The Surprising Power of Belief Propagation

Elchanan Mossel

June 12, 2015

Why do you want to know about BP

- It’s a popular algorithm.
- We will talk abut its analysis.
- Many open problems.
- Connections to:
  - Random graphs.
  - Recursions of Random Variables.
  - The Cavity and Replica Methods from Physics.
  - Random Matrices.
  - ...

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- Goal of Belief Propagation: Compute marginals:

$$p(x_v = a)??$$

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- Example of Block model $(\theta = (a-b)/a+b)$.

$$\eta_{v \to u} := \frac{\prod_{w \neq u, (w,v) \in E}(1 + \theta \eta_{w \to v}) - \prod_{w \neq u, (w,v) \in E}(1 - \theta \eta_{w \to v})}{\prod_{w \neq u, (w,v) \in E}(1 + \theta \eta_{w \to v}) + \prod_{w \neq u, (w,v) \in E}(1 - \theta \eta_{w \to v})}$$

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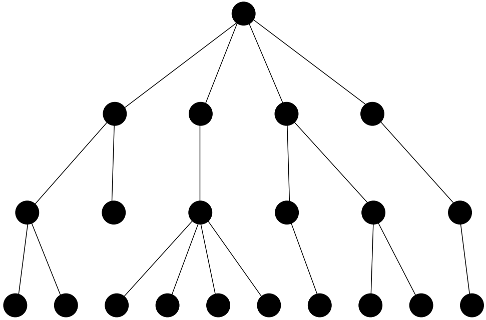

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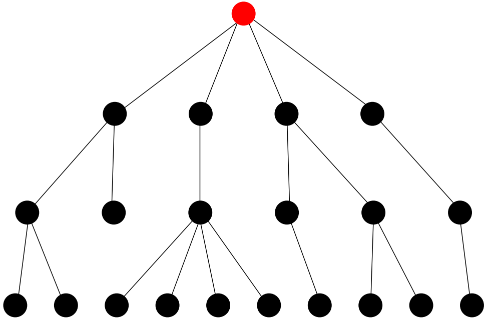

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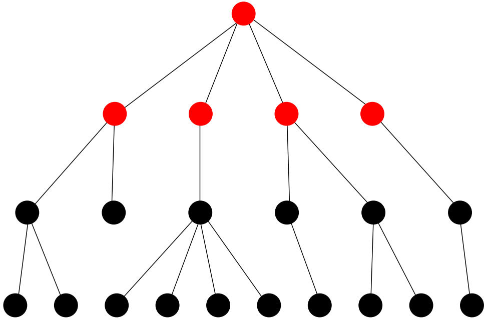

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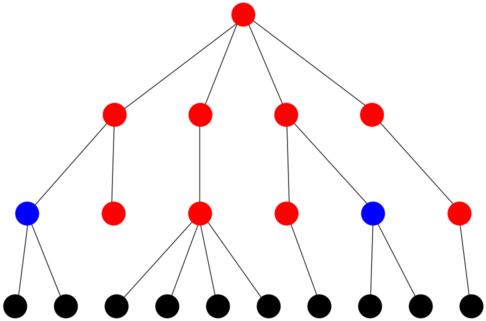

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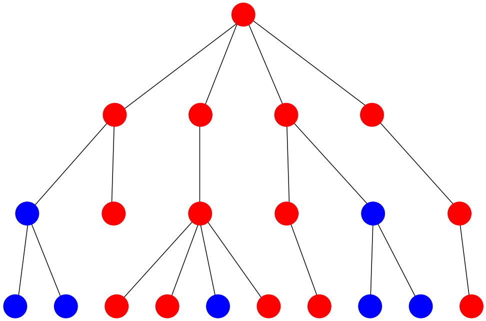

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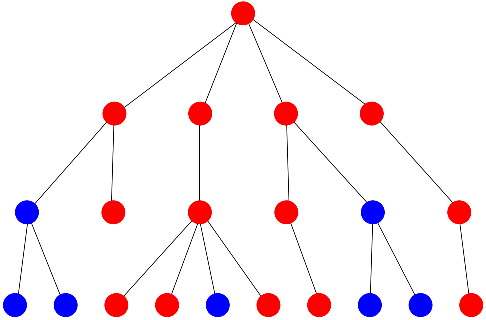

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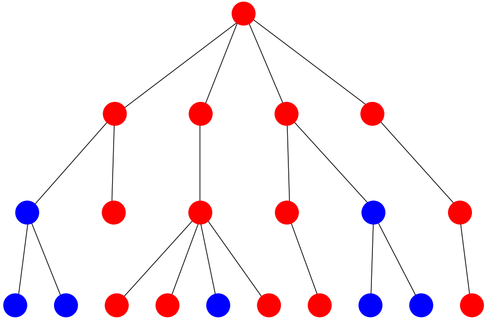

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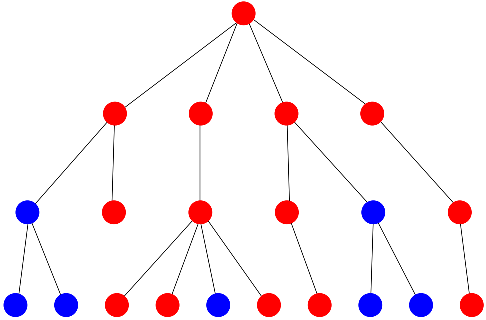

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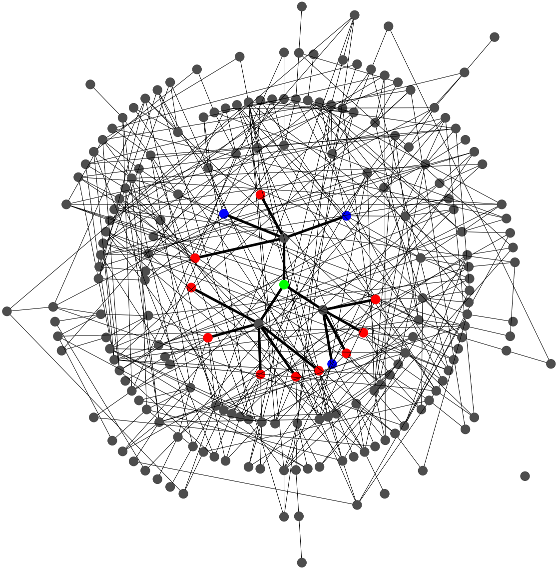
Nice tools: recursions of random variables, information inequalities etc.

What we proved in pictures



Theorem (Mossel-Neeman-Sly-12)

Given $G \setminus B(v, r)$ it is possible to guess the status of v better than random as $r \to \infty$ iff $(a-b)^2 > 2(a+b)$

Q: Why is this a Theorem?
A: Not obvious that non-neighbors provide diminishing information.
Note: The proof further shows that for *any* values of a, b, Belief Propagation maximizes the probability of guessing the color of r.

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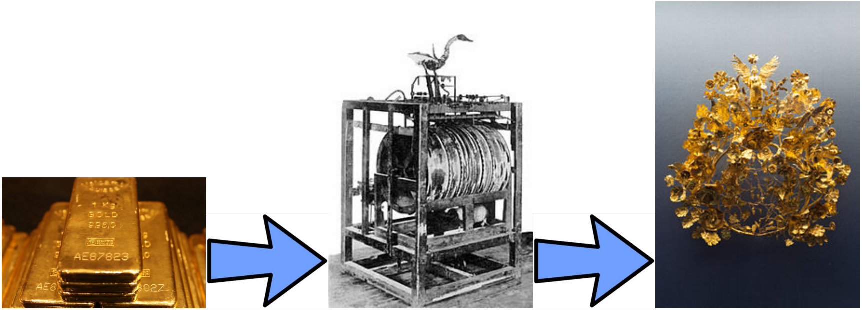

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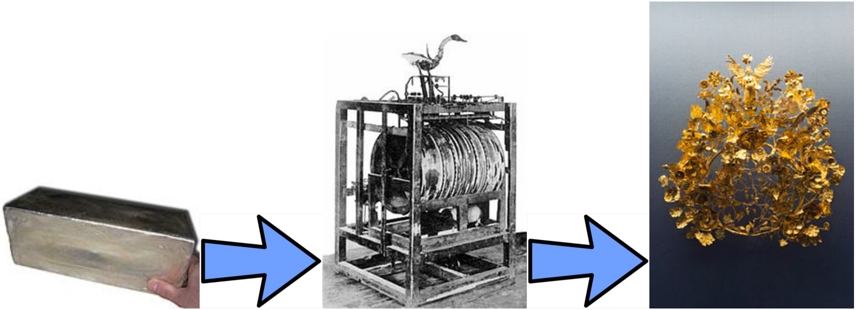
- If G is not a forest then $T(G)$ is infinite ...

BP on tree-like graphs and local information



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- Luby-Mitzenmacher-Shokrollahi-88
- Spielman-00,Richardson-Shokrollahi-Urbanke-01.
- Recent breakthrough: spatially coupled codes - achieve capacity efficiently - Kudekar-Richardson Urnabke.

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Conjecture - this is true for all a and b

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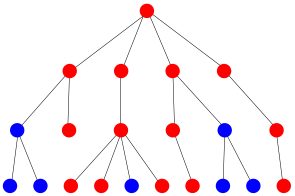

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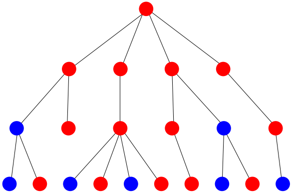

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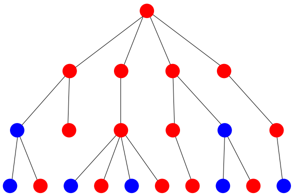

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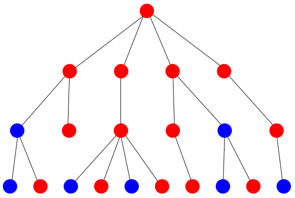

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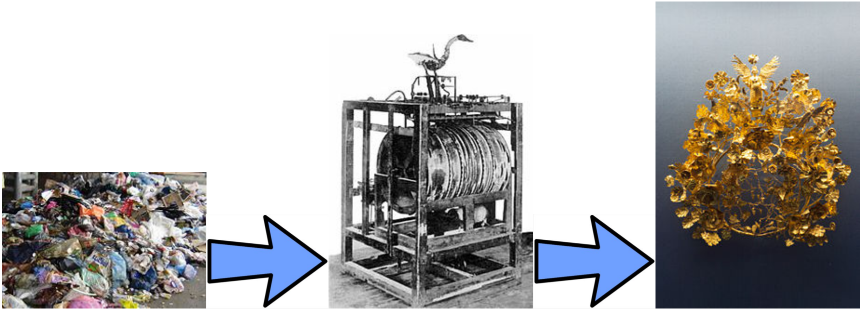
Strong property of a non-linear dynamical system (stronger than non-ergodicity, "robust reconstruction" etc. (Janson-M-04).

BP on tree-like graphs without local information



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- Note: **no prior information on any node.**
- Conjecture (Decelle, Krzakala, Moore and Zdeborova): ”Belief-Propagation” is the optimal algorithm.
- and ... possible to do better than random iff $(a - b)^2 > 2(a + b)$.

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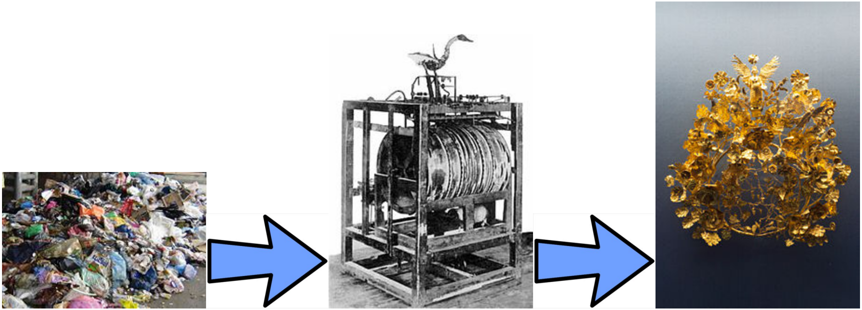
- Note: graph is very sparse - cannot hope to recover clusters exactly.

BP on tree-like graphs without local information



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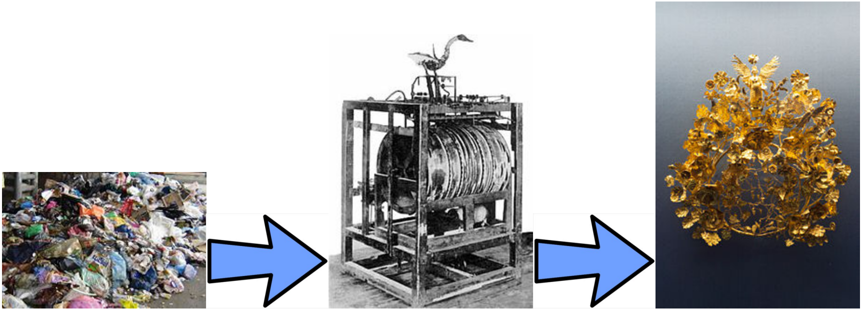

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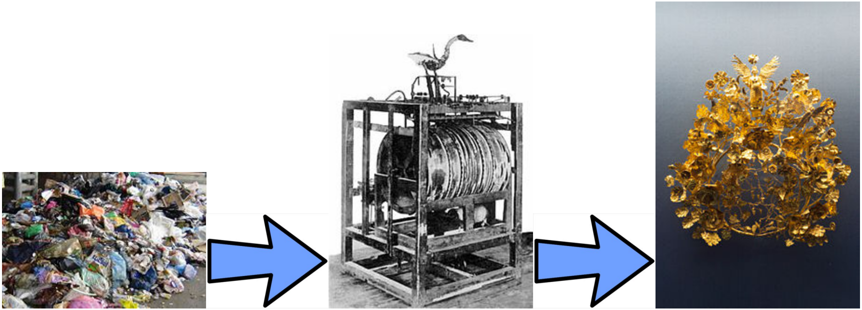

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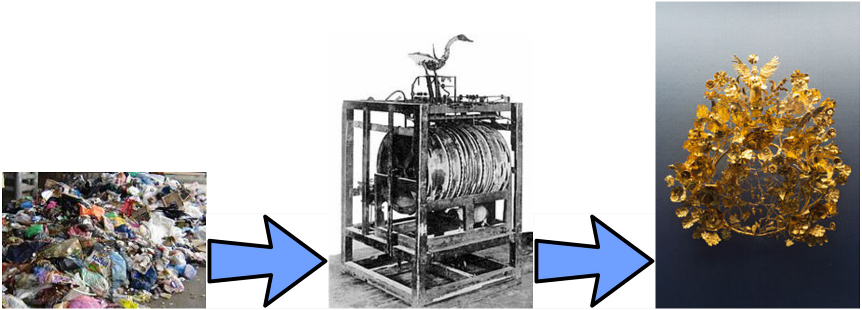

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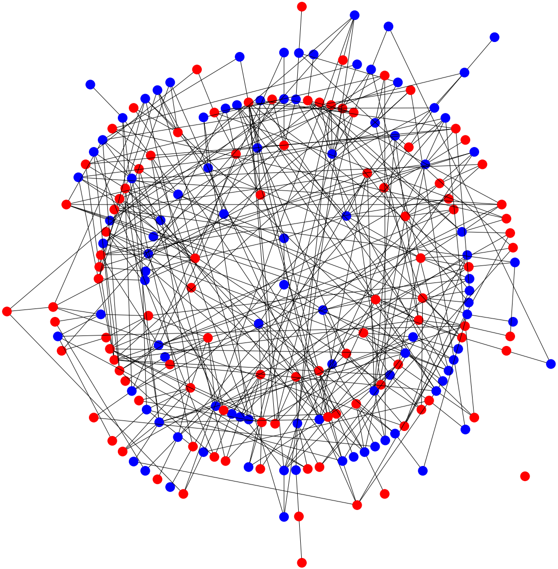
- Is randomization needed?

The Block Model in pictures

A sample from the model



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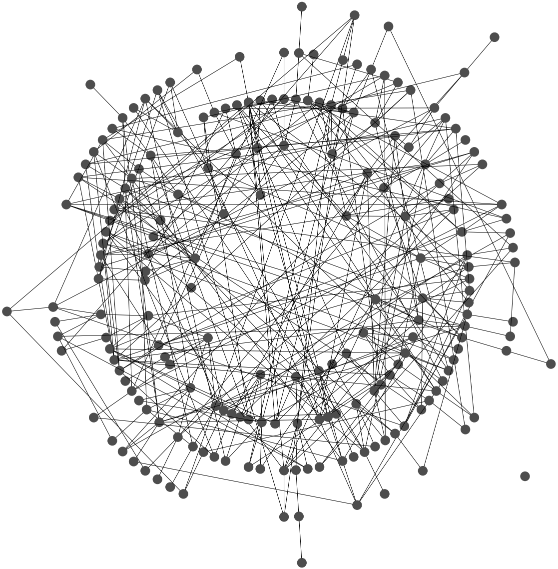
The data (one sample!)



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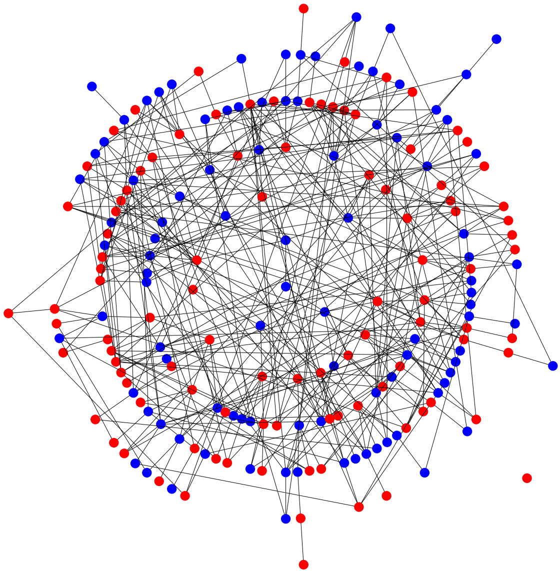
What we want to Infer



The Conjecture is Correct - Part 1

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- More later.

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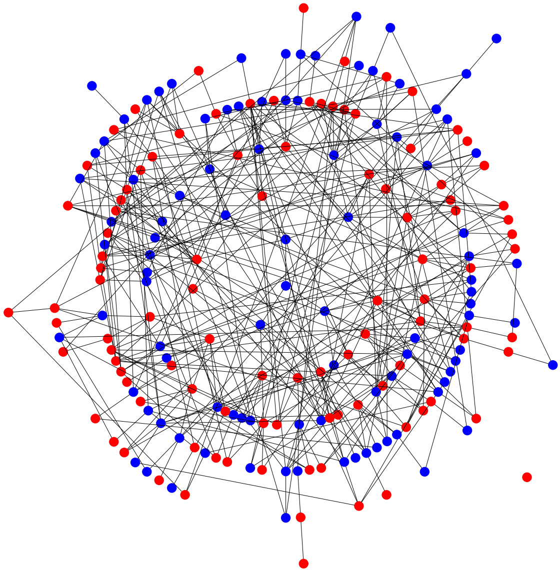

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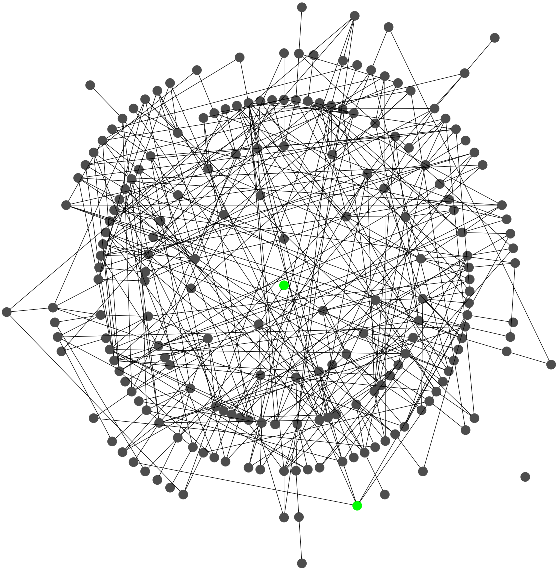

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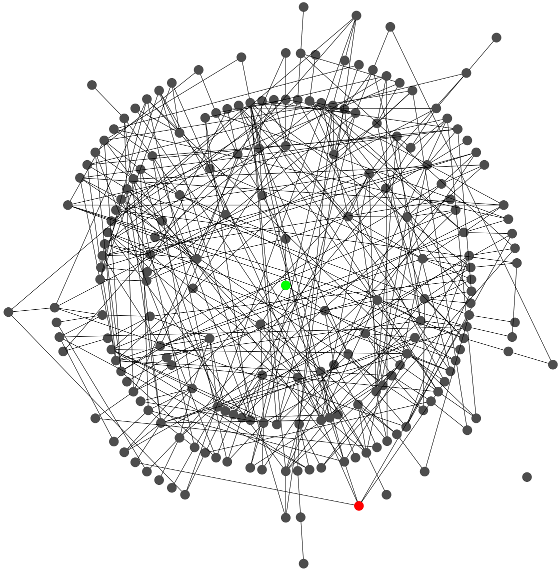

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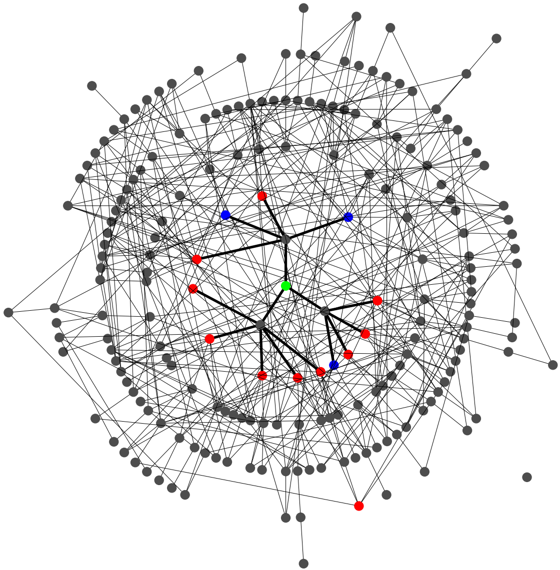

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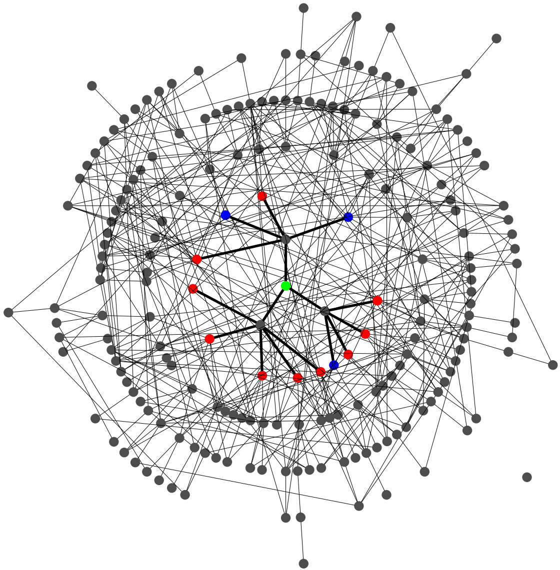

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- Note: Thm 4 improves on a very long line of research in computer science and statistics including Boppana (87) Dyer and Freeze (89), Jerrum and Sorkin (89), Carson and Impagliazzo (01) and Condon and Karp (01).

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- Note: conjectured linear algebra algorithm is deterministic.

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- Bordenave-Lelarge-Massoulie 15: 2nd eigenvector works!

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- Study it and conjecture it's optimality.

Zeta functions on graphs

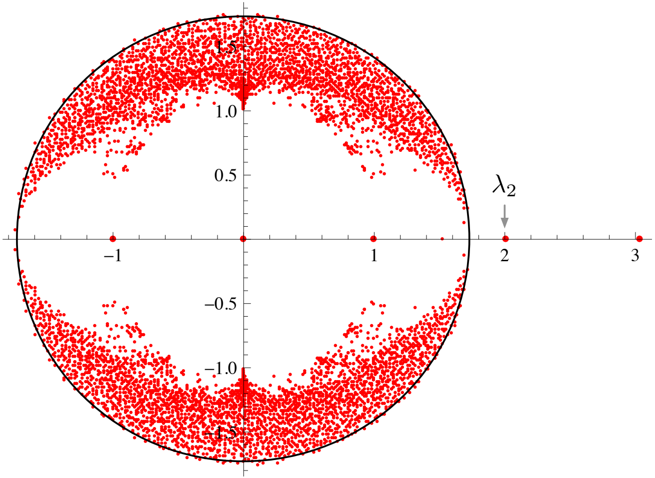
1. Hashimoto-89: Introduced a graph analogue of Zeta functions of p-adic algebraic varieties:

   $$Z(u, f) = \exp\Big(\sum_{\ell=1}^{\infty} \sum_{C \in X_\ell} \frac{f(C)}{\ell} u^\ell\Big),$$

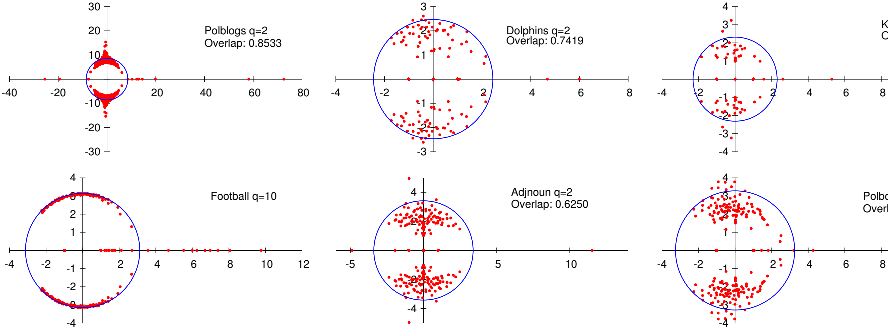
   where $X_\ell =$ set of closed non backtracking loops of length ℓ and $f(C) = \prod_{e \in C} f(e)$.
2. Proved that $Z(f, u)$ is a rational function of u.
3. Asked: how much $Z(f, u)$ is revealing about the graph ...

The Eigenvalues of N

$$\frac{a+b}{2} = 3, \quad \frac{a-b}{2} = 2, \quad \sqrt{\frac{a+b}{2}} = 1.732...$$



The spectrum on real networks



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- Optimal "non-backtracking" expansion.

Performance on Real Networks

- $R = N$.
- $L =$ normalized laplacian (random walk matrix).

network name	BP overlap	sign of vector 2 of **R**	k-means of **R**	sign of vector 2 of $\mathbf{L}_{sym}$	k-means of $\mathbf{L}_{sym}$
words	*	**0.9107**	0.875	0.5625	0.5714
political blogs	0.5167	0.9313	0.6383	**0.9542**	0.9476
karate club	0.5588	**1**	**1**	0.9706	**1**
dophin	**0.9838**	0.8710	0.96774	0.9677	**0.9839**
brsmall	*	0.6548	**0.69345**	0.6235	0.6687
brcorp	*	0.6993	0.72631	**0.7332**	0.6993
adjnoun	0.5625	0.8125	**0.8214**	0.5446	0.5357

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- Massoulie gets symmetric matrix. MNS - almost linear time.

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- Typically expect computational threshold to be different than information threshold.
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- More challenging: BP and Survey Propagation for satisfiability problems.
- How to let linear algebra algorithms utilize local information?