k=0}^{\infty} kp_k = 1$, and condition the tree to have total progeny $n$.

Let $T_n^{\mathrm{GW}}$ be the family tree associated with this process (thought of as a rooted plane tree with $n$ vertices).



# Combinatorial trees

By taking different offspring distributions, we can obtain various different natural combinatorial models:

- Poisson(1) corresponds to the uniform random tree (once we forget the planar order and give the tree a uniform labelling).
- Geometric(1/2) gives a uniform plane tree.
- $p_0 = 1/2$, $p_2 = 1/2$ gives a uniform (complete) binary tree (as long as $n$ is odd).

# The finite-variance case

**Theorem (Aldous (1993); Le Gall (2005))**

*Suppose* $\sigma^2 := \sum_{k=2}^{\infty}(k-1)^2 p_k < \infty$*. Then*

$$\frac{1}{\sqrt{n}} T_n^{GW} \xrightarrow{d} c_\sigma \mathcal{T}_2 \quad \textit{as } n \to \infty$$

*where* $\mathcal{T}_2$ *is Aldous' Brownian continuum random tree and* $c_\sigma$ *is a non-negative constant. (The convergence is in the sense of the Gromov–Hausdorff distance.)*

# Infinite variance

What if the offspring distribution does not have finite variance? It is natural to consider offspring distributions such that $p_k \sim k^{-1-\alpha}$ for $\alpha \in (1, 2)$ (or, more generally, distributions in the domain of attraction of a stable law of parameter $\alpha$).

# The infinite-variance case

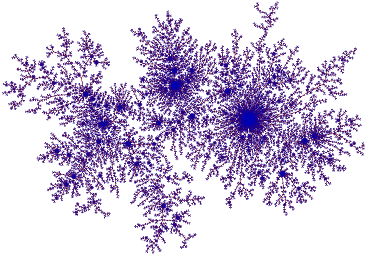
**Theorem (Duquesne & Le Gall (2002); Duquesne (2003))**

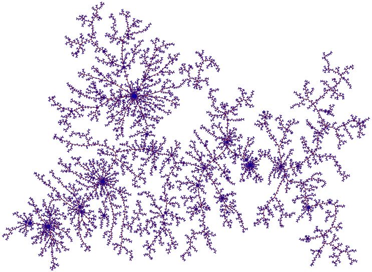
*Suppose that $(p_k)_{k \geq 0}$ lies in the domain of attraction of a stable law of index $\alpha \in (1,2)$. Then as $n \to \infty$,*

$$\frac{1}{n^{1-1/\alpha}} T_n^{GW} \xrightarrow{d} c_\alpha \mathcal{T}_\alpha,$$

*where $\mathcal{T}_\alpha$ is the stable tree of parameter $\alpha$ and $c_\alpha$ is a non-negative constant. (The convergence is in the sense of the Gromov–Hausdorff distance.)*

# The stable trees

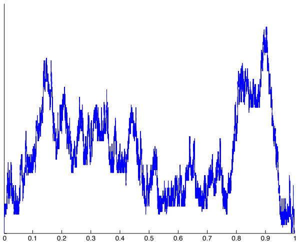


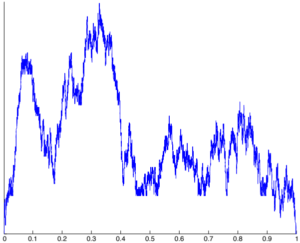


[Pictures by Igor Kortchemski]

# The stable trees

The stable trees also possess a functional encoding (although the excursions concerned are rather more involved to describe).

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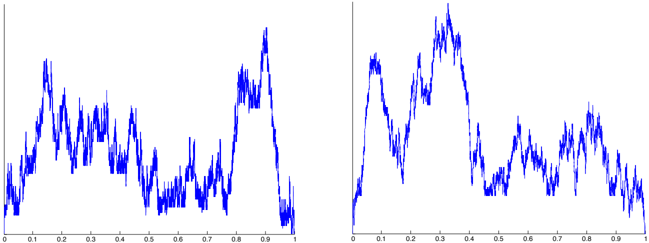


[Pictures by Igor Kortchemski]

An important difference between the stable trees for $\alpha \in (1, 2)$ and the Brownian CRT is that the Brownian CRT is binary. The stable trees, on the other hand, have only branch-points of infinite degree.

# A uniform measure

The principal theme of the rest of this talk is how to give a (relatively) simple description of the stable trees (and how to use it to get at their distributional properties).

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For $\alpha \in (1, 2]$, the stable tree $\mathcal{T}_\alpha$ is naturally endowed with a "uniform" probability measure $\mu_\alpha$, which is the limit of the discrete uniform measure on $T_n^{\mathsf{GW}}$. It turns out that $\mu_\alpha$ is supported by the set of leaves of $\mathcal{T}_\alpha$.

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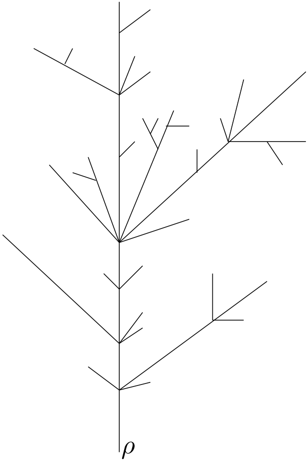
For $\alpha \in (1, 2]$, the stable tree $\mathcal{T}_\alpha$ is naturally endowed with a "uniform" probability measure $\mu_\alpha$, which is the limit of the discrete uniform measure on $T_n^{\mathrm{GW}}$. It turns out that $\mu_\alpha$ is supported by the set of leaves of $\mathcal{T}_\alpha$.

Aldous' theory of continuum random trees tells us that we can characterize the laws of such trees via sampling.

# Reduced trees

Let $X_1, X_2, \ldots$ be leaves sampled independently from $\mathcal{T}_\alpha$ according to $\mu_\alpha$, and let $\mathcal{T}_{\alpha,n}$ be the subtree spanned by the root $\rho$ and $X_1, \ldots, X_n$:



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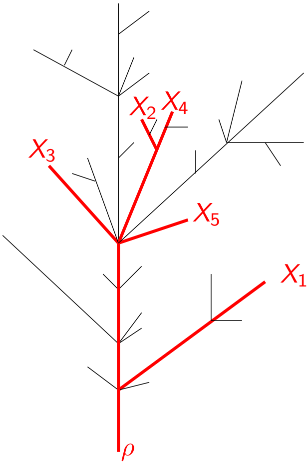

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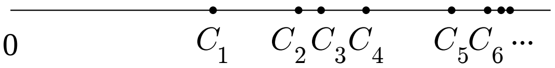
The laws of $(\mathcal{T}_{\alpha,n}, n \geq 1)$ (the random finite-dimensional distributions) are sufficient to fully specify the law of $\mathcal{T}_\alpha$.

Moreover,

$$\mathcal{T}_\alpha = \overline{\bigcup_{n \geq 1} \mathcal{T}_{\alpha,n}}.$$

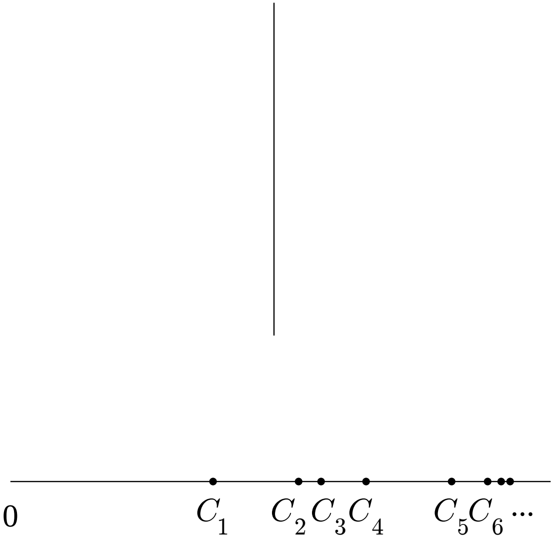
# Reminder: Aldous' line-breaking construction of the Brownian CRT

Let $C_1, C_2, \ldots$ be the points of an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity $t\ dt$.



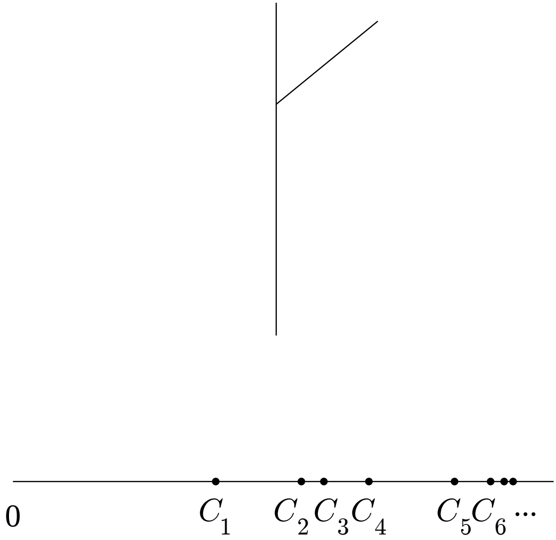
# Line-breaking construction

$\tilde{\mathcal{T}}_1$



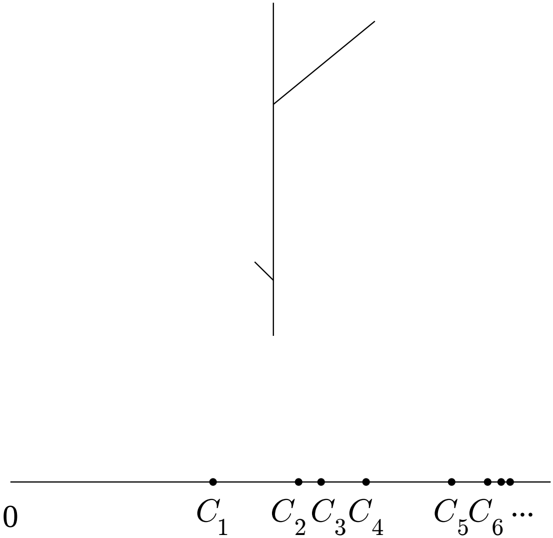
# Line-breaking construction

$\tilde{\mathcal{T}}_2$



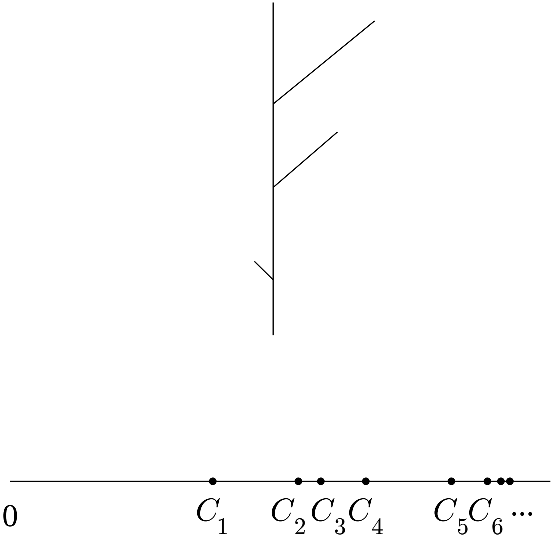
# Line-breaking construction

$\tilde{\mathcal{T}}_3$



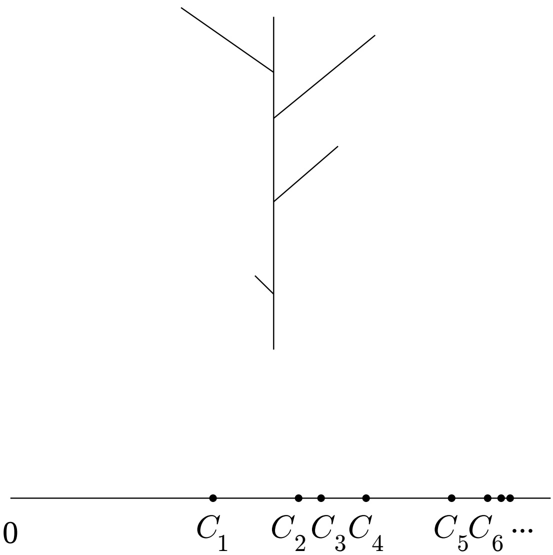
# Line-breaking construction

$\tilde{\mathcal{T}}_4$



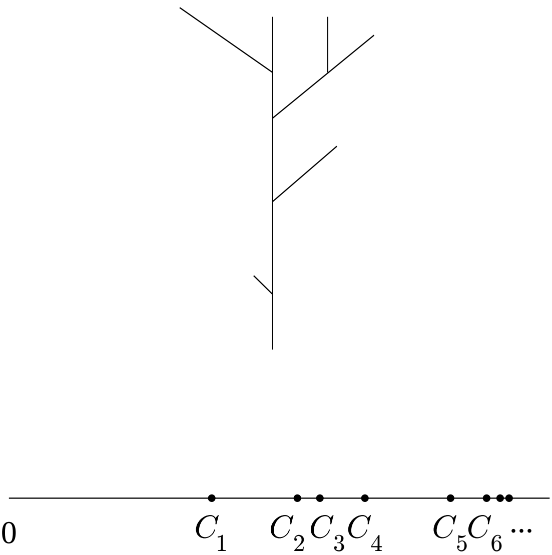
# Line-breaking construction

$\tilde{\mathcal{T}}_5$



# Line-breaking construction

$\tilde{\mathcal{T}}_6$



# Line-breaking construction

It turns out that the line-breaking construction precisely gives the random finite-dimensional distributions for the Brownian CRT, i.e.

$$(\tilde{\mathcal{T}}_n, n \geq 1) \stackrel{d}{=} \left(\frac{1}{\sqrt{2}}\mathcal{T}_{2,n}, n \geq 1\right).$$

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Question: does there exist a similar line-breaking construction for the stable trees with $\alpha \in (1, 2)$?

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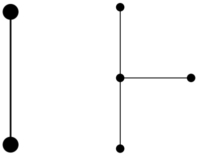

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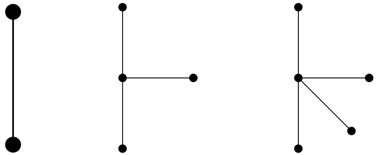

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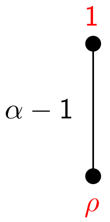

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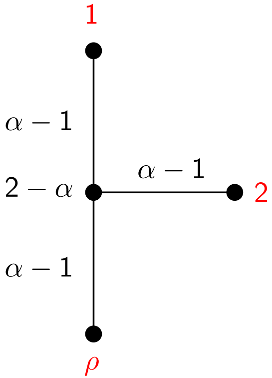

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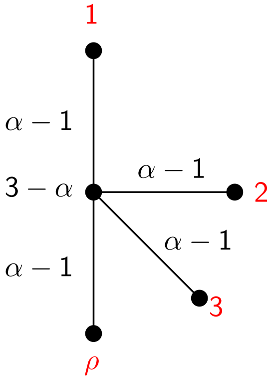

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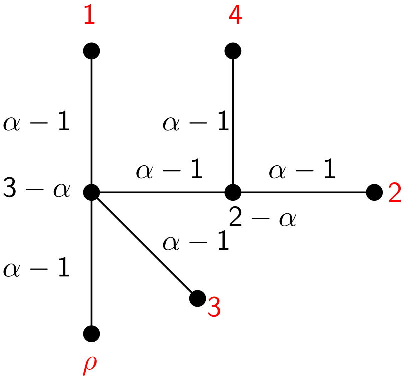

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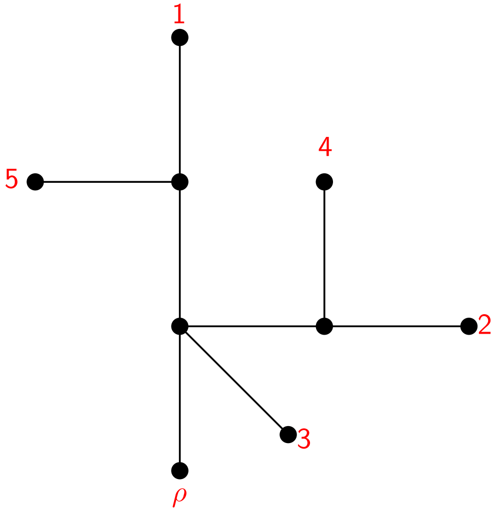

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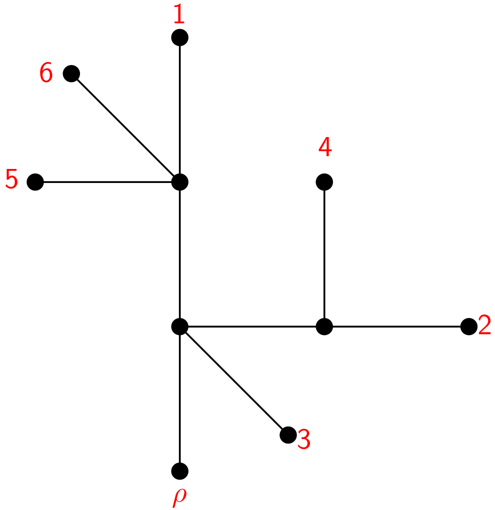

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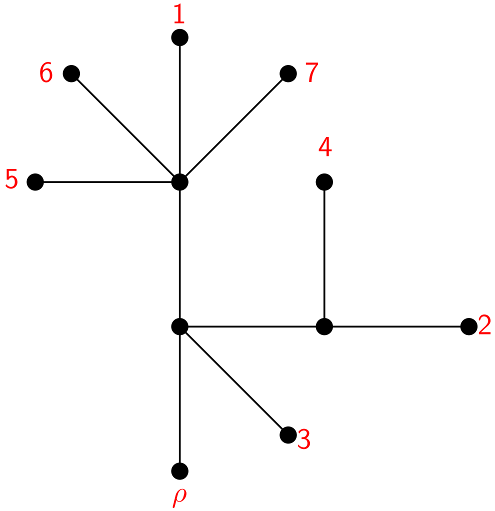

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Then

$$(\tilde{T}_n, n \geq 1) \stackrel{d}{=} (T_{\alpha,n}, n \geq 1).$$

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Our new line-breaking construction gives a nested sequence of continuous trees which converge a.s. to $\mathcal{T}_\alpha$ without any need for rescaling.

# The generalized Mittag-Leffler distribution

For $\beta \in (0,1)$, let $\sigma_\beta$ be a stable random variable with Laplace transform

$$\mathbb{E}\left[\exp(-\lambda\sigma_\beta)\right] = \exp(-\lambda^\beta), \quad \lambda \geq 0.$$

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Say that a non-negative random variable $M$ has the generalized Mittag-Leffler distribution with parameters $\beta \in (0,1)$ and $\theta > -\beta$, and write $M \sim \mathrm{ML}(\beta, \theta)$, if

$$\mathbb{E}\left[f(M)\right] = C_{\beta,\theta}\mathbb{E}\left[\sigma_\beta^{-\theta} f\left(\sigma_\beta^{-\beta}\right)\right].$$

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If $\beta = 1/2$ and $n \geq 1$, $\mathrm{ML}(1/2, n - 1/2) = 2\sqrt{\mathrm{Gamma}(n, 1)}$.

# A generalized Pólya urn scheme

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Pick a colour with probability proportional to its weight in the urn.

- If black is picked, add $1/\beta$ to the black weight.
- If red is picked, add $1 - 1/\beta$ to the black weight and 1 to the red weight.

Let $R_n$ be the weight of red at step $n$. Then [Janson (2006)],

$$n^{-\beta} R_n \overset{\text{a.s.}}{\to} W \sim \mathrm{ML}(\beta, \theta).$$

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At each subsequent step,

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- with probability proportional to $W_n$ add the new edge elsewhere; this yields $W_{n+1} = W_n + \alpha$.

(We always add weight $\alpha$ to the whole tree.)

# Urns in Marchal's algorithm

Then $(D_n, n \geq 1)$ behaves exactly as the red weight in the generalized Pólya urn with $\beta = \theta = 1 - 1/\alpha$. It follows that

$$\frac{1}{n^{1-1/\alpha}} D_n \xrightarrow{d} \mathrm{ML}(1 - 1/\alpha, 1 - 1/\alpha)$$

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This suggests that the first stick in any line-breaking construction should have length distributed as $\mathrm{ML}(1 - 1/\alpha, 1 - 1/\alpha)$.

# A Markov chain

We define an increasing $\mathbb{R}_+$-valued process which will play a role similar to that of the inhomogeneous Poisson process in the Brownian case.

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We define an increasing $\mathbb{R}_+$-valued process which will play a role similar to that of the inhomogeneous Poisson process in the Brownian case.

Let $(M_n, n \geq 1)$ be a Markov chain such that

- $M_n \sim \mathrm{ML}(1 - 1/\alpha, n - 1/\alpha)$ for $n \geq 1$.
- The backward transition fron $M_{n+1}$ to $M_n$ is given by

$$M_n = M_{n+1}\ \beta_n,$$

where $\beta_n$ is independent of $M_{n+1}$ and

$$\beta_n \sim \mathrm{Beta}\left(\frac{(n+1)\alpha - 2}{\alpha - 1}, \frac{1}{\alpha - 1}\right).$$

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*If* $\alpha = 2$, $(M_n, n \geq 1)$ *are the ordered points of an inhomogeneous Poisson process on* $\mathbb{R}_+$ *with intensity* $\frac{t}{2}dt$.

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## Sketch proof.

It suffices to show that $(M_n^2/4, n \geq 1)$ are the ordered points of a Poisson process of rate 1. But
$M_n \sim \mathrm{ML}(1/2, n - 1/2) = 2\sqrt{\mathrm{Gamma}(n, 1)}$ and so
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**Lemma**

*If $\alpha = 2$, $(M_n, n \geq 1)$ are the ordered points of an inhomogeneous Poisson process on $\mathbb{R}_+$ with intensity $\frac{t}{2}dt$.*

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The relationship between successive points encoded in
$M_n = \beta_n M_{n+1}$ where $\beta_n \sim \mathrm{Beta}(2n, 1)$ gives exactly the right dependence structure. □

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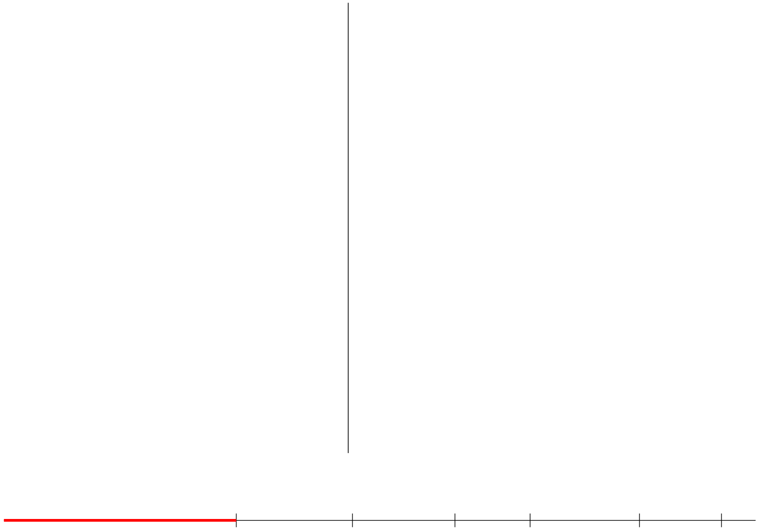
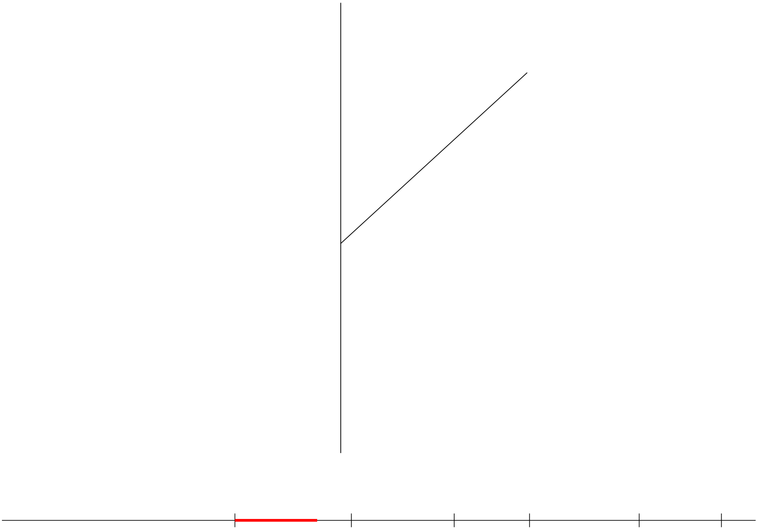
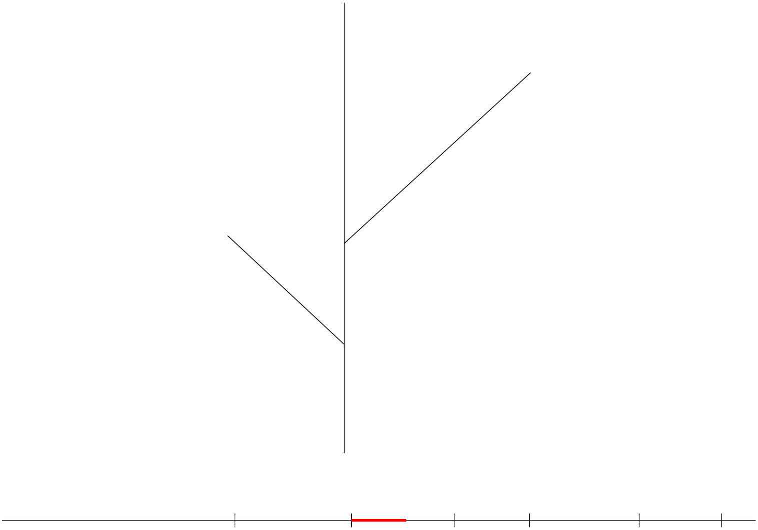
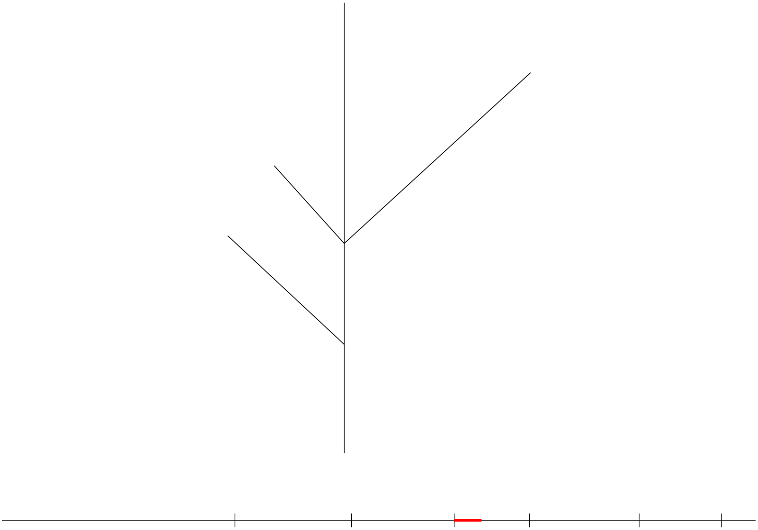
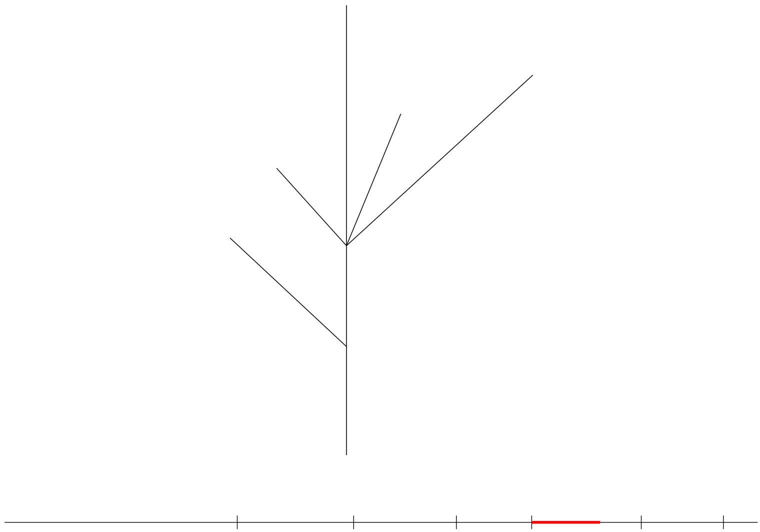
  3. If we select the edges in 2, glue the new branch at a uniform point along $\tilde{\mathcal{T}}_n$.
  4. If we select the branchpoints in 2, pick a branchpoint at random in such a way that a branchpoint of degree $d \geq 3$ is chosen with probability proportional to $d - 1 - \alpha$. Then glue the new branch to the selected branchpoint.

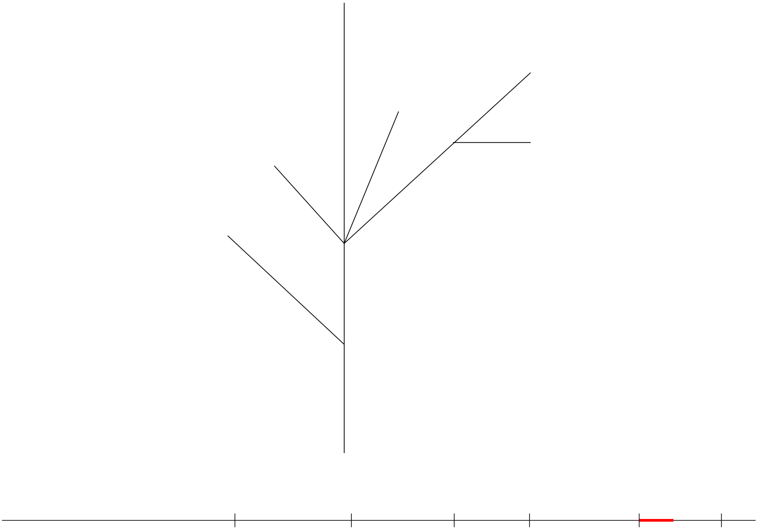


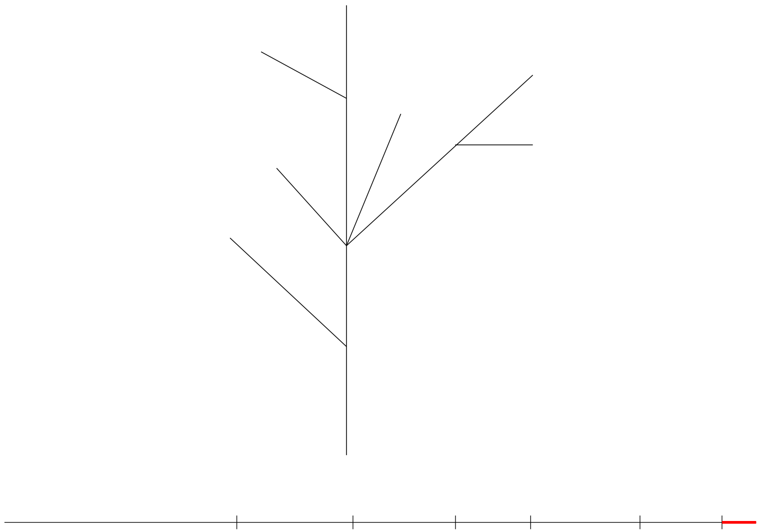


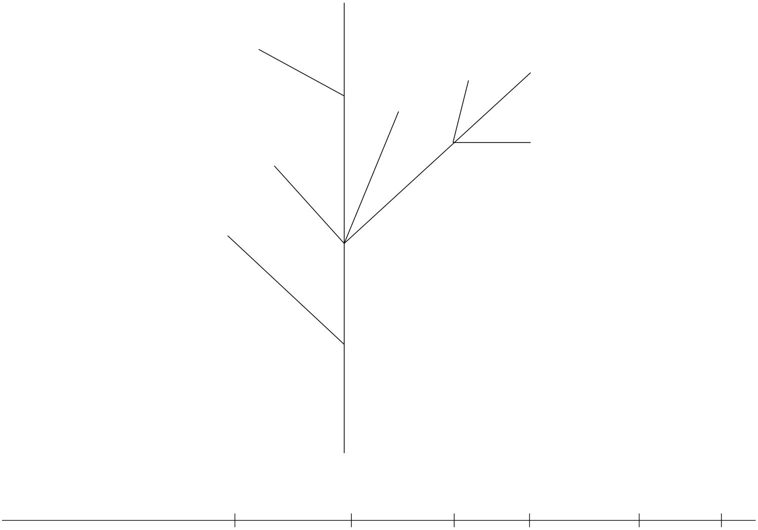


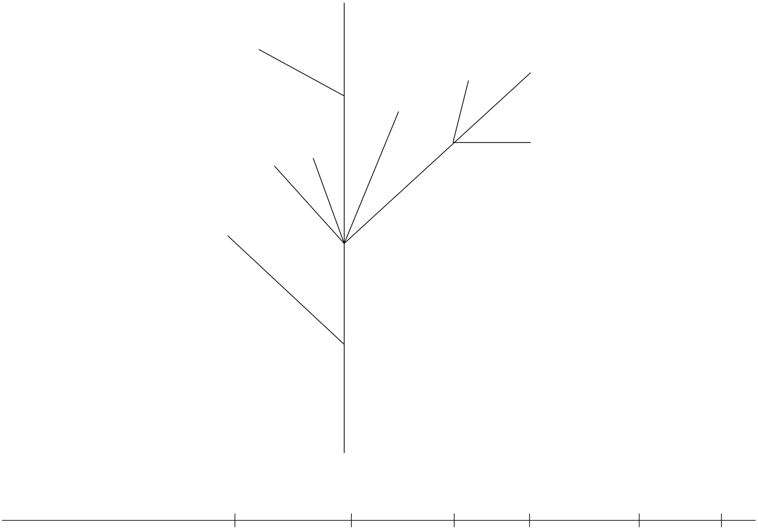


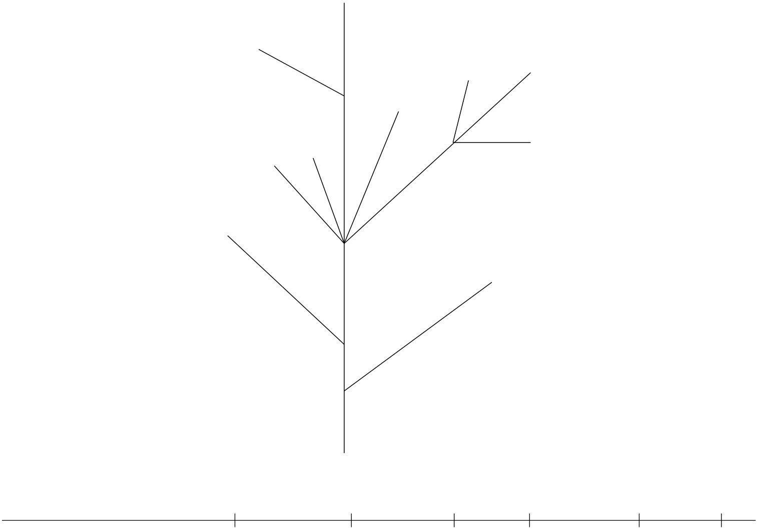


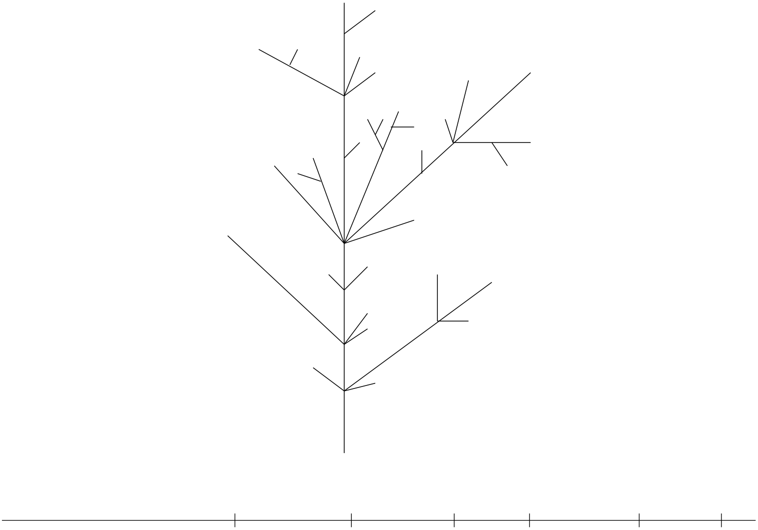












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# Line-breaking constructions

Theorem (Haas & G.)

*Let* $(\tilde{\mathcal{T}}_n, n \geq 1)$ *be the sequence of trees produced by either version of the construction. Then*

$$(\tilde{\mathcal{T}}_n, n \geq 1) \stackrel{d}{=} (\mathcal{T}_{\alpha,n}, n \geq 1)$$

*and, therefore,*

$$\mathcal{T}_\alpha \stackrel{d}{=} \overline{\bigcup_{n \geq 1} \tilde{\mathcal{T}}_n}.$$

# Remarks

In the case $\alpha = 2$, we have Beta $\left(1, \frac{2-\alpha}{\alpha-1}\right) = \text{Beta}(1, 0)$. We interpret this as $B_n = 1$ almost surely for all $n \geq 1$. Then we recover (a scaled version of) Aldous' Poisson line-breaking construction of the Brownian CRT.

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The tree-shapes $(\tilde{T}_n, n \geq 1)$ of $(\tilde{\mathcal{T}}_n, n \geq 1)$ perform Marchal's algorithm.

# Consequences: distributional results for $(\mathcal{T}_{\alpha,n}, n \geq 1)$

**Edge-lengths**:

Let $\mathrm{t}$ be a discrete rooted tree with $n \geq 2$ leaves and $k$ edges. Then conditionally on $T_{\alpha,n} = \mathrm{t}$, the sequence of edge-lengths of $\mathcal{T}_{\alpha,n}$ has the same distribution as

$$M_n \cdot \beta_k \cdot (D_1, D_2, \ldots, D_k),$$

where these random variables are independent and

$$M_n \sim \mathrm{ML}(1 - 1/\alpha, n - 1/\alpha)$$

$$\beta_k \sim \mathrm{Beta}\left(k, \frac{n\alpha - 1}{\alpha - 1}\right)$$

$$(D_1, D_2, \ldots, D_k) \sim \mathrm{Dir}(1, 1, \ldots, 1).*$$

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*Dirichlet distribution: $\mathrm{Dir}(a_1, \ldots, a_n)$ has density

$$\frac{\Gamma(a_1 + \ldots + a_n)}{\prod_{i=1}^n \Gamma(a_i)} x_1^{a_1 - 1} \ldots x_n^{a_n - 1}$$

with respect to Lebesgue measure on

$$\left\{(x_1, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i = 1\right\}.$$

# Consequences: distributional results for $(\mathcal{T}_{\alpha,n}, n \geq 1)$

**Total length of the conditioned tree**:
Conditionally on $T_{\alpha,n}$ having $k$ edges, the total length of the tree $\mathcal{T}_{\alpha,n}$ has the same distribution as

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where these random variables are independent and $M_n \sim \mathrm{ML}(1 - 1/\alpha, n - 1/\alpha)$ and $\beta_k \sim \mathrm{Beta}(k, \frac{n\alpha-1}{\alpha-1})$.

# Consequences: distributional results for $(\mathcal{T}_{\alpha,n}, n \geq 1)$

**Total length of the unconditioned tree**:
The total length of the tree $\mathcal{T}_{\alpha,n}$ has the same distribution as

$$M_n \cdot \left( \prod_{j=1}^{n-1} \beta_j + \sum_{i=1}^{n-1} B_i (1-\beta_i) \prod_{j=i+1}^{n-1} \beta_j \right),$$

where the random variables on the right-hand side are mutually independent and such that

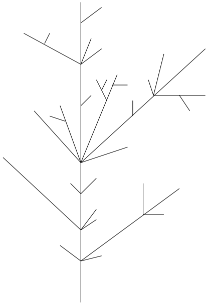
$$M_n \sim \mathrm{ML}(1-1/\alpha, n-1/\alpha)$$

$$\beta_i \sim \mathrm{Beta}\left( \frac{(i+1)\alpha - 2}{\alpha - 1}, \frac{1}{\alpha-1} \right), \quad i \geq 1$$

$$B_1, B_2, \ldots, B_n \sim \mathrm{Beta}\left(1, \frac{2-\alpha}{\alpha-1}\right).$$

# Open problem

Does there exist a discrete version of our line-breaking construction (à la Aldous' construction of the uniform random tree)?



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# Beta-Gamma algebra

The proof relies heavily on the following distributional facts.

- If $B \sim \text{Beta}(a, b)$ and $G \sim \text{Gamma}(a + b, 1)$ are independent then
$$G \times (B, 1 - B) \overset{d}{=} (G_1, G_2),$$
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  and is independent of $G_1 + G_2 \sim \text{Gamma}(a + b, 1)$.
- Let $\mathbf{D} = (D_1, D_2, \ldots, D_n) \sim \text{Dir}(a_1, a_2, \ldots, a_n)$ and $\mathbb{P}(I = i | \mathbf{D}) = D_i$. Then, conditionally on the event $\{I = i\}$, we have

$$(D_1, \ldots, D_i, \ldots, D_n) \sim \text{Dir}(a_1, \ldots, a_i + 1, \ldots, a_n).$$

# An idea of the proof (of version (II))

The key point is that, conditionally on the shapes $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n$ (with $\tilde{T}_n$ having $k$ edges and $\ell$ internal vertices), the edge-lengths and vertex weights are such that

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This can be proved inductively.

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If we pick a co-ordinate which corresponded to an edge, it now has parameter 2. Splitting that co-ordinate with an independent uniform gives back 2 co-ordinates with parameter 1.

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Whether we picked an edge or a vertex, we now want to add one co-ordinate equal to 1 (representing the new edge) and either a co-ordinate equal to $\frac{2-\alpha}{\alpha-1}$ (for a new vertex) or an additional weight to the existing vertex whose weight we already biased: $\frac{d-1-\alpha}{\alpha-1} + 1 + \frac{2-\alpha}{\alpha-1} = \frac{(d+1)-1-\alpha}{\alpha-1}$.

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This is the role of $(B_n, 1 - B_n) \sim \mathrm{Beta}(1, \frac{2-\alpha}{\alpha-1})$.

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The $\beta_n$ factor is precisely what is needed to rescale the Dirichlet vector in order to accommodate the extra co-ordinates we added.

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