Graphs with degree constraints

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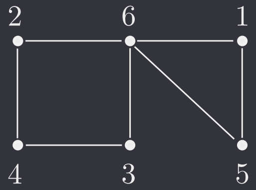
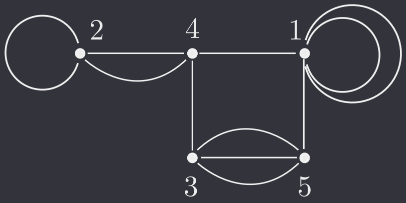
Introduction

$SG_{n,m}^{(D)}$ denotes the set of simple labelled graphs with n vertices, m edges and all degrees in $D \subset \mathbb{Z}_{\geq 0}$.
(Example: D contains the even integers.)



$MG_{n,m}^{(D)}$ multigraphs, *i.e.* loops and multiedges allowed.



Introduction

Examples

- regular graphs $D = \{d\}$ (Bender Canfield 1978),
- minimum degree constraint $D = \mathbb{Z}_{\geq \delta}$ (Pittel Wormald 2003),
- Euler graphs $D = \{2n \,|n \geq 0\}$ (Read 1962, Robinson 1969).

Motivations

- expand the analytic combinatorics of graphs,
- asymptotics of connected graphs when m is proportional to n (Bender Canfield McKay 1990).

Related works

- configuration model (Wormald 1978, Bollobás 1980),
- graphs with a given degree sequence (Bender Canfield 1978),
- symmetric matrices with constant row sum (Chyzak Mishna Salvy 2005).

Analytic combinatorics

We assume $|D| \geq 2$. The generating function of the set D is

$$\mathrm{Set}_D(z) = \sum_{d \in D} \frac{z^d}{d!}.$$

Radius of convergence 0 *e.g.* cubic multigraphs $\sum_\ell \frac{(6\ell)!}{288^\ell (3\ell)!} \frac{z^{2\ell}}{(2\ell)!}$.

Large Powers Theorem (Flajolet Sedgewick 2009) saddle-point method.
Derives the asymptotics of

$$[z^{2m}] A(z) \, \mathrm{Set}_D(z)^n$$

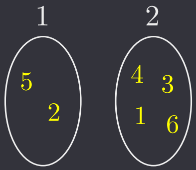
when $\min(D) < \lim \frac{2m}{n} < \max(D)$.

Multigraphs with degree constraints

Random multigraph with $n = 2$ vertices, $m = 3$ edges.

$(2, 1), (2, 2), (1, 2)$

Compensation factor $\kappa(G) = \frac{\text{orderings}(G)}{2^m m!}$, equal to 1 iff G is simple
(Flajolet Knuth Pittel 1989, Janson Knuth Łuczak Pittel 1993).
The total weight of $\mathcal{F}$ is $\sum_{G \in \mathcal{F}} \kappa(G)$.



Ordering on n vertices $\leftrightarrow$ sequence of n labelled sets.

$$\sum_{G \in \mathrm{MG}_{n,m}^{(D)}} \text{orderings}(G) = (2m)!\, [x^{2m}]\, \mathrm{Set}_D(x)^n$$

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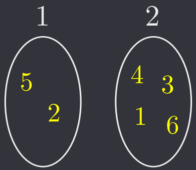

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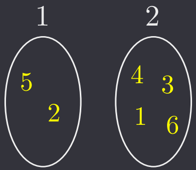

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Inclusion-exclusion

(1) Mark all multiedges and loops

$$\mathrm{MG}^{(D)}_{n,m}(u,v) = \sum_{G \in \mathrm{MG}^{(D)}_{n,m}} \kappa(G) u^{\text{marked multiedges}} v^{\text{marked loops}},$$

$$\mathrm{MG}^{(D)}_{n,m}(0,0) = \sum_{G \in \mathrm{SG}^{(D)}_{n,m}} \kappa(G) = |\,\mathrm{SG}^{(D)}_{n,m}\,|.$$

(2) Mark some multiedges and loops to obtain $\mathrm{MG}^{(D)}_{n,m}(u+1, v+1)$.

$$(\ ,\ ), (\ ,\ ), (\ ,\ ), (\ ,\ ), (\ ,\ )$$

- introduce marked multiedges and loops in the ordering,
- complete with normal edges, so that the ordering is in $\mathrm{MG}^{(D)}_{n,m}$.

Problem the marked edges may intersect in complicated ways.

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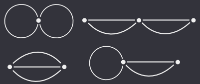
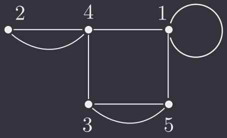
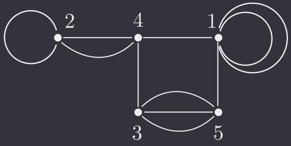
Marked multigraphs

$$\mathrm{MG}^{(D)}_{n,m} = \left\{ \begin{array}{l}\text{Each vertex belongs} \\ \text{to at most one loop} \\ \text{or one double edge.}\end{array} \right\} \uplus \left\{ \text{There exists a “bad” vertex.} \right\}$$







(3) Mark some multiegdes and loops, such that no vertex belongs to two marked edges.

$$\sum_{G \in \mathrm{MG}^{(D)}_{n,m}} \kappa(G)(-1)^{\text{marked multiedges}}(-1)^{\text{marked loops}}$$

$$= |\,\mathrm{SG}^{(D)}_{n,m}\,| + \text{negligible.}$$

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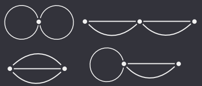

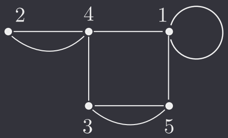

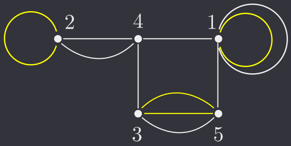

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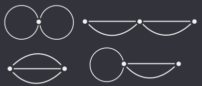
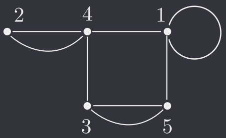
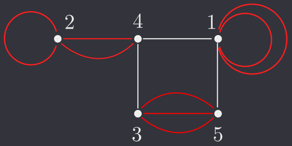
$\mathsf{MG}_{n,m}^{(D)} = \Big\{$ Each vertex belongs to at most one loop or one double edge. $\Big\} \uplus \Big\{$ There exists a "bad" vertex.  $\Big\}$





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$\sum_{k,\ell\geq 0}$ $k = 1, \ell = 1$

$\binom{n}{2k,\ell,n-2k-\ell}$ $\{2,3\},\{1\}$

$\frac{(2k)!}{2^k k!}$ $\{2,3\}$

$(2k)!2^k\ell!$ $\{(3,2),(2,3)\},\{(1,1)\}$

$\binom{m}{2k,\ell,m-2k-\ell}$

$(2m-4k-2\ell)![x^{2m-4k-2\ell}]$
$\mathrm{Set}_{D-2}(x)^{2k+\ell}\,\mathrm{Set}_D(x)^{n-2k-\ell}$ $(2,1),(1,2)$

$(-1)^k(-1)^\ell$

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$(2,1),(3,2),(1,2),(1,1),(2,3)$

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Simple graphs with degree constraints

$$\frac{(2m)!}{2^m m!}[x^{2m}]\left(\sum_{k,\ell\geq 0}\frac{a_{n,2k+\ell}a_{m,2k+\ell}}{a_{2m,4k+2\ell}}\frac{(-W(x)^2)^k}{k!}\frac{(-W(x))^\ell}{\ell!}\right)\mathrm{Set}_D(x)^n,$$

where $a_{n,j}=\frac{n!}{(n-j)!n^j}$ and $W(x)=\frac{n}{4m}\frac{x^2\,\mathrm{Set}_{D-2}(x)}{\mathrm{Set}_D(x)}$.

Result, after the simplification $a_{n,j}\sim 1$,

$$|\,\mathrm{SG}_{n,m}^{(D)}\,|=\frac{(2m)!}{2^m m!}[x^{2m}]e^{-W(x)^2-W(x)}\,\mathrm{Set}_D(x)^n\left(1+O(n^{-1})\right).$$

Application Euler graphs, with $\mathrm{Set}_D(x)=\cosh(x)$

$$\frac{(2m)!}{2^m m!}\frac{2e^{-\left(\frac{n\zeta^2}{4m}\right)^2-\frac{n\zeta^2}{4m}}}{\sqrt{2\pi n\zeta\Phi'(\zeta)}}\frac{\cosh(\zeta)^n}{\zeta^{2m}},$$

where $\Phi(\zeta)=\zeta\tanh(\zeta)=\frac{2m}{n}$.

Connected graphs with large excess

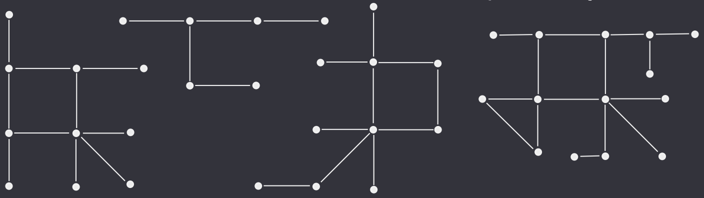
Excess $k = m - n = \text{edges} - \text{vertices}$.

Asymptotics of connected graphs

- when $k = o(n^{1/3})$ (Wright 1980),
- when $k \to \infty$ (Bender Canfield McKay 1990),
  (Pittel Wormald 2005),
  (van der Hofstad Spencer 2006).

Erdős Rényi 1960 When $k \to \infty$, w.h.p. a graph without tree nor unicycle component is connected.

Graphs without trees = graphs with minimum degree ≥ 2
with vertices replaced by rooted trees.



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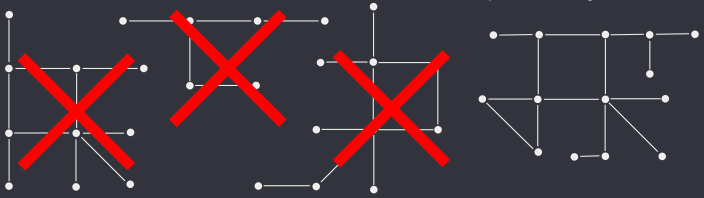

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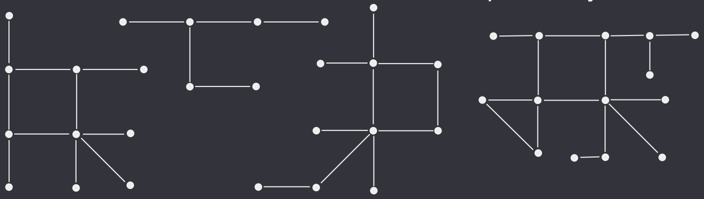

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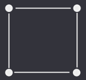

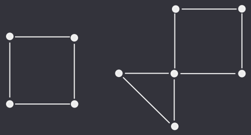

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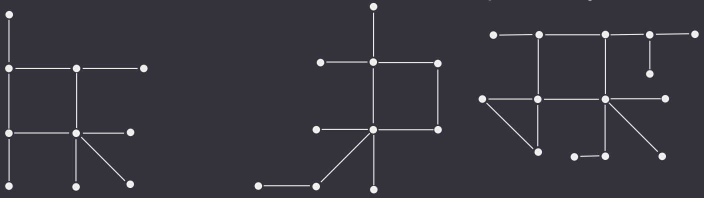

Connected graphs with large excess

GF of simple graphs with minimum deg ≥ 2 and excess k

$$\mathsf{SG}_k^{(\geq 2)}(z) = \sum_{n \geq 0} |\,\mathsf{SG}_{n,m=k+n}^{(\geq 2)}\,| \frac{z^n}{n!} \sim \frac{(2k)!}{2^k k!}[x^{2k}] \frac{e^{-W(x)^2 - W(x)}}{\left(1 - 2\frac{\mathsf{Set}_{\geq 2}(x)}{x^2} z\right)^{k+\frac{1}{2}}}.$$

GF of graphs without trees $\mathsf{SG}_k^{(\geq 2)}(T(z))$.

GF of graphs without trees nor unicycles $\mathsf{SG}_k^{(\geq 2)}(T(z))e^{-V(z)}$, where $V(z)$ is the generating function of unicycles.

Connected graphs with n vertices and excess k, proportionnal to n

$$\sim \frac{n!(2k)!}{2^k k!}[z^n x^{2k}] \frac{e^{-W(x)^2 - W(x)} \sqrt{1 - T(z)}\, e^{\frac{T(z)}{2} + \frac{T(z)^2}{4}}}{\left(1 - 2\frac{e^x - 1 - x}{x^2} T(z)\right)^{k+\frac{1}{2}}}$$

Future extensions

Graphs where each vertex v has a set of allowed degrees D_v,

asymptotics when $m = O(n \log(n))$,

complete asymptotic expansion,

hypergraphs with degree constraints,

structure of random graphs with degree constraints,

structure of random graphs when $\lim \frac{m}{n} > \frac{1}{2}$.