Uniform Sampling of Subshifts of Finite Type

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It is a subset of $\mathcal{A}^{\mathbb{Z}^d}$, which is shift-invariant.

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Fibonacci / golden mean / hard-core (or hard-square) subshift

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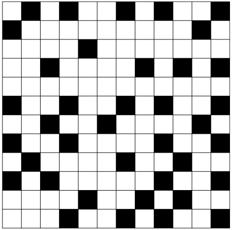

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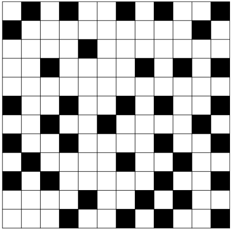

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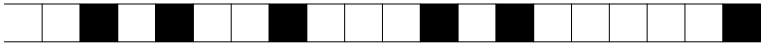

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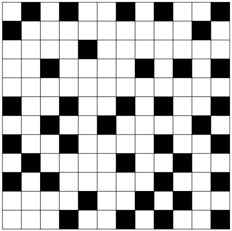

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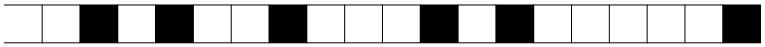

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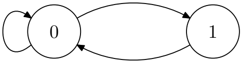

Graph of allowed transitions in one-dimension

Let $\mathcal{A}$ be an alphabet with n letters, and let $A \in \mathcal{M}_n(\{0,1\})$.

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One-dimensional subshift of finite type

The **subshift of finite type** associated to A is the set Σ_A of words $w \in \mathcal{A}^{\mathbb{Z}}$ such that if $A_{i,j} = 0$, w does not contain the pattern ij.

$$A_{i,j} = \begin{cases} 1 \text{ if } ij \text{ is an allowed pattern,} \\ 0 \text{ if } ij \text{ is a forbidden pattern.} \end{cases}$$

$$\Sigma_A = \{w \in \mathcal{A}^{\mathbb{Z}}; \forall k \in \mathbb{Z}, A_{w_k, w_{k+1}} = 1\}.$$

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In what follows, we assume that the matrix A is irreducible and aperiodic.

From Perron-Frobenius theory, the matrix A has a real **eigenvalue** $\lambda > 0$ such that $|\mu| \leq \lambda$ for any other eigenvalue μ.
Furthermore, there is a unique choice of $r_1, \ldots, r_n \geq 0$ such that $\sum_{i=1}^{n} r_i = 1$ and

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Definition of the Parry measure

The **Parry measure** is the (shift-invariant) Markov measure π on $\mathcal{A}^{\mathbb{Z}}$ of transition matrix P defined, for any $i, j \in \mathcal{A}$, by

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For a word $a_1 \ldots a_k \in \mathcal{A}^k$, $\pi(a_1 \ldots a_k) = \pi(a_1) P_{a_1,a_2} \ldots P_{a_{k-1},a_k}$.

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Proposition

Let μ_k be the uniform measure on allowed patterns of length $2k+1$, centered at position 0.
The sequence μ_k converges (weakly) to π on $\mathcal{A}^{\mathbb{Z}}$.

Markov-uniform property

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The Parry measure is **Markov-uniform**: for given $k \geq 1$ and $a, b \in \mathcal{A}$, the value

$$\pi(awb)$$

does not depend on the word $w \in \mathcal{A}^k$ such that awb is allowed.

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Proof. By definition, $P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}$. If awb is allowed, then:

$$\pi(awb) = \pi_a P_{a,w_1} P_{w_1,w_2} \ldots P_{w_{k-1},w_k} P_{w_k,b}$$

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$$= \frac{\pi_a r_b}{\lambda^{k+1} r_a}.$$

Measure of maximal entropy

Theorem

Let $\mathcal{M}_{\Sigma_A}$ be the set of translation invariant measures on the SFT Σ_A, and let $\pi \in \mathcal{M}_{\Sigma_A}$. The following properties are equivalent.

(i) π is the **Parry measure** associated to Σ_A,

(ii) π is a **Markov-uniform** measure on Σ_A,

(iii) π is the measure of **maximal entropy** of Σ_A,

(iv) the entropy of π is equal to the **topological entropy** $h(\Sigma_A)$.

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On $\mathbb{Z}^d$, the equivalence between *(ii)*, *(iii)*, and *(iv)* can be extended to a class of multi-dimensional SFT.

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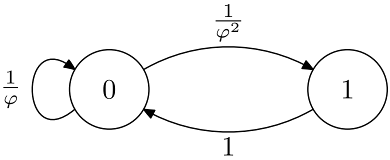
But there can be several measures satisfying these properties...

Let $\mathcal{A} = \{0, 1\}$. The **one-dimensional Fibonacci SFT** is the set of words that do not contain two consecutive 1's. It is given by:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its topological entropy is equal to $\log \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

The Parry measure is the Markov measure given by



with $\pi_0 = \frac{\varphi^2}{1+\varphi^2}$ and $\pi_1 = \frac{1}{1+\varphi^2}$.

The Parry measure of the Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability $r_0 = \frac{1}{\varphi}$ and a 1 with probability $r_1 = \frac{1}{\varphi^2}$,
- rejecting the 1's creating forbidden patterns.

First rejection sampling

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where $\mathcal{S}(i) = \{j \in \mathcal{A}; A_{i,j} = 1\}$ is the set of succesors of i.

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Proposition [J. Mairesse - I. Marcovici]

For **confluent** SFT, the Parry measure can be generated by independent draws of letters and deletion of forbidden patterns.

Coloring trees...

SFT defined on trees

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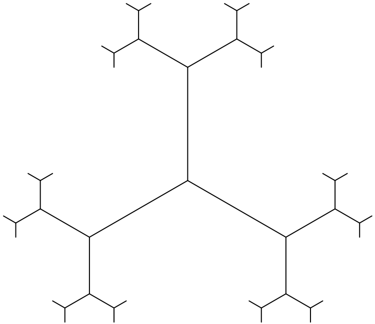

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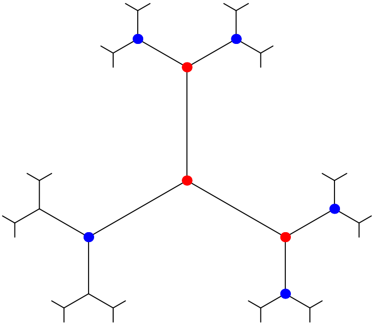

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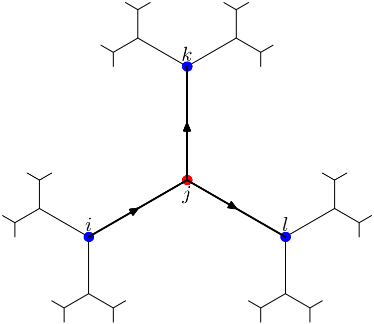

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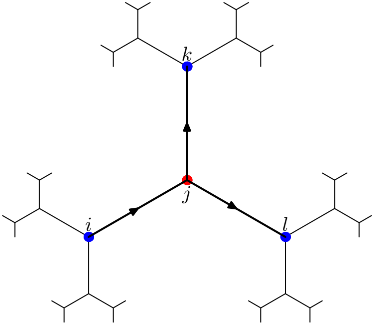

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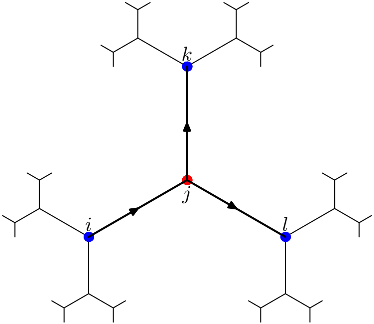

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For given i, k, l, we want this value to be independent of the letter j such that the pattern is allowed.

Idea 2: like for the Parry measure, choose P under the form:

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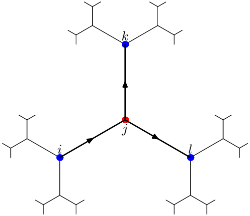

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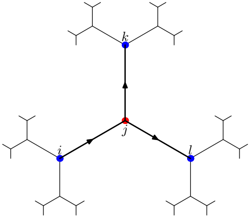
Let us try to choose $(r_i)_{i \in \mathcal{A}}$ such that:

$$\sum_{s \in \mathcal{A}} A_{j,s} r_s = \lambda {r_j}^{1/2},$$

for any $j \in \mathcal{A}$!



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For a tree of degree $d+1$, the problem is to find a probability distribution $(r_i)_{i\in\mathcal{A}}$ such that for some $\lambda > 0$,

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Proposition

Let A be an irreducible non-negative matrix, and let $d \geq 1$. There exist $\lambda > 0$ and $r_1, \ldots, r_n > 0$ satisfying $\sum_{i=1}^{n} r_i = 1$ and:

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Proof. Fixed point theorem.

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Remark. λ and $(r_i)_{i\in\mathcal{A}}$ may not be unique.

Proposition

If the distribution of probability $(r_i)_{i\in\mathcal{A}}$ satisfies

$$A\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1^{1/d} \\ \vdots \\ r_n^{1/d} \end{pmatrix}$$

for some $\lambda > 0$, then the Markov chain given by:

$$P_{i,j} = A_{i,j}\frac{r_j}{\sum_{s\in\mathcal{A}} A_{i,s} r_s} = A_{i,j}\frac{r_j}{\lambda r_i^{1/d}},$$

defines a **Markov-uniform** measure on the SFT Σ_A.

We search $P = \begin{pmatrix} \alpha & 1-\alpha \\ 1 & 0 \end{pmatrix}$, such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \lambda \begin{pmatrix} \alpha^{1/d} \\ (1-\alpha)^{1/d} \end{pmatrix}.$$

For any $d \geq 1$, there exists a unique solution, given by $r_0 = \alpha$ and $r_1 = 1 - \alpha$, where α is the unique positive solution of the equation

$$\alpha^{d+1} = 1 - \alpha.$$

For $d = 1$, we recover $r_0 = \frac{1}{\varphi}$ and $r_1 = \frac{1}{\varphi^2}$.

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$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \lambda \begin{pmatrix} \alpha^{1/d} \\ (1-\alpha)^{1/d} \end{pmatrix}.$$

For any $d \geq 1$, there exists a unique solution, given by $r_0 = \alpha$ and $r_1 = 1-\alpha$, where α is the unique positive solution of the equation

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Interpretation of these measures in terms of entropy?

Sampling using probabilistic cellular automata

One-dimensional Fibonacci SFT

Consider a configuration distributed according to the Parry measure π of the Fibonacci SFT.

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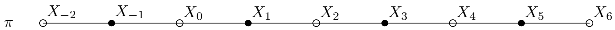
By the **Markov-uniform** property, the new sequence is still distributed according to π.

One-dimensional Fibonacci SFT



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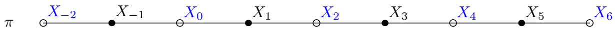

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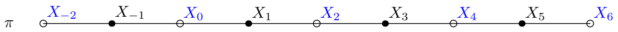

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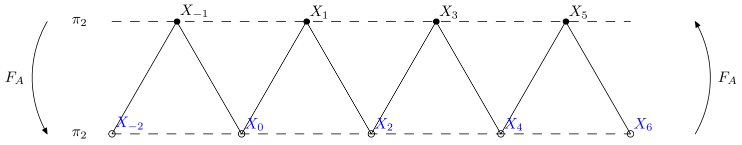

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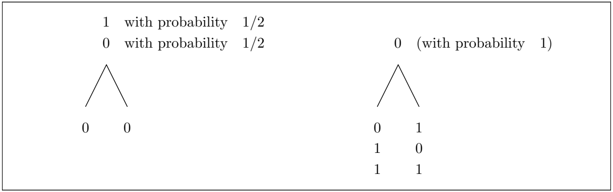
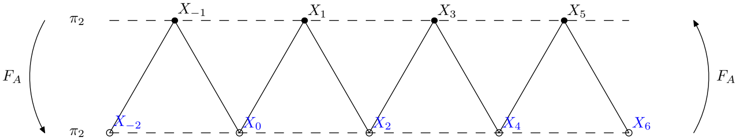
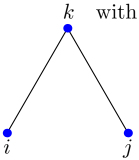




Proposition

The projection π_2 of the Parry measure on odd (resp. even) sites is an invariant measure of the probabilistic cellular automaton.

For a general one-dimensional SFT Σ_A, we consider the PCA F_A defined by:



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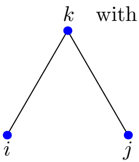

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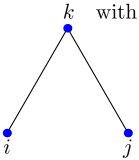

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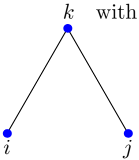

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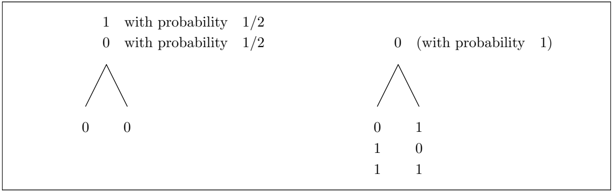
- Monte Carlo method
- Perfect sampling via coupling from the past

PCA family of the Fibonacci SFT



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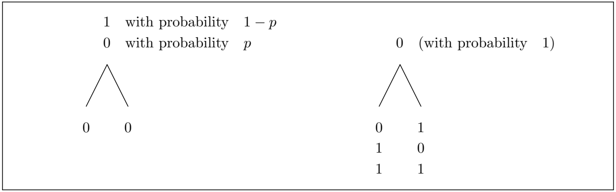

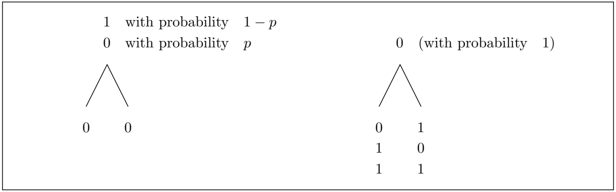

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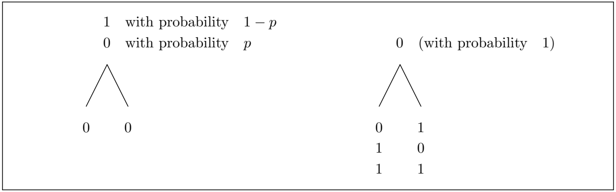

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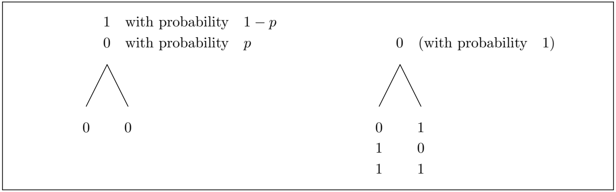

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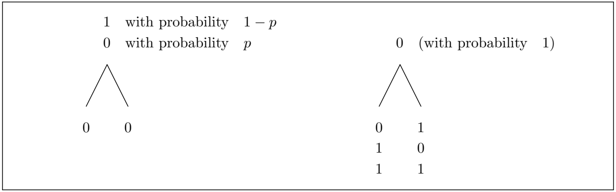

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And how to be sure that the coupling from the past algorithm will stop in finite time?

Conclusion

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