Most trees are short and fat.

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AofA'15
Strobl

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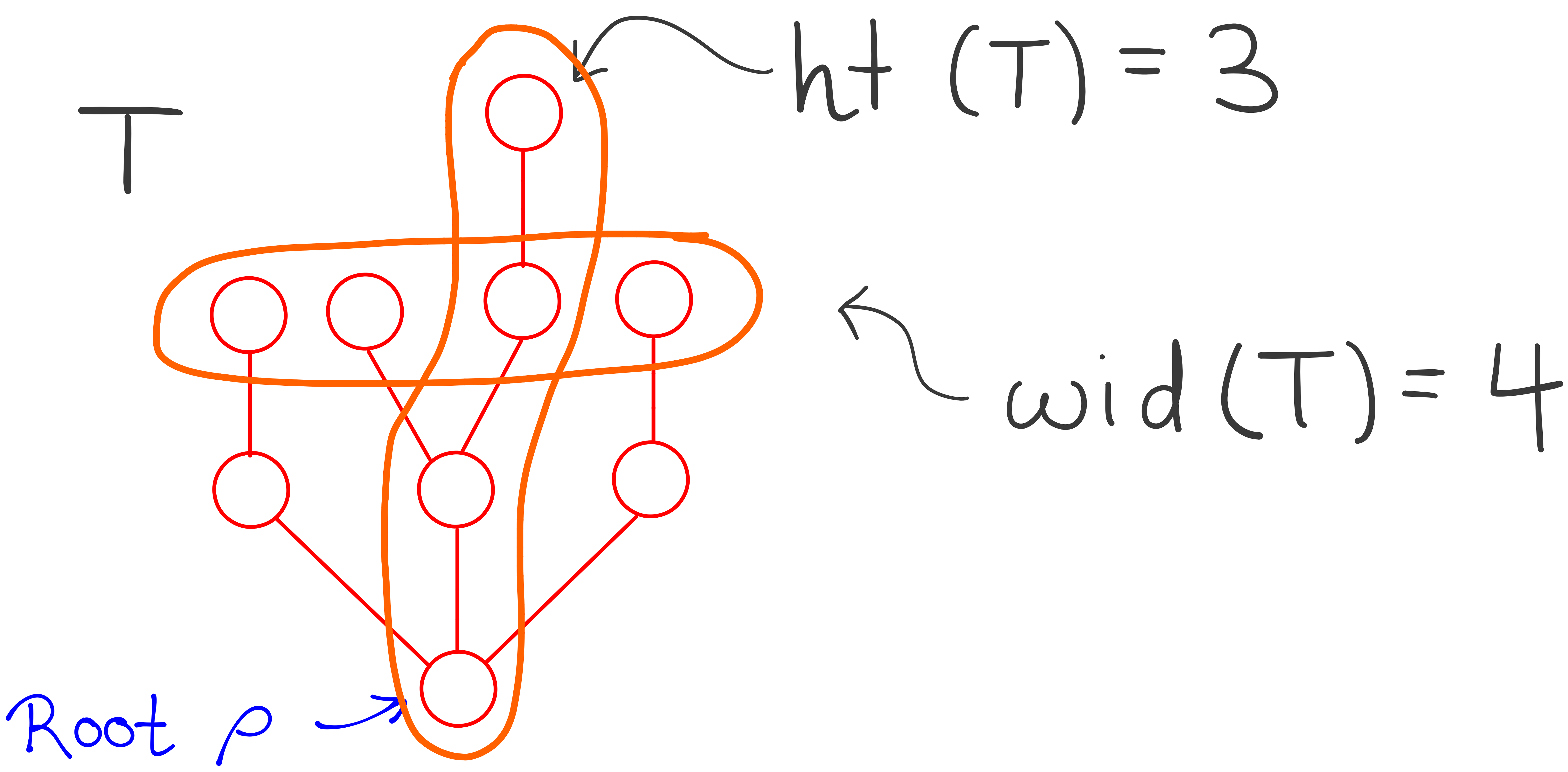
Trees:



Height: Greatest distance from any node to the root } ht(T)

Width: Greatest # nodes on a single level. } wid(T)

Main Results

Fix any r.v. C with $\sum_{k \geq 0} \mathbb{P}(C=k) = 1$, write $p_k = \mathbb{P}(C=k)$.

Let T be $GW(C)$ distributed.

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There is a universal constant $\delta > 0$ s.t.

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Remark: Let $\sigma=\#$ nodes of T. If $\mathbb{E}C>1$ then $\mathbb{P}(\sigma=\infty)>0$, and

$$\mathbb{P}(ht(T)=wid(T)=\infty \mid \sigma=\infty)=1.$$

Also, given that $\sigma<\infty$, the cond. dist. of T is $GW(\hat{C})$ where $\mathbb{P}(\hat{C}=1)=p_1$.

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Heuristic: GW trees satisfy $wid(T)\cdot ht(T)\simeq vol(T):=\sigma$

Implies "$ht > C^2 \cdot wid$" $\hat{=}$ "$ht^2 \geq C^2\cdot vol$"

so $$\mathbb{P}\left(ht(T)>\frac{k}{\sqrt{1-p_1}}\sqrt{vol(T)}\right)\leq \exp(-\delta k^2)$$

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Implies "$ht>C^2 wid$" $\cong$ "$ht^2\geq C^2\cdot vol$" so $\mathbb{P}\left(ht(T)>\frac{k}{\sqrt{1-p_1}}\sqrt{vol(T)}\right)\leq \exp(-\delta k^2)$

Theorem: $$\mathbb{P}\left(ht(T)\geq\frac{k}{\sqrt{1-p_1}}vol(T)^{1/2}\right)\leq \exp\left(-\delta\left(\frac{k}{\log k}\right)^2\right)$$

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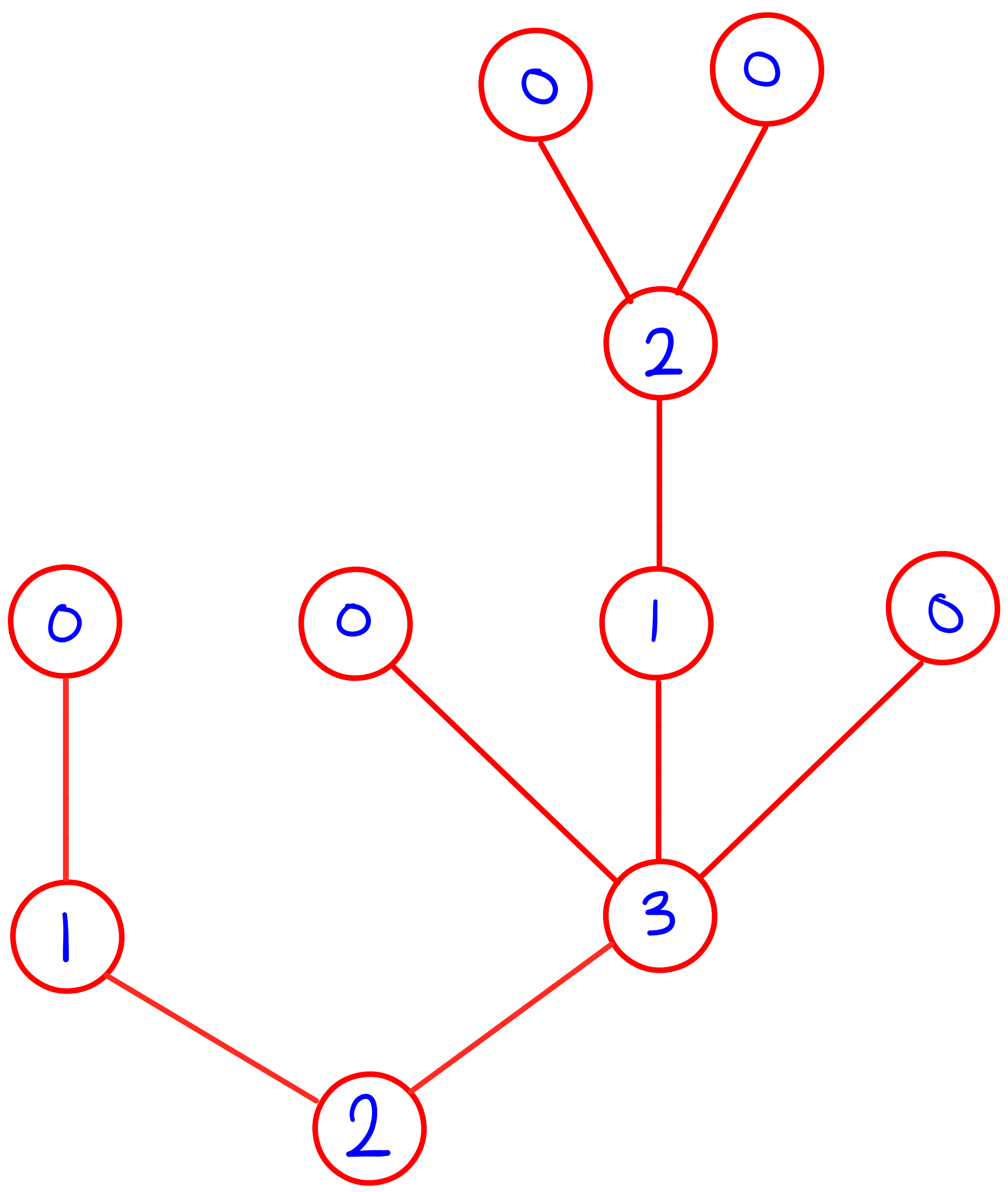
Example: $(2, 1, 3, 0, 0, 1, 0, 2, 0, 0, 1, 4, 0, \ldots)$



Random Plane Trees: Galton-Watson Trees

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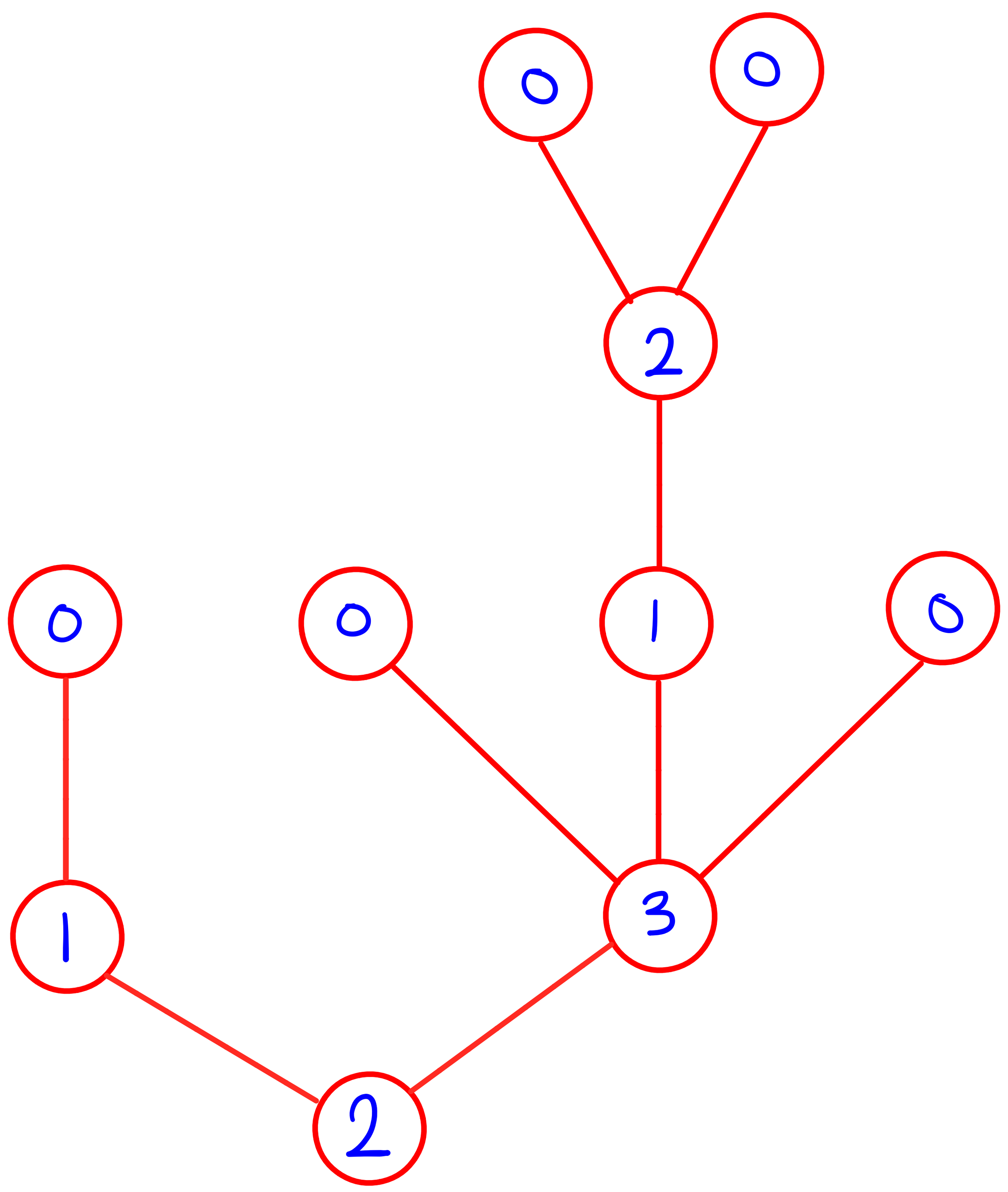

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Trees with n nodes have $n-1$ edges.

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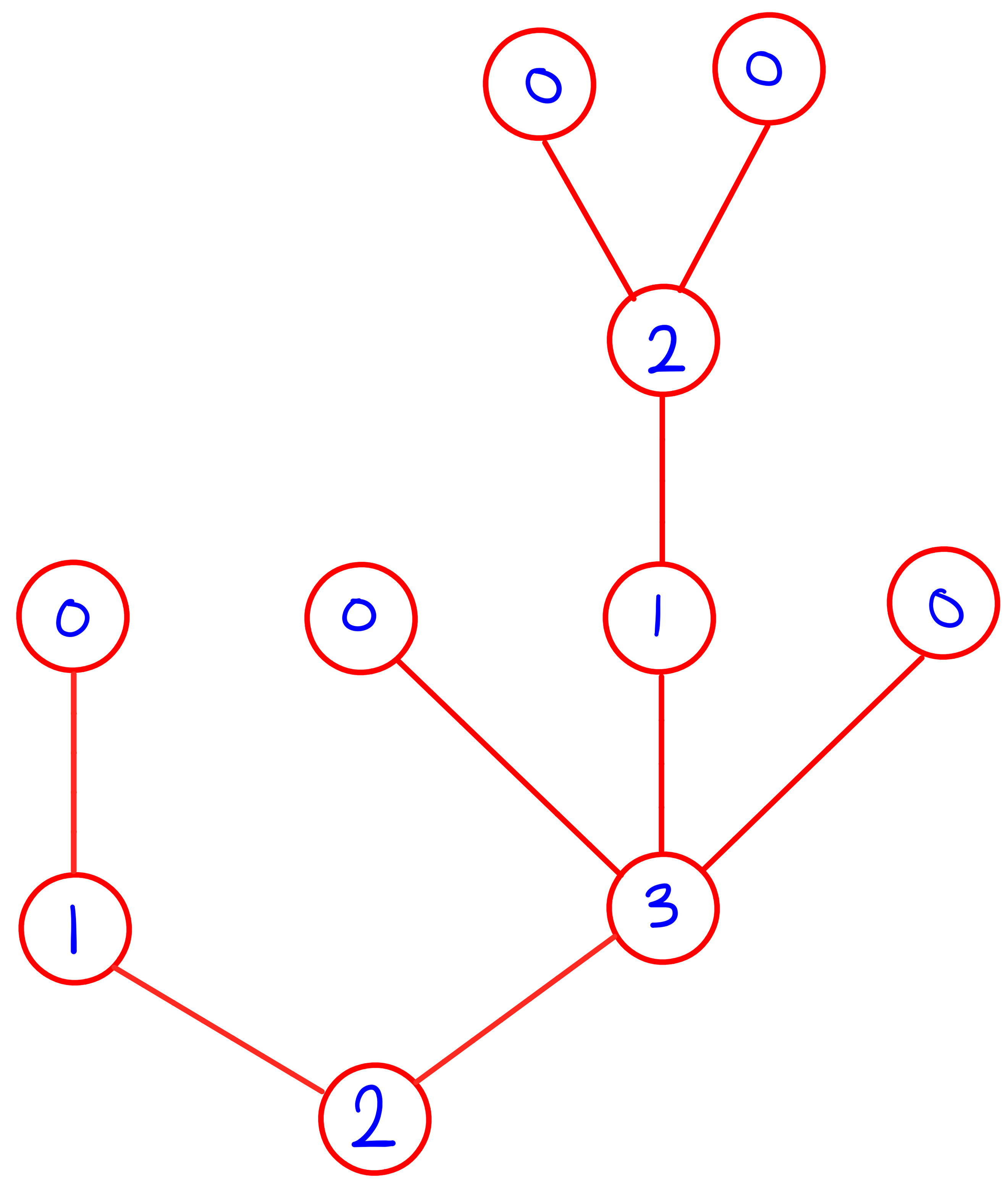

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#nodes discovered by $\underbrace{\text{time } n}_{n \text{ steps of BFS}}$.

$$= 1 + \sum_{i=1}^{n} C_i$$

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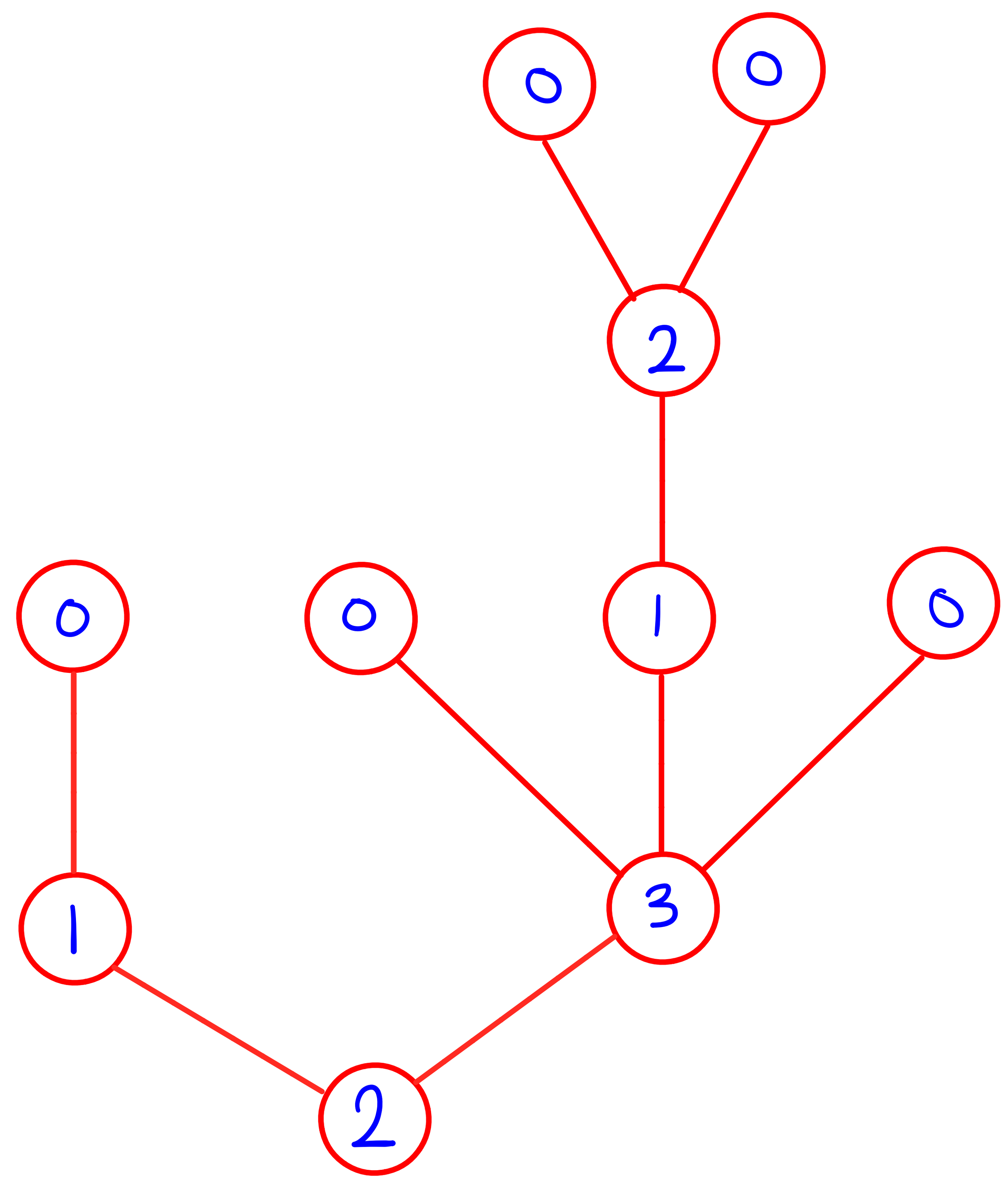

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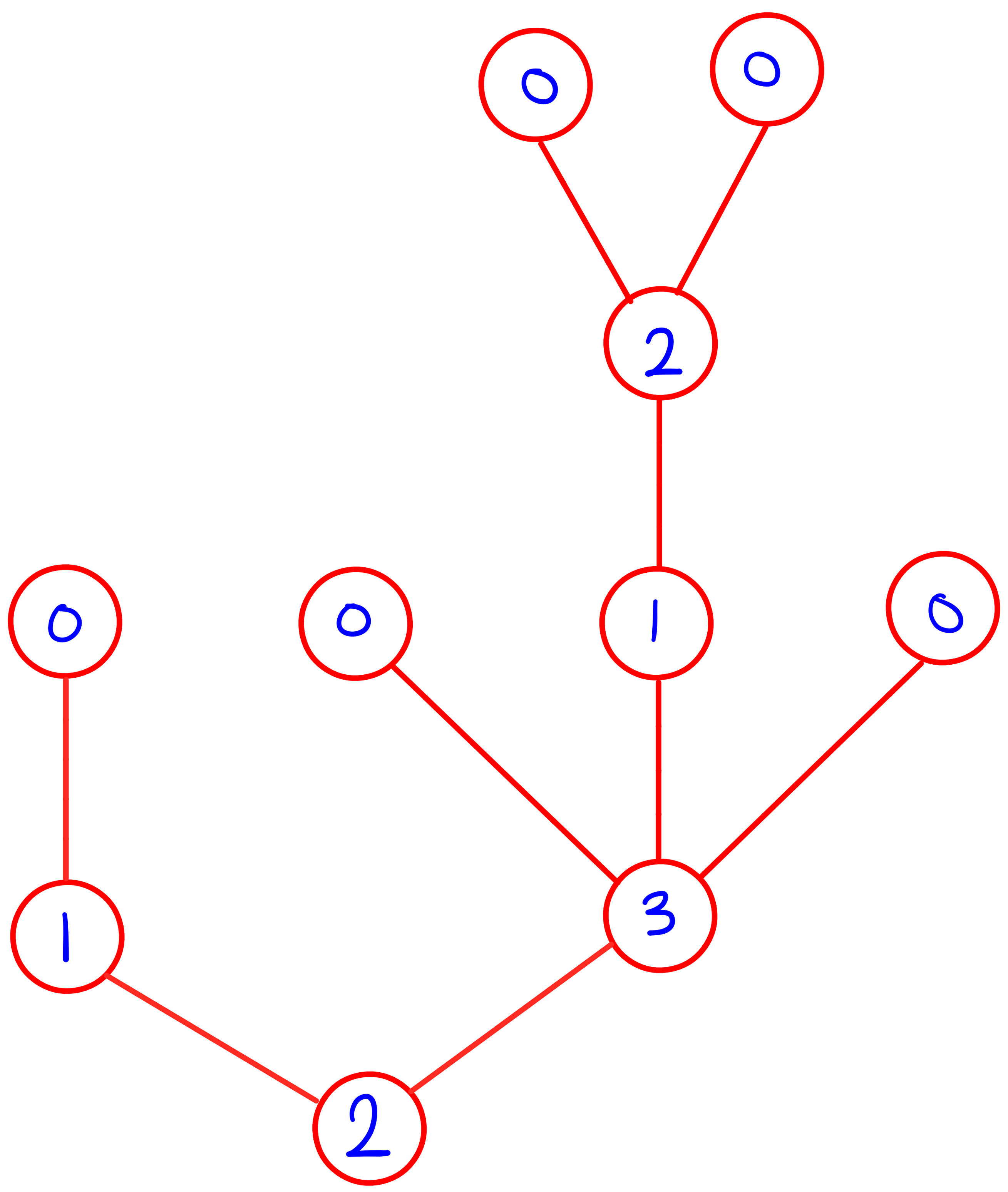

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Size = # vertices $=: \sigma$

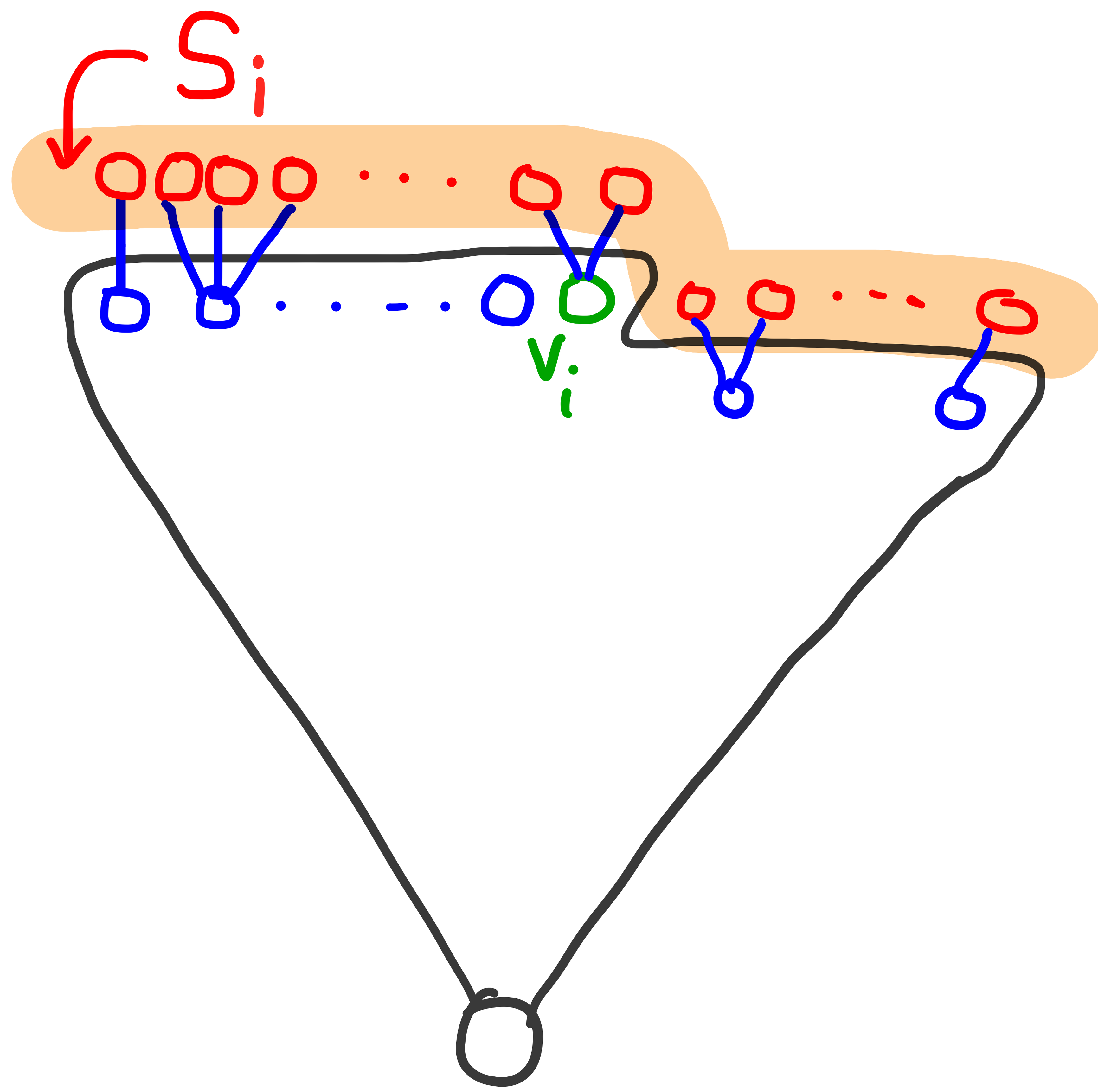
$$= \inf\left\{ n : 1 + \sum_{i=1}^{n} C_i = n \right\}.$$

Setup

$1 + \sum_{j=1}^{i} C_j$ = # nodes discovered by time i.

Let $S_i = 1 + \sum_{j=1}^{i} (C_j - 1)$

= # nodes in "BFS queue" at time i



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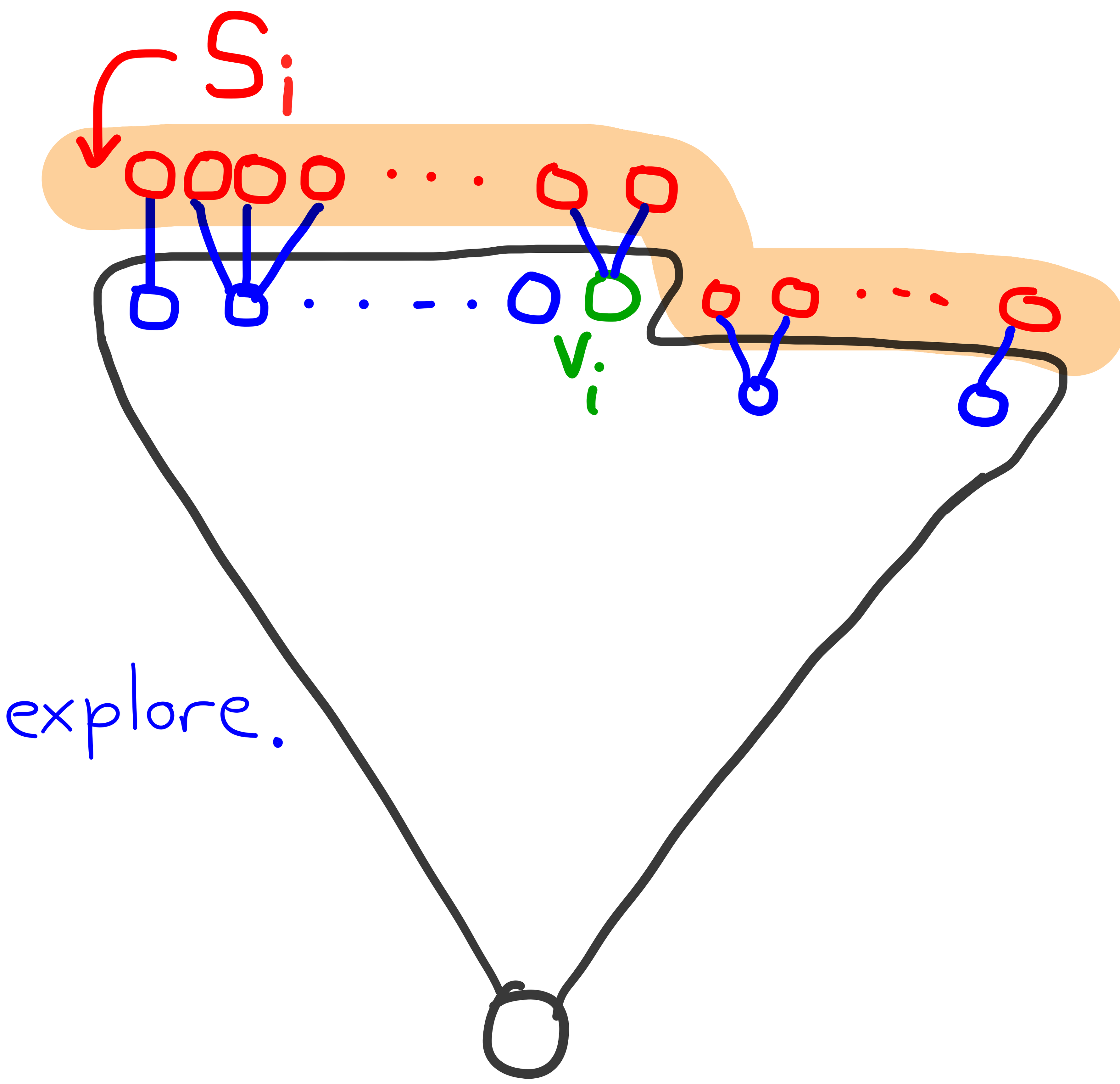

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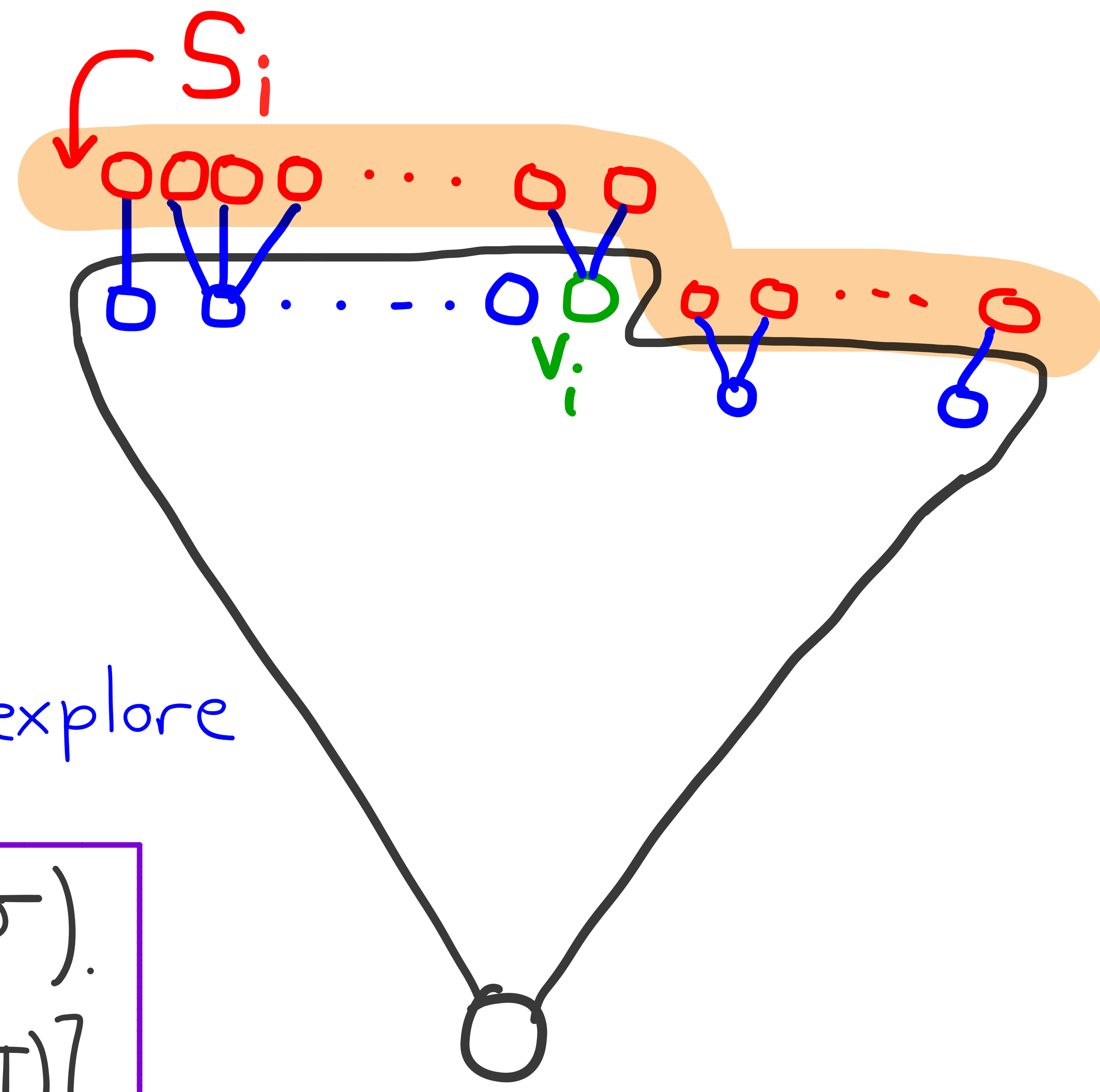

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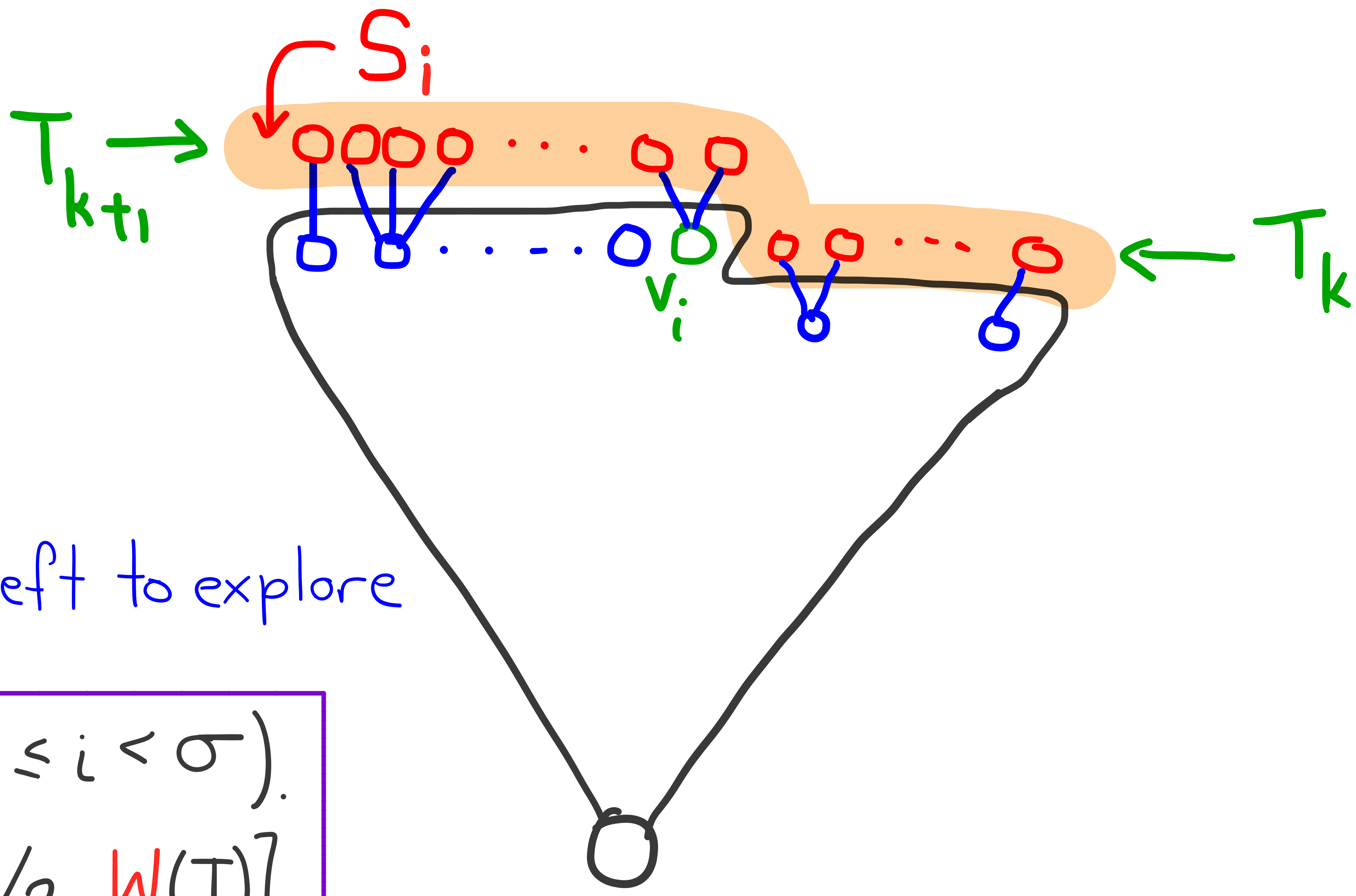

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Proof: During BFS on level k, "exploration queue" $\subset T_k \cup T_{k+1}$; and $= |T_k|$ at start of level k. ∎

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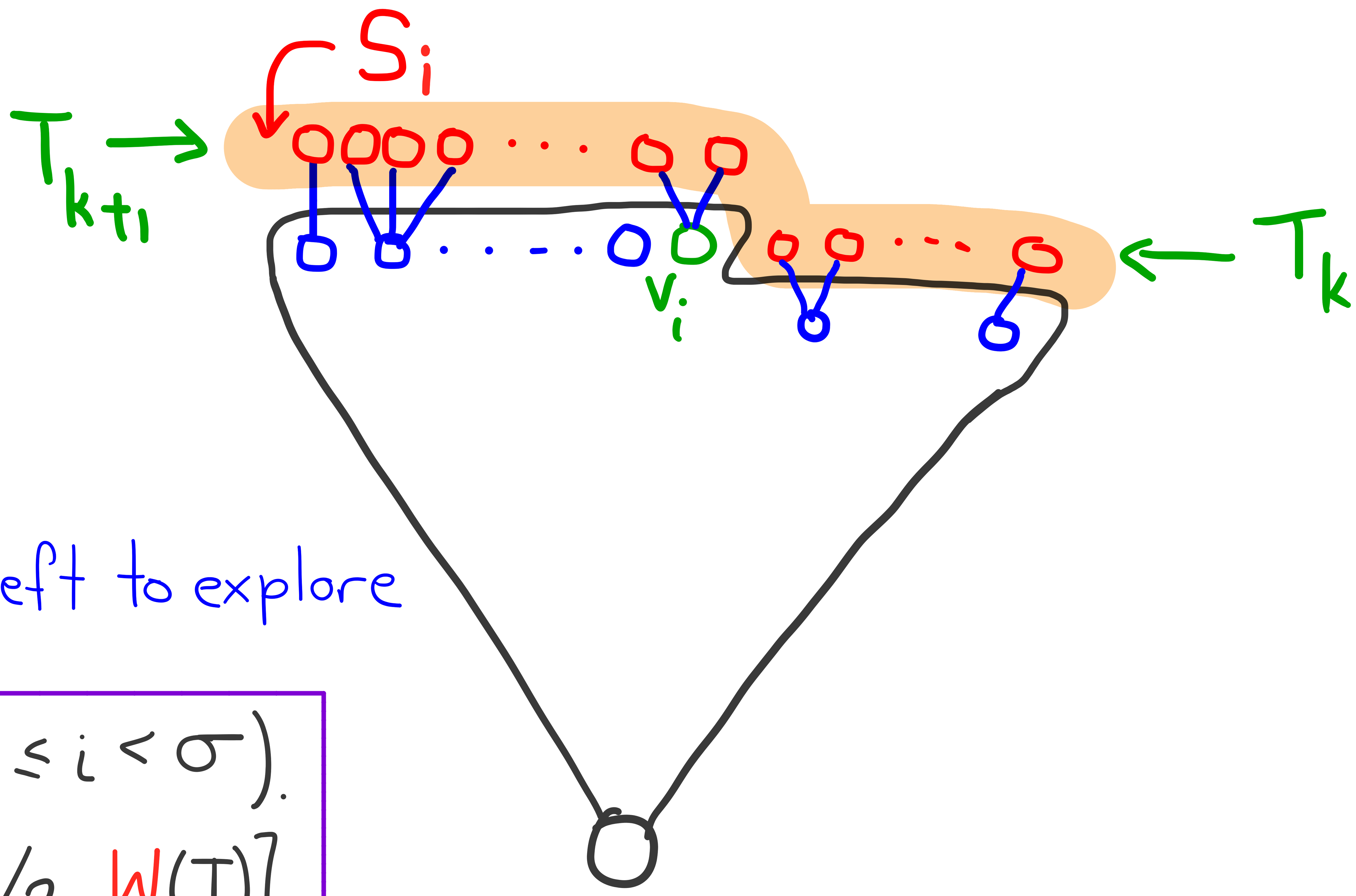

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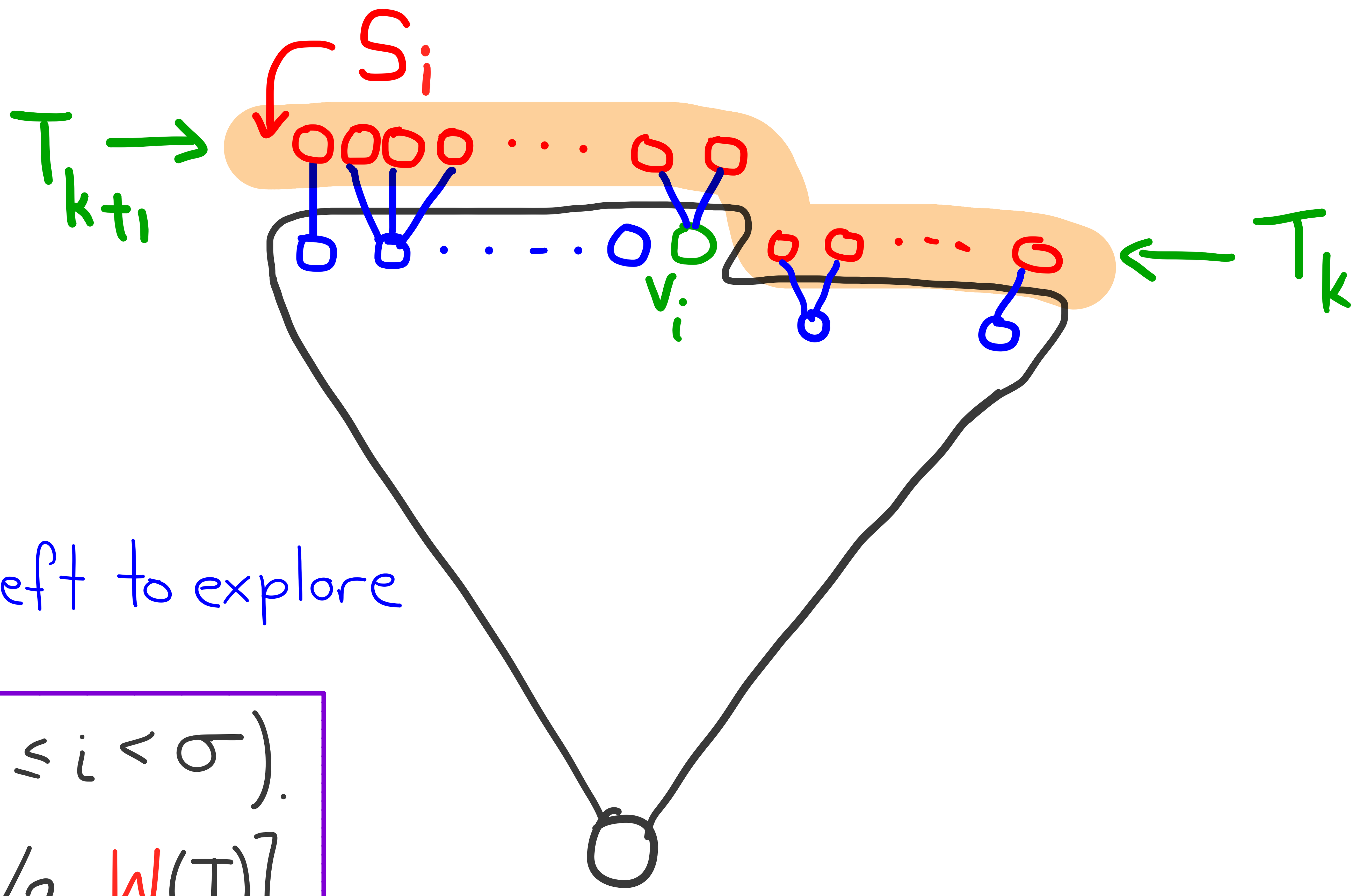

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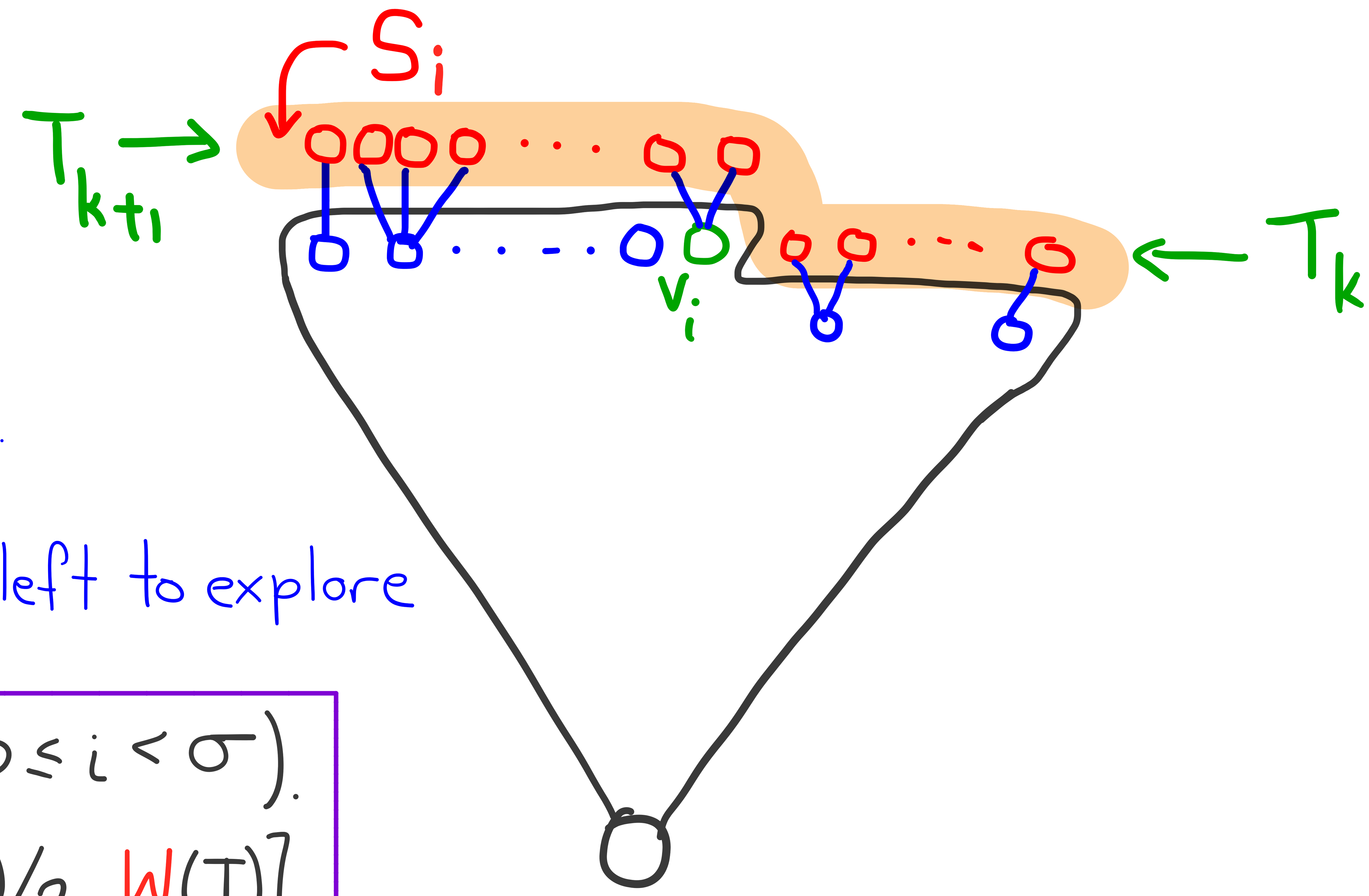

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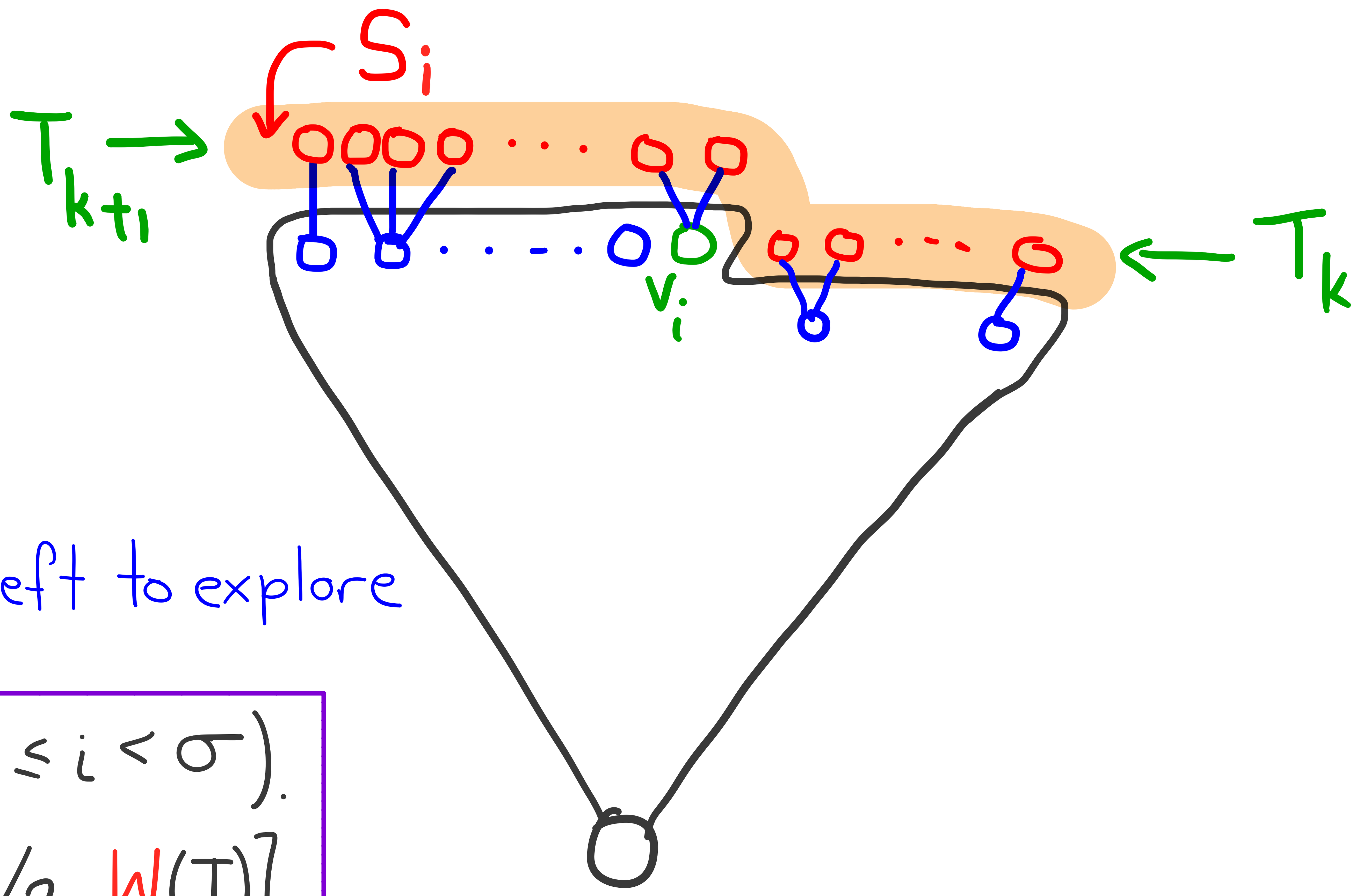

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Prop: $ht(T) \le 3H(T)$.

Corollary Suffices to prove $\mathbb{P}\left(H(T) \ge \frac{k}{1-p_1} W(T)\right) \le e^{-\delta k}$; thm. follows.

$W(\sigma) = \max(S_i, 0 \le i < \sigma)$ $H(\sigma) = \sum_{i=1}^{\sigma} \frac{1}{S_i}$ Aim: $\mathbb{P}\left(H(\sigma) \ge \frac{k}{1-p_1} W(\sigma)\right) \le e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j \approx 2^\ell$ ("scale 2^ℓ") for $j \in \{i, ..., i+2^\ell\}$, have

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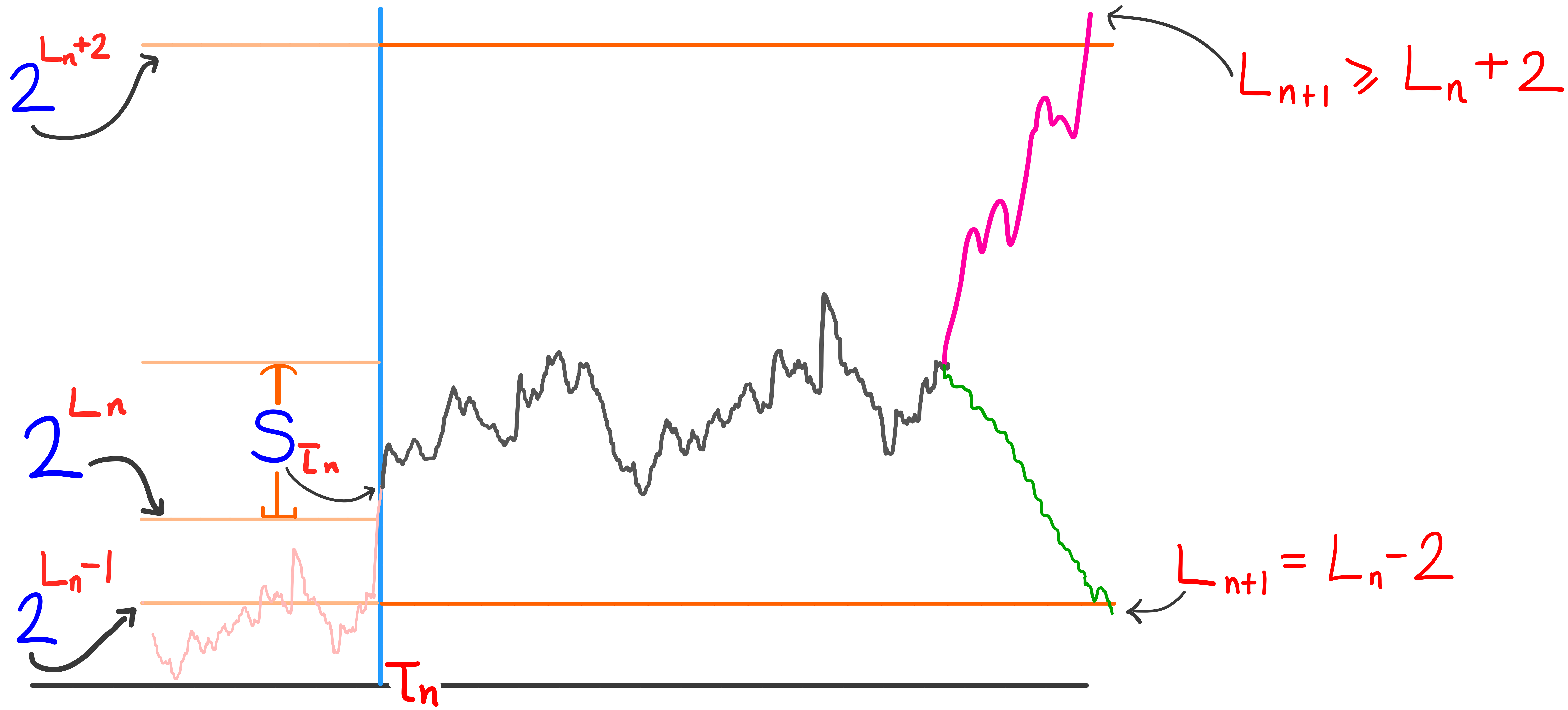

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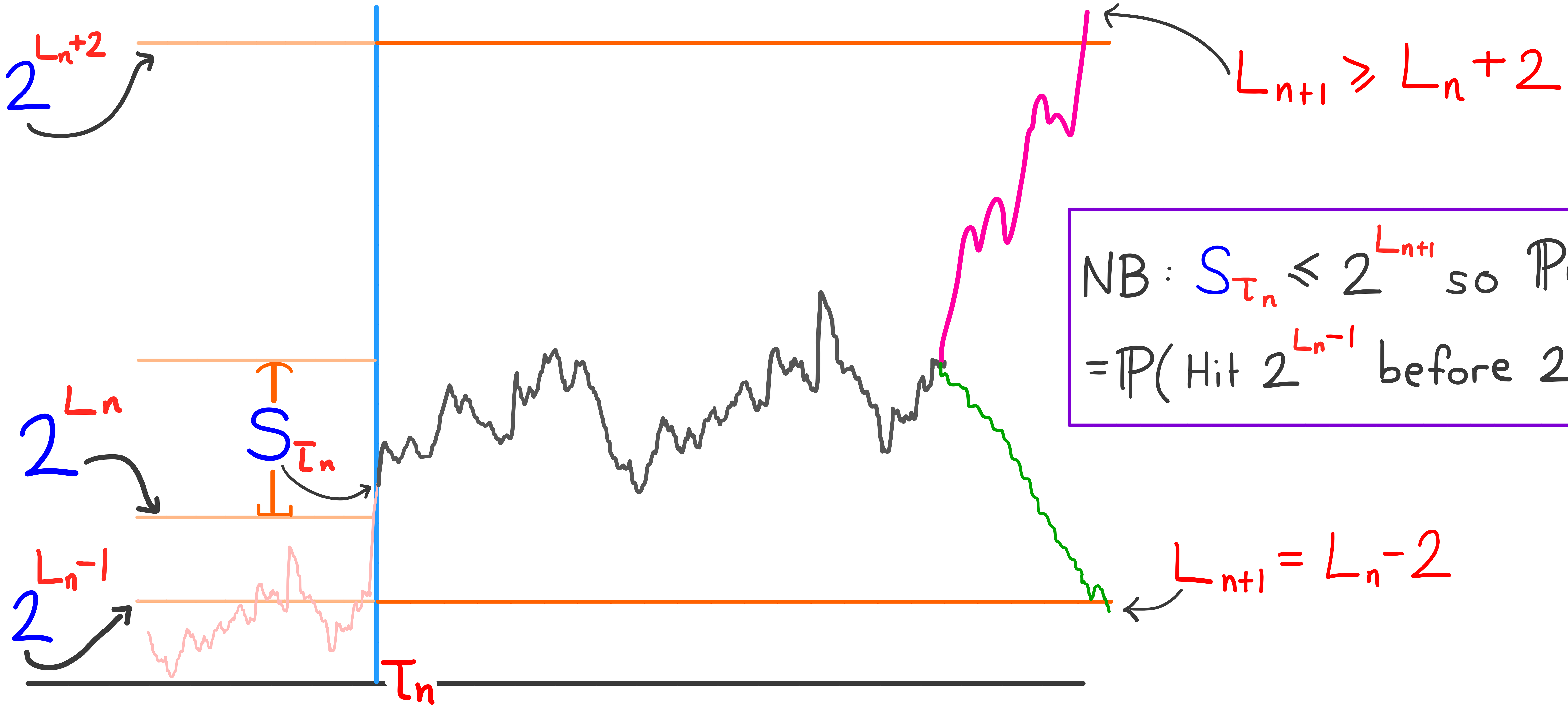

NB: $S_{\tau_n} \leq 2^{L_{n+1}}$ so $\mathbb{P}(L_{n+1} < L_n)$
$= \mathbb{P}(\text{Hit } 2^{L_n-1} \text{ before } 2^{L_n+2}) > \frac{1}{2}.$

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(a) Thm (Lévy; Doeblin; Kolmogorov; Rogozin; Le Cam; Esséen; Kesten):

With $p = \max p_i$, have $\boxed{\max_k \mathbb{P}(S_n = k) \le \frac{Cp}{\sqrt{n(1-p)}}}$ $C > 0$ universal.

"Any random walk spreads out over $\ge \sqrt{n}$ values by time n". Here $\sqrt{n} \simeq 2^\ell$.

$W(\sigma) = \max(S_i, 0 \le i < \sigma)$ $H(\sigma) = \sum_{i=1}^{\sigma} \frac{1}{S_i}$ **Aim**: $\mathbb{P}\left(H(\sigma) \ge \frac{k}{1-p_1} W(\sigma)\right) \le e^{-\delta k}$

Key Tool: Decomposition into scales.

When $S_j \approx 2^\ell$ ("scale 2^ℓ") for $j \in \{i, \dots, i+2^\ell\}$, have

$$H(i+2^\ell) - H(i) = \sum_{j=i+1}^{i+2^\ell} \frac{1}{S_j} \simeq 2^\ell \cdot \frac{1}{2^\ell} = 1.$$

So bound (a) time to change scales,

(b) "# visits to scales" $= (M(\ell), \ell \ge 1)$

(a) Thm (Lévy; Doeblin; Kolmogorov; Rogozin; Le Cam; Esséen; Kesten):

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(b) **Fact**: Given that $M(\ell) \ne 0$, $M(\ell)$ dominated by sum of 2 Geom$(\frac{1}{2})$ r.v.s; $\Rightarrow \mathbb{P}(M(\ell) > k \mid M(\ell) > 0) \le 2^{-k/2}$.

Proof via upcrossings. ∎

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NB: Here should have $\delta = \delta(p_0, p_1)$

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To stochastically maximize $ht(T_{\vec{n}})$ among sequences with $n_0 = k$, $n_1 = 0$,

choose the seq. $(k, 0, k-1, 0, \ldots)$


IMPACASTORINA
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Thank you!

- Claimed theorem with dependence only on p_1 proved it with dependence on $p = \max p_i$.

Fix: requires more careful "disperson" bound for our setting.

(Idea: If subcritical then $p_{max} = p_0$ or p_1; if p_0 close to 1 then either very subcritical or make large jumps.)