The number of spanning trees of random 2-trees

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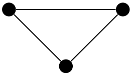
They are a special case of k-trees (same principle, but we start with a complete graph of order $k+1$ and attach a new complete graph K_{k+1} to an existing clique of order k). 1-trees are just ordinary trees.

Example: construction of a 2-tree



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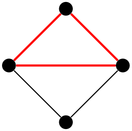

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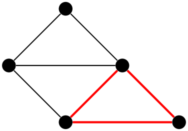

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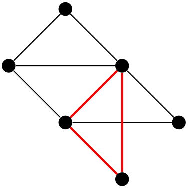

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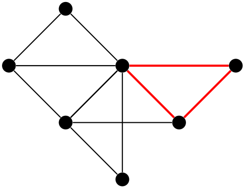

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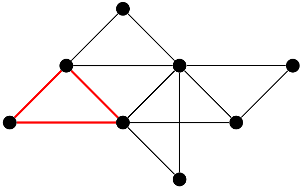

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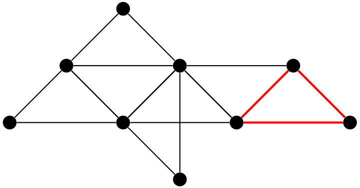

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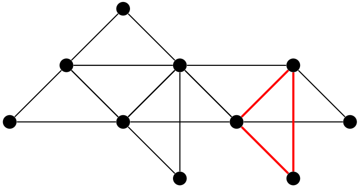

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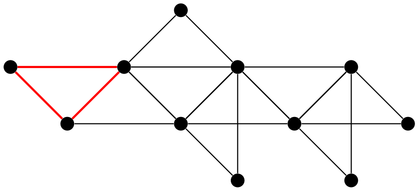

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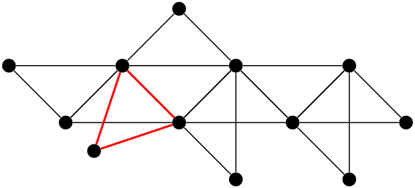

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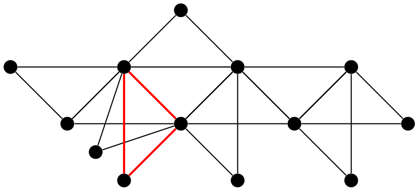

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- The number of spanning trees of random 2-trees (using various different models of randomness) was recently considered by Xiao and Zhao, who made several conjectures on the growth (based on simulations).

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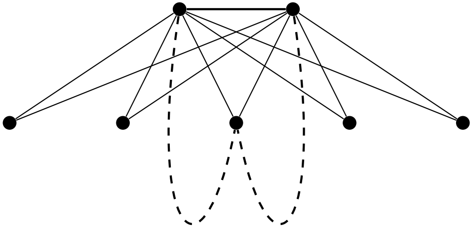
- Ehrenmüller and Rué studied spanning trees of 2-trees (as well as series-parallel graphs and 2-connected series-parallel graphs) in another very recent paper and determined the average number of spanning trees in random labelled 2-trees asymptotically.

Edge-rooted 2-trees

It will often be convenient to regard 2-trees as rooted at an edge; this edge is part of a number of triangles, each of which has two edge-rooted 2-trees attached to it (one on each of the other two edges).



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- Uniform "labelled" 2-trees: 2-trees with triangles labelled from 1 to n (corresponds to labelled trees)
- Uniform "binary" 2-trees: no edge may be part of more than two triangles (corresponds to random binary trees)
- Uniform "plane" 2-trees: edge-rooted 2-trees, the different triangles that belong to the root edge are ordered left to right (corresponds to random plane trees)

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Random attachment models

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- Uniform restricted attachment: an edge is selected uniformly at random among those that are not yet part of two triangles (corresponds to binary increasing trees)
- Preferential attachment: each edge is chosen with probability proportionate to the number of triangles it belongs to (corresponds to plane-oriented recursive trees)

A useful decomposition

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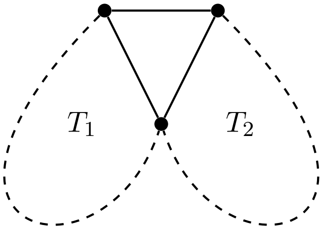
- $\rho(T)$ denotes the number of spanning trees that contain the root edge,
- $\sigma(T) = \tau(T) - \rho(T)$ denotes the number of spanning trees that do not contain the root edge.

A useful decomposition

Lemma

Let T be an edge-rooted 2-tree whose root is part of a single triangle. The edge-rooted sub-2-trees attached on the two other sides of this triangle are denoted by T_1 and T_2 respectively. Then we have

$$\rho(T) = \tau(T_1)\rho(T_2) + \rho(T_1)\tau(T_2) \quad \text{and} \quad \sigma(T) = \tau(T_1)\tau(T_2).$$

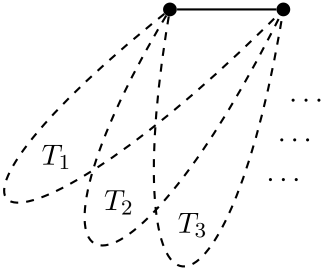


A useful decomposition

Lemma

Let T be an edge-rooted 2-tree with k triangles containing the root edge. The edge-rooted sub-2-trees containing those triangles are denoted by $T_1, T_2, \ldots, T_k$ respectively. Then we have

$$\rho(T) = \prod_{j=1}^{k} \rho(T_j) \quad \text{and} \quad \sigma(T) = \prod_{j=1}^{k} \rho(T_j) \sum_{j=1}^{k} \frac{\sigma(T_j)}{\rho(T_j)}.$$



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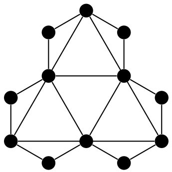
- The minimum is obtained for the "star": n triangles that share a common edge; this 2-tree has $(n+2)2^{n-1}$ spanning trees.
- The maximum is obtained for any "path", where every further triangle is attached to one of the two edges added in the previous step (remarkably, it does not matter which). The number of spanning trees of such a 2-tree is the Fibonacci number F_{2n+2} ($F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$).

Extremal values

If edges cannot be part of more than one triangle, the "complete binary" 2-tree becomes minimal (in the asymptotic sense): the number of spanning trees of any such 2-tree with n triangles is $\Omega(\alpha^n)$, where

$$\alpha = 4 \prod_{k=1}^{\infty} (1 - 2^{-k})^{2^{-k}} \approx 2.5747573641\ldots,$$

and this is attained in the limit by complete binary 2-trees. The figure shows the complete binary 2-tree of level 3. Starting with a single triangle, we attach a triangle to each of the outer edges at each step.



Translation to generating functions

As an example for the generating functions approach, we consider uniformly random 2-trees (with the triangles labelled). It is useful to consider edge-rooted 2-trees where the root edge is only part of one triangle as an auxiliary structure. We also consider the sides and vertices of every triangle as distinguishable (to facilitate counting).

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The generating function Y for counting such 2-trees satisfies

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Furthermore, 2-trees with labelled triangles rooted at a triangle have generating function

$$\overline{Y}(x) = xe^{3Y(x)},$$

and $n![x^n]\overline{Y}(x) = (3n)(2n+1)^{n-2}$.

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Clearly, $T(x) = R(x) + S(x)$, and we also have

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Similar systems of functional or differential equations can be found for all six models.

Combinatorial surprises

In the uniform binary case, this approach surprisingly even leads to an explicit formula:

Proposition

The total number of spanning trees in all binary 2-trees consisting of n triangles and rooted at one of the triangles is $\frac{6\cdot 4^n}{n+2}\binom{3n/2}{n+1}$, and there are $\frac{3}{n+2}\binom{2n}{n+1}$ such 2-trees, so the average number of spanning trees in uniformly random binary 2-trees is

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There is also a bijective correspondence to certain classes of connected noncrossing graphs.

Combinatorial surprises

Proposition

Let T be a 2-tree consisting of n triangles. If a triangle is attached to an edge of T that is selected uniformly at random to obtain a new 2-tree T', then

$$\mathbb{E}(\tau(T')) = \frac{5n+3}{2n+1}\tau(T).$$

Corollary

If a sequence $T_1, T_2, \ldots$ of 2-trees is constructed according to the uniform random attachment model, then

$$X_n = \prod_{j=1}^{n} \frac{2j-1}{5j-2}\tau(T_n)$$

is a martingale. In particular, $\mathbb{E}(\tau(T_n)) = \prod_{j=1}^{n} \frac{5j-2}{2j-1}$.

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- obtained from a spanning tree of T that contains the edge where the new triangle was attached, namely by replacing this edge with the two new edges. Any given spanning tree of T contains $n+1$ of the $2n+1$ edges, so we get an additional expected value of $\frac{n+1}{2n+1}\tau(T)$.

Application of singularity analysis

For our other models, we have to rely on singularity analysis of generating functions to obtain the asymptotic behaviour. Let us return to the functional equations that we obtained in the case of uniform labelled 2-trees:

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We immediately obtain $S(x)/R(x) = (S(x)+1)/2$, thus $R(x) = 2S(x)/(S(x)+1)$. This is plugged back in:

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The dominant singularity is easily established as the solution of the simultaneous equations $s = x(s+1)^2 e^{4s/(s+1)}$ and $1 = x\frac{d}{ds}(s+1)^2 e^{4s/(s+1)}$, which is given by $s = \sqrt{5}-2$ and $x = (\sqrt{5}-1)e^{\sqrt{5}-3}/8$.

Application of singularity analysis

From the asymptotic expansion of $S(x)$ at the dominant singularity, one also obtains the behaviour of all other generating functions. The final result reads as follows:

Proposition

The average number of spanning trees in uniform labelled 2-trees is asymptotically equal to $2e^{1-\sqrt{5}/2}\sqrt{1-\frac{1}{\sqrt{5}}} \cdot \big((1+\sqrt{5})e^{2-\sqrt{5}}\big)^n$.

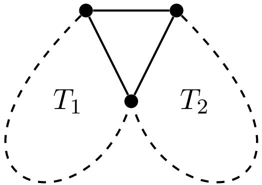
Application of singularity analysis

Other random models are treated in a similar way. The following growth constants are obtained:

- Uniform labelled: $(1+\sqrt{5})e^{2-\sqrt{5}} \approx 2.55561$
- Uniform binary: $3\sqrt{3}/2 \approx 2.59808$
- Uniform plane: $8(7\sqrt{7}-10)/27 \approx 2.52452$
- Uniform attachment: $5/2 = 2.5$
- Uniform restricted attachment: $1/(\log 4 - 1) \approx 2.58870$
- Preferential attachment: $8/(\log 27) \approx 2.42730$

Additive functionals

We would also like to say more about the distribution of the number of spanning trees. Consider binary 2-trees and their decomposition:



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$$\tau(T) = \tau(T_1)\tau(T_2) + \tau(T_1)\rho(T_2) + \rho(T_1)\tau(T_2).$$

Additive functionals

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can also be written as

$$\log \tau(T) = \log \tau(T_1) + \log \tau(T_2) + \kappa(T),$$

with $\kappa(T) = \log(1 + \rho(T_1)/\tau(T_1) + \rho(T_2)/\tau(T_2))$. We can thus regard τ as an *additive functional* with toll function κ. This function is easily seen to be bounded, which allows us to invoke a theorem of Janson on additive functionals of Galton-Watson trees:

Theorem

Let $\mathcal{T}_n$ denote a uniformly random binary 2-tree consisting of n triangles. The normalised logarithm of $\tau(\mathcal{T}_n)$ converges in probability to a constant:

$$\frac{\log(\tau(\mathcal{T}_n))}{n} \xrightarrow{p} C \approx 0.95.$$

Random weak limits

Suppose that $G_1, G_2, \ldots$ is a sequence of (possibly random) finite graphs. A probability mearure ρ on rooted infinite graphs is called the *random weak limit* of this sequence if the probability that the ball of radius R around a randomly chosen vertex of G_n is a fixed finite rooted graph H converges to the probability given by ρ as $n \to \infty$ for any fixed R and H.

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A famous theorem of Lyons states that $\log(\tau(G_n))/|G_n|$ (the *tree entropy*) converges in probability to a constant depending only on ρ if the sequence of graphs G_n has a random weak limit with bounded expected average degree.

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Making use of Lyons's results, we find that the tree entropy of random 2-trees converges to a constant depending on the specific model. However, this does not imply a central limit theorem.

Further work

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- Prove limit laws for other types of graphs, e.g. subcritical graph classes (which include for instance cacti, outerplanar graphs, series-parallel graphs, ...).