A line-breaking construction of the stable trees

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Uniform random trees

Let \( \mathbb{T}_n \) be the set of unordered trees on \( n \) vertices labelled by \([n] := \{1, 2, \ldots, n\}\).

Write \( T_n \) for a tree chosen uniformly from \( \mathbb{T}_n \).
Uniform random trees

Let $\mathbb{T}_n$ be the set of unordered trees on $n$ vertices labelled by $[n] := \{1, 2, \ldots, n\}$.

Write $T_n$ for a tree chosen uniformly from $\mathbb{T}_n$.

What happens as $n$ grows?
An algorithm due to Aldous

In order to study $T_n$, it’s useful to have a way of building it.

1. Start from the vertex labelled 1.
2. For $2 \leq i \leq n$, connect vertex $i$ to vertex $V_i$ such that
   \[ V_i = \begin{cases} 
   i - 1 & \text{with probability } 1 - (i - 2)/(n - 1) \\
   \text{uniform on } \{1, 2, \ldots, i - 2\} & \text{otherwise.}
   \end{cases} \]
3. Take a uniform random permutation of the labels.
Aldous’ algorithm

Consider $n = 10$. 

1
Aldous’ algorithm

\[ V_2 = 1 \text{ with probability 1} \]
Aldous’ algorithm

\[ V_3 = \begin{cases} 
1 & \text{with probability } 1/9 \\
2 & \text{with probability } 8/9 
\end{cases} \]
Aldous’ algorithm

\[ V_4 = \begin{cases} 
    j & \text{with probability } 1/9, \ 1 \leq j \leq 2 \\
    3 & \text{with probability } 7/9
\end{cases} \]
Aldous’ algorithm

\[ V_5 = \begin{cases} 
  j & \text{with probability } \frac{1}{9}, \ 1 \leq j \leq 3 \\
  4 & \text{with probability } \frac{6}{9} 
\end{cases} \]
Aldous’ algorithm

\[ V_6 = \begin{cases} 
  j & \text{with probability } \frac{1}{9}, \ 1 \leq j \leq 4 \\
  5 & \text{with probability } \frac{5}{9} 
\end{cases} \]
Aldous’ algorithm

\[ V_7 = \begin{cases} 
  j & \text{with probability } 1/9, \ 1 \leq j \leq 5 \\
  6 & \text{with probability } 4/9 
\end{cases} \]
Aldous’ algorithm

\[ V_8 = \begin{cases} 
    j & \text{with probability } 1/9, \ 1 \leq j \leq 6 \\
    7 & \text{with probability } 3/9
\end{cases} \]
Aldous’ algorithm

\[ V_9 = \begin{cases} 
  j & \text{with probability } 1/9, \ 1 \leq j \leq 7 \\
  8 & \text{with probability } 2/9 
\end{cases} \]
Aldous’ algorithm

\[ V_{10} = \begin{cases} j & \text{with probability } \frac{1}{9}, \ 1 \leq j \leq 8 \\ 9 & \text{with probability } \frac{1}{9} \end{cases} \]
Aldous’ algorithm

Permute.
Typical distances

Consider the tree before we permute. Let

\[ L_n = \inf\{i \geq 2 : V_{i+1} \neq i\} \].

We can use \( L_n \) to give us an idea of typical distances in the tree.

In our example, \( L_{10} = 4 \):

```
1  2  3  4
  \ \ \ \\
  5  6  7  8
  \ \ \ \\
  9
```
Typical distances

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$$V_i = \begin{cases} i - 1 & \text{with probability } 1 - (i - 2)/(n - 1) \\ \text{uniform on } \{1, 2, \ldots, i - 2\} & \text{otherwise.} \end{cases}$$

$$L_n = \inf \{i \geq 2 : V_{i+1} \neq i\}$$

Proposition

As $n \to \infty$,

$$\mathbb{P} \left( n^{-1/2} L_n > x \right) \to \exp(-x^2/2).$$
Proof

\[ \mathbb{P} \left( n^{-1/2} L_n > x \right) = \mathbb{P} \left( L_n \geq \lfloor xn^{1/2} \rfloor + 1 \right) \]

\[ = \mathbb{P} \left( 2 \rightarrow 1, 3 \rightarrow 2, \ldots, \lfloor xn^{1/2} \rfloor + 1 \rightarrow \lfloor xn^{1/2} \rfloor \right) \]

\[ = 1 \cdot \left( 1 - \frac{1}{n-1} \right) \left( 1 - \frac{2}{n-1} \right) \cdots \left( 1 - \frac{\lfloor xn^{1/2} \rfloor - 1}{n-1} \right). \]
Proof

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\[ = P \left( 2 \rightarrow 1, 3 \rightarrow 2, \ldots, \lfloor xn^{1/2} \rfloor + 1 \rightarrow \lfloor xn^{1/2} \rfloor \right) \]

\[ = 1 \cdot \left( 1 - \frac{1}{n-1} \right) \left( 1 - \frac{2}{n-1} \right) \ldots \left( 1 - \frac{\lfloor xn^{1/2} \rfloor - 1}{n-1} \right). \]

So

\[ - \log P \left( n^{-1/2} L_n > x \right) = - \sum_{i=1}^{\lfloor xn^{1/2} \rfloor - 1} \log \left( 1 - \frac{i}{n-1} \right) \]

\[ \approx \sum_{i=1}^{\lfloor xn^{1/2} \rfloor - 1} \frac{i}{n} = \frac{\lfloor xn^{1/2} \rfloor (\lfloor xn^{1/2} \rfloor - 1)}{2n} \sim \frac{x^2}{2}. \]
Once we have built this first stick of consecutive labels, we pick a uniform starting point along that stick and attach a new stick with a random length, and so on.
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Imagine now that edges in the tree have length 1. The proposition suggests that rescaling edge-lengths by $n^{-1/2}$ will give some sort of limit for the whole tree. The limiting version of the algorithm is as follows.
Line-breaking construction

Let $E_1, E_2, \ldots$ be independent Exponential$(1/2)$ r.v.’s and set

$$C_k = \sqrt{\sum_{i=1}^{k} E_i}.$$  (Equivalently, let $C_1, C_2, \ldots$ be the points of an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity $t \, dt$.)

![Diagram showing line-breaking construction with points $C_1, C_2, C_3, C_4, C_5, C_6, \ldots$]
Line-breaking construction

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(Note that $\mathbb{P}(C_1 > x) = \mathbb{P}(E_1 > x^2) = \exp(-x^2/2)$.)
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(Note that $\mathbb{P}(C_1 > x) = \mathbb{P}(E_1 > x^2) = \exp(-x^2/2)$.)

- Consider the line-segments $[0, C_1), [C_1, C_2), \ldots$.
- Start from $[0, C_1)$ and proceed inductively.
- For $i \geq 2$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.
Line-breaking construction
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The Brownian continuum random tree

[Picture by Igor Kortchemski]
The scaling limit of the uniform random tree

**Theorem (Aldous (1991); Le Gall (2005))**

Then

\[
\frac{1}{\sqrt{n}} T_n \xrightarrow{d} cT_2 \quad \text{as } n \to \infty
\]

where \( T_2 \) is Aldous’ *Brownian continuum random tree* and \( c \) is a non-negative constant. (The convergence is in the sense of the Gromov–Hausdorff distance.)
Trees as metric spaces

The vertices of $T_n$ come equipped with a natural metric: the graph distance.

We write $\frac{1}{\sqrt{n}} T_n$ for the metric space given by the vertices of $T_n$ with the graph distance divided by $\sqrt{n}$. 
Measuring the distance between metric spaces

Suppose that \((X, d)\) and \((X', d')\) are compact metric spaces.
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A correspondence \(R\) is a subset of \(X \times X'\) such that for every \(x \in X\), there exists \(x' \in X'\) with \((x, x') \in R\) and vice versa.
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Measuring the distance between metric spaces

The distortion of $R$ is

$$\text{dis}(R) = \sup \{ |d(x, y) - d'(x', y')| : (x, x'), (y, y') \in R \}.$$
(\(X, d\)) and (\(X', d'\)) are at Gromov-Hausdorff distance less than \(\epsilon > 0\) if there exists a correspondence \(R\) between \(X\) and \(X'\) such that \(\text{dis}(R) < 2\epsilon\). Write
\[
d_{\text{GH}}((X, d), (X', d')) < \epsilon.
\]
The Brownian CRT

Why Brownian continuum random tree?

Because $\mathcal{T}_2$ can be obtained by a glueing operation performed on the standard Brownian excursion, $(e(t), 0 \leq t \leq 1)$. 
The Brownian CRT
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[Pictures by Igor Kortchemski]
Critical Galton–Watson trees

Consider a Galton–Watson branching process with offspring distribution \((p_k)_{k \geq 0}\).

Suppose that the offspring distribution is critical i.e. \(\sum_{k=0}^{\infty} kp_k = 1\), and condition the tree to have total progeny \(n\).

Let \(T_n^{GW}\) be the family tree associated with this process (thought of as a rooted plane tree with \(n\) vertices).
Combinatorial trees

By taking different offspring distributions, we can obtain various different natural combinatorial models:

- Poisson(1) corresponds to the uniform random tree (once we forget the planar order and give the tree a uniform labelling).
- Geometric(1/2) gives a uniform plane tree.
- $p_0 = 1/2$, $p_2 = 1/2$ gives a uniform (complete) binary tree (as long as $n$ is odd).
The finite-variance case

Theorem (Aldous (1993); Le Gall (2005))
Suppose $\sigma^2 := \sum_{k=2}^{\infty} (k - 1)^2 p_k < \infty$. Then

$$\frac{1}{\sqrt{n}} T^G_{n} \xrightarrow{d} c_\sigma \mathcal{T}_2 \quad \text{as } n \to \infty$$

where $\mathcal{T}_2$ is Aldous’ Brownian continuum random tree and $c_\sigma$ is a non-negative constant. (The convergence is in the sense of the Gromov–Hausdorff distance.)
Infinite variance

What if the offspring distribution does not have finite variance? It is natural to consider offspring distributions such that $p_k \sim k^{-1-\alpha}$ for $\alpha \in (1,2)$ (or, more generally, distributions in the domain of attraction of a stable law of parameter $\alpha$).
The infinite-variance case

Theorem (Duquesne & Le Gall (2002); Duquesne (2003))

Suppose that \((p_k)_{k \geq 0}\) lies in the domain of attraction of a stable law of index \(\alpha \in (1, 2)\). Then as \(n \to \infty\),

\[
\frac{1}{n^{1-1/\alpha}} T_n^{GW} \overset{d}{\to} c_\alpha \mathcal{T}_\alpha,
\]

where \(\mathcal{T}_\alpha\) is the stable tree of parameter \(\alpha\) and \(c_\alpha\) is a non-negative constant. (The convergence is in the sense of the Gromov–Hausdorff distance.)
The stable trees

[Pictures by Igor Kortchemski]
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An important difference between the stable trees for $\alpha \in (1, 2)$ and the Brownian CRT is that the Brownian CRT is binary. The stable trees, on the other hand, have only branch-points of infinite degree.
The principal theme of the rest of this talk is how to give a (relatively) simple description of the stable trees (and how to use it to get at their distributional properties).
A uniform measure

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For $\alpha \in (1, 2]$, the stable tree $T_\alpha$ is naturally endowed with a “uniform” probability measure $\mu_\alpha$, which is the limit of the discrete uniform measure on $T_{n}^{\text{GW}}$. It turns out that $\mu_\alpha$ is supported by the set of leaves of $T_\alpha$. 
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Aldous’ theory of continuum random trees tells us that we can characterize the laws of such trees via sampling.
Reduced trees

Let $X_1, X_2, \ldots$ be leaves sampled independently from $\mathcal{T}_\alpha$ according to $\mu_\alpha$, and let $\mathcal{T}_{\alpha,n}$ be the subtree spanned by the root $\rho$ and $X_1, \ldots, X_n$: 
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Characterising the law of a stable tree

$T_{\alpha,n}$ can be thought of in two parts: its tree-shape $T_{\alpha,n}$ (a rooted unordered tree with $n$ labelled leaves) and its edge-lengths.
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The laws of $(\mathcal{T}_{\alpha,n}, n \geq 1)$ (the random finite-dimensional distributions) are sufficient to fully specify the law of $\mathcal{T}_{\alpha}$. 
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The laws of $(\mathcal{T}_{\alpha,n}, n \geq 1)$ (the random finite-dimensional distributions) are sufficient to fully specify the law of $\mathcal{T}_{\alpha}$.

Moreover,

$$\mathcal{T}_{\alpha} = \bigcup_{n \geq 1} \mathcal{T}_{\alpha,n}.$$
Reminder: Aldous’ line-breaking construction of the Brownian CRT

Let $C_1, C_2, \ldots$ be the points of an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity $t \, dt$. 

\[0 \quad C_1 \quad C_2 C_3 C_4 \quad C_5 C_6 \ldots\]
Line-breaking construction

$\tilde{T}_1$
Line-breaking construction
\[ \tilde{T}_2 \]
Line-breaking construction

$\tilde{T}_3$
Line-breaking construction

\[ \widetilde{T}_4 \]
Line-breaking construction

\[ \widetilde{T}_5 \]
Line-breaking construction

\( \tilde{T}_6 \)
It turns out that the line-breaking construction precisely gives the random finite-dimensional distributions for the Brownian CRT, i.e.

\[(\tilde{T}_n, n \geq 1) \overset{d}{=} \left( \frac{1}{\sqrt{2}} \mathcal{T}_{2,n}, n \geq 1 \right)\].

Question: does there exist a similar line-breaking construction for the stable trees with \(\alpha \in (1, 2)\)?
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**Question:** does there exist a similar line-breaking construction for the stable trees with \(\alpha \in (1, 2)\)?
Marchal’s algorithm

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- Start from a single edge, rooted at one end-point and with the other end-point labelled 1.
- At all subsequent steps, assign edges weight $\alpha - 1$ and vertices of degree $d \geq 3$ weight $d - 1 - \alpha$. 

▶ At step $n$, pick an edge or a vertex with probability proportional to their weights.
▶ If we pick an edge, subdivide it into two edges and attach the leaf labelled $n$ to the middle vertex we just created.
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\[ \alpha - 1 \]
Marchal’s algorithm
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\[\alpha - 1, \rho, 3 - \alpha, \alpha - 1, 2, 3, \alpha - 1, 3\]
Marchal’s algorithm
Marchal’s algorithm
Marchal’s algorithm
Marchal’s algorithm

ρ

1

6

5

2

3

4

7
Marchal’s algorithm

Then

\[(\tilde{T}_n, n \geq 1) \overset{d}{=} (T_{\alpha,n}, n \geq 1).\]

(The \(\alpha = 2\) case is Rémy’s algorithm (1985) for building a uniform binary rooted tree with \(n\) labelled leaves.)
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Moreover,

\[\frac{1}{n^{1-1/\alpha}} \tilde{T}_n \overset{a.s.}{\Rightarrow} c'_\alpha T_\alpha\]

as \(n \to \infty\) [Curien-Haas (2013)].
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Moreover,
\[
\frac{1}{n^{1-1/\alpha}} \tilde{T}_n \overset{a.s.}{\rightarrow} c'_{\alpha} \mathcal{T}_\alpha
\]
as \(n \to \infty\) [Curien-Haas (2013)].

Our new line-breaking construction gives a nested sequence of continuous trees which converge a.s. to \(\mathcal{T}_\alpha\) without any need for rescaling.
The generalized Mittag-Leffler distribution

For $\beta \in (0, 1)$, let $\sigma_\beta$ be a stable random variable with Laplace transform

$$E\left[\exp(-\lambda \sigma_\beta)\right] = \exp(-\lambda^\beta), \quad \lambda \geq 0.$$
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Say that a non-negative random variable $M$ has the generalized Mittag-Leffler distribution with parameters $\beta \in (0, 1)$ and $\theta > -\beta$, and write $M \sim \text{ML}(\beta, \theta)$, if

$$\mathbb{E} [f(M)] = C_{\beta, \theta} \mathbb{E} \left[ \sigma_\beta^{-\theta} f \left( \sigma_\beta^{-\beta} \right) \right].$$

for all suitable test-functions $f$. 

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for all suitable test-functions $f$. The law of $M$ is characterized by its moments:

$$\mathbb{E} \left[ M^k \right] = \frac{\Gamma(\theta)\Gamma(\theta/\beta + k)}{\Gamma(\theta/\beta)\Gamma(\theta + k\beta)}$$

for any $k \geq 1$. 
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Say that a non-negative random variable $M$ has the generalized Mittag-Leffler distribution with parameters $\beta \in (0, 1)$ and $\theta > -\beta$, and write $M \sim ML(\beta, \theta)$, if

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\[ \mathbb{E} \left[ M^k \right] = \frac{\Gamma(\theta) \Gamma(\theta/\beta + k)}{\Gamma(\theta/\beta) \Gamma(\theta + k\beta)} \]

for any $k \geq 1$.

If $\beta = 1/2$ and $n \geq 1$, $ML(1/2, n - 1/2) = 2\sqrt{\text{Gamma}(n, 1)}$. 
A generalized Pólya urn scheme

ML(β, θ) arises as an almost sure limit in the context of a generalized Pólya urn scheme.
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Start with weight 0 on black and weight \(\theta/\beta\) on red.

Pick a colour with probability proportional to its weight in the urn.

- If black is picked, add \(1/\beta\) to the black weight.
- If red is picked, add \(1 - 1/\beta\) to the black weight and 1 to the red weight.

Let \(R_n\) be the weight of red at step \(n\). Then [Janson (2006)],

\[
\lim_{n \to \infty} n^{-\beta} R_n \overset{\text{a.s.}}{\to} W \sim \text{ML}(\beta, \theta).
\]
Urns in Marchal’s algorithm

Idea: there are many such urns embedded in Marchal’s algorithm!
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Consider the distance $D_n$ between the root and the leaf labelled 1. The associated weight is $(\alpha - 1)D_n$. Let $W_n$ be the remaining weight in the rest of the tree.

$D_1 = 1$ and $W_1 = 0$. 
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At each subsequent step, 

$D_{n+1} = D_n + 1$, the associated weight increases by $\alpha - 1$, and $W_{n+1} = W_n + (2 - \alpha) + (\alpha - 1) = W_n + 1$;

with probability proportional to $W_n$ add the new edge elsewhere; this yields $W_{n+1} = W_n + \alpha$.

(We always add weight $\alpha$ to the whole tree.)
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$D_1 = 1$ and $W_1 = 0$.

At each subsequent step,

- with probability proportional to $(\alpha - 1)D_n$, we pick one of the $D_n$ edges between the root and 1 to split. Then, $D_{n+1} = D_n + 1$, the associated weight increases by $\alpha - 1$, and $W_{n+1} = W_n + (2 - \alpha) + (\alpha - 1) = W_n + 1$;

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At each subsequent step,

- with probability proportional to $(\alpha - 1)D_n$, we pick one of the $D_n$ edges between the root and 1 to split. Then, $D_{n+1} = D_n + 1$, the associated weight increases by $\alpha - 1$, and $W_{n+1} = W_n + (2 - \alpha) + (\alpha - 1) = W_n + 1$;
- with probability proportional to $W_n$ add the new edge elsewhere; this yields $W_{n+1} = W_n + \alpha$.

(We always add weight $\alpha$ to the whole tree.)
Then \((D_n, n \geq 1)\) behaves exactly as the red weight in the generalized Pólya urn with \(\beta = \theta = 1 - 1/\alpha\). It follows that

\[
\frac{1}{n^{1-1/\alpha}} D_n \xrightarrow{d} \mathcal{ML}(1 - 1/\alpha, 1 - 1/\alpha)
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as \(n \to \infty\).
Urns in Marchal’s algorithm

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This suggests that the first stick in any line-breaking construction should have length distributed as \(\text{ML}(1 - 1/\alpha, 1 - 1/\alpha)\).
A Markov chain

We define an increasing \( \mathbb{R}_+ \)-valued process which will play a role similar to that of the inhomogeneous Poisson process in the Brownian case.
A Markov chain

We define an increasing $\mathbb{R}_+\text{-valued}$ process which will play a role similar to that of the inhomogeneous Poisson process in the Brownian case.

Let $(M_n, n \geq 1)$ be a Markov chain such that

1. $M_n \sim \text{ML}(1 - 1/\alpha, n - 1/\alpha)$ for $n \geq 1$.
2. The backward transition from $M_{n+1}$ to $M_n$ is given by

$$M_n = M_{n+1} \beta_n,$$

where $\beta_n$ is independent of $M_{n+1}$ and

$$\beta_n \sim \text{Beta}\left(\frac{(n+1)\alpha - 2}{\alpha - 1}, \frac{1}{\alpha - 1}\right).$$
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Lemma

If $\alpha = 2$, $(M_n, n \geq 1)$ are the ordered points of an inhomogeneous Poisson process on $\mathbb{R}_+$ with intensity $\frac{t}{2} dt$. 
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Sketch proof.

It suffices to show that $(M_n^2/4, n \geq 1)$ are the ordered points of a Poisson process of rate 1. But

$M_n \sim ML(1/2, n - 1/2) = 2 \sqrt{\text{Gamma}(n, 1)}$ and so

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Lemma

If $\alpha = 2$, $(M_n, n \geq 1)$ are the ordered points of an inhomogeneous Poisson process on $\mathbb{R}_+$ with intensity $\frac{t}{2} \, dt$.

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The relationship between successive points encoded in $M_n = \beta_n M_{n+1}$ where $\beta_n \sim \text{Beta}(2n, 1)$ gives exactly the right dependence structure.
Line-breaking construction of the stable tree (I)

Start with $M_1$ and set $L_1 = M_1$. Let $\tilde{T}_1$ be the tree consisting of a line-segment of length $L_1$.

For $n \geq 1$, given $\tilde{T}_n$ (which has total length $L_n$):

1. Let $B_{n+1} \sim \text{Beta}(1, 2 - \alpha)$ be independent of everything we have already constructed. We will glue a new branch of length $(M_n + 1 - M_n) \cdot B_{n+1}$ onto $\tilde{T}_n$, at a point to be specified; let $L_{n+1} = L_n + (M_n + 1 - M_n) \cdot B_{n+1}$ be the new total length.

2. In order to find where to glue the new branch, we first select either the set of edges of $\tilde{T}_n$, with probability $L_n / M_n$, or the set of branchpoints of $\tilde{T}_n$, with probability $1 - L_n / M_n$.

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Theorem (Haas & G.)

Let \((\tilde{T}_n, n \geq 1)\) be the sequence of trees produced by either version of the construction. Then

\[(\tilde{T}_n, n \geq 1) \overset{d}{=} (\mathcal{T}_\alpha,n, n \geq 1)\]

and, therefore,

\[\mathcal{T}_\alpha \overset{d}{=} \bigcup_{n \geq 1} \tilde{T}_n.\]
In the case $\alpha = 2$, we have $\text{Beta} \left( 1, \frac{2-\alpha}{\alpha-1} \right) = \text{Beta}(1, 0)$. We interpret this as $B_n = 1$ almost surely for all $n \geq 1$. Then we recover (a scaled version of) Aldous’ Poisson line-breaking construction of the Brownian CRT.
Remarks

In the case $\alpha = 2$, we have $\text{Beta}\left(1, \frac{2-\alpha}{\alpha-1}\right) = \text{Beta}(1, 0)$. We interpret this as $B_n = 1$ almost surely for all $n \geq 1$. Then we recover (a scaled version of) Aldous’ Poisson line-breaking construction of the Brownian CRT.

The tree-shapes $(\tilde{T}_n, n \geq 1)$ of $(\tilde{T}_n, n \geq 1)$ perform Marchal’s algorithm.
Consequences: distributional results for \((T_{\alpha,n}, n \geq 1)\)

**Edge-lengths:**
Let \(t\) be a discrete rooted tree with \(n \geq 2\) leaves and \(k\) edges. Then conditionally on \(T_{\alpha,n} = t\), the sequence of edge-lengths of \(T_{\alpha,n}\) has the same distribution as

\[
M_n \cdot \beta_k \cdot (D_1, D_2, \ldots, D_k),
\]

where these random variables are independent and

\[
M_n \sim \text{ML}(1 - 1/\alpha, n - 1/\alpha)
\]

\[
\beta_k \sim \text{Beta} \left( k, \frac{n\alpha - 1}{\alpha - 1} \right)
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*Dirichlet distribution: Dir(a_1, \ldots, a_n) has density

\[
\frac{\Gamma(a_1 + \ldots + a_n)}{\prod_{i=1}^{n} \Gamma(a_i)} x_1^{a_1-1} \ldots x_n^{a_n-1}
\]

with respect to Lebesgue measure on

\[
\{(x_1, \ldots, x_n) \in [0, 1]^n : \sum_{i=1}^{n} x_i = 1\}.
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**Total length of the conditioned tree:**
Conditionally on \(T_{\alpha,n}\) having \(k\) edges, the total length of the tree \(T_{\alpha,n}\) has the same distribution as

\[ M_n \cdot \beta_k, \]

where these random variables are independent and \(M_n \sim \text{ML}(1 - 1/\alpha, n - 1/\alpha)\) and \(\beta_k \sim \text{Beta}(k, \frac{n\alpha - 1}{\alpha - 1})\).
Consequences: distributional results for \( (\mathcal{T}_{\alpha,n}, n \geq 1) \)

**Total length of the unconditioned tree:**
The total length of the tree \( \mathcal{T}_{\alpha,n} \) has the same distribution as

\[
M_n \cdot \left( \prod_{j=1}^{n-1} \beta_j + \sum_{i=1}^{n-1} B_i(1 - \beta_i) \prod_{j=i+1}^{n-1} \beta_j \right),
\]

where the random variables on the right-hand side are mutually independent and such that

\[
M_n \sim \text{ML}(1 - 1/\alpha, n - 1/\alpha)
\]

\[
\beta_i \sim \text{Beta}\left(\frac{(i + 1)\alpha - 2}{\alpha - 1}, \frac{1}{\alpha - 1}\right), \quad i \geq 1
\]

\[
B_1, B_2, \ldots, B_n \sim \text{Beta}\left(1, \frac{2 - \alpha}{\alpha - 1}\right).
\]
Open problem

Does there exist a discrete version of our line-breaking construction (à la Aldous’ construction of the uniform random tree)?
A line-breaking construction of the stable trees, joint with Bénédicte Haas,
Beta-Gamma algebra

The proof relies heavily on the following distributional facts.

- If $B \sim \text{Beta}(a, b)$ and $G \sim \text{Gamma}(a + b, 1)$ are independent then

  $$G \times (B, 1 - B) \overset{d}{=} (G_1, G_2),$$

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  \[ \left( \frac{G_1}{G_1 + G_2}, \frac{G_2}{G_1 + G_2} \right) \overset{d}{=} (B, 1 - B) \]
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and is independent of \( G_1 + G_2 \sim \text{Gamma}(a + b, 1) \).

- Let \( \mathbf{D} = (D_1, D_2, \ldots, D_n) \sim \text{Dir}(a_1, a_2, \ldots, a_n) \) and \( \mathbb{P}(I = i|\mathbf{D}) = D_i \). Then, conditionally on the event \( \{I = i\} \), we have

\[
(D_1, \ldots, D_i, \ldots, D_n) \sim \text{Dir}(a_1, \ldots, a_i + 1, \ldots, a_n).
\]
An idea of the proof (of version (II))

The key point is that, conditionally on the shapes $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n$ (with $\tilde{T}_n$ having $k$ edges and $\ell$ internal vertices), the edge-lengths and vertex weights are such that

$$(L_1^{(n)}, \ldots, L_k^{(n)}, W_1^{(n)}, \ldots, W_\ell^{(n)})$$

$$\overset{d}{=} \text{ML}(1 - 1/\alpha, n - 1/\alpha) \times \text{Dir} \left( 1, \ldots, 1, \frac{d_1 - 1 - \alpha}{\alpha - 1}, \ldots, \frac{d_\ell - 1 - \alpha}{\alpha - 1} \right)$$

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This can be proved inductively.
An idea of the proof (of version (II))

\[(L_1^{(n)}, \ldots, L_k^{(n)}, W_1^{(n)}, \ldots, W_{\ell}^{(n)})\]

\[\overset{d}{=} \text{ML}(1 - 1/\alpha, n - 1/\alpha) \times \text{Dir} \left(1, \ldots, 1, \frac{d_1 - 1 - \alpha}{\alpha - 1}, \ldots, \frac{d_{\ell} - 1 - \alpha}{\alpha - 1}\right)\]

Recall that we add our new branch either at a node or somewhere uniformly chosen along the edges. So we pick an edge or a vertex with probability proportional to its weight.
An idea of the proof (of version (II))

\((L_1^{(n)}, \ldots, L_k^{(n)}, W_1^{(n)}, \ldots, W_\ell^{(n)})\)

\[\overset{d}{=} \text{ML}(1 - 1/\alpha, n - 1/\alpha) \times \text{Dir}\left(1, \ldots, 1, \frac{d_1 - 1 - \alpha}{\alpha - 1}, \ldots, \frac{d_\ell - 1 - \alpha}{\alpha - 1}\right)\]

Recall that we add our new branch either at a node or somewhere uniformly chosen along the edges. So we pick an edge or a vertex with probability proportional to its weight.

This amounts to taking a size-biased pick from amongst the co-ordinates of the Dirichlet vector, and has the effect of adding 1 to the corresponding parameter.
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This amounts to taking a size-biased pick from amongst the co-ordinates of the Dirichlet vector, and has the effect of adding 1 to the corresponding parameter.

If we pick a co-ordinate which corresponded to an edge, it now has parameter 2. Splitting that co-ordinate with an independent uniform gives back 2 co-ordinates with parameter 1.
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Whether we picked an edge or a vertex, we now want to add one co-ordinate equal to 1 (representing the new edge) and either a co-ordinate equal to \(\frac{2-\alpha}{\alpha-1}\) (for a new vertex) or an additional weight to the existing vertex whose weight we already biased:

\[
\frac{d-1-\alpha}{\alpha-1} + 1 + \frac{2-\alpha}{\alpha-1} = \frac{(d+1)-1-\alpha}{\alpha-1}.
\]
An idea of the proof (of version (II))

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\[\overset{d}{=} \text{ML}(1 - 1/\alpha, n - 1/\alpha) \times \text{Dir} \left( 1, \ldots, 1, \frac{d_1 - 1 - \alpha}{\alpha - 1}, \ldots, \frac{d_\ell - 1 - \alpha}{\alpha - 1} \right)\]

Whether we picked an edge or a vertex, we now want to add one co-ordinate equal to 1 (representing the new edge) and either a co-ordinate equal to \(\frac{2 - \alpha}{\alpha - 1}\) (for a new vertex) or an additional weight to the existing vertex whose weight we already biased:
\[\frac{d - 1 - \alpha}{\alpha - 1} + 1 + \frac{2 - \alpha}{\alpha - 1} = \frac{(d+1) - 1 - \alpha}{\alpha - 1}.
\]

This is the role of \((B_n, 1 - B_n) \sim \text{Beta}(1, \frac{2-\alpha}{\alpha-1})\).
An idea of the proof (of version (II))

\((L_1^{(n)}, \ldots, L_k^{(n)}, W_1^{(n)}, \ldots, W_\ell^{(n)})\)

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Recall that

\[ M_n = M_{n+1} \beta_n. \]
An idea of the proof (of version (II))

\[ (L_1^{(n)}, \ldots, L_k^{(n)}, W_1^{(n)}, \ldots, W_\ell^{(n)}) \]

\[ \overset{d}{=} \text{ML}(1 - 1/\alpha, n - 1/\alpha) \times \text{Dir} \left( 1, \ldots, 1, \frac{d_1 - 1 - \alpha}{\alpha - 1}, \ldots, \frac{d_\ell - 1 - \alpha}{\alpha - 1} \right) \]

Recall that

\[ M_n = M_{n+1} \beta_n. \]

The \( \beta_n \) factor is precisely what is needed to rescale the Dirichlet vector in order to accommodate the extra co-ordinates we added.
A line-breaking construction of the stable trees, joint with Bénédicte Haas,