Limit distributions for Urn models with multiple drawings

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Pólya-Eggenberger urns: Setup

Urn contains $W = n$ white and $B = m$ black balls.

- Every discrete time steps a ball is drawn at **random**:
  $p_{\text{white}} = \frac{n}{n+m}$, $p_{\text{black}} = \frac{m}{n+m}$.
- Color inspection:
  White - a white and b black balls are added/removed; Black - c white and d black balls are added/removed; $a, b, c, d \in \mathbb{Z}$.
- 2×2 ball replacement matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

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$$M = \begin{array}{c} \\ \{W\} \\ \{B\} \end{array} \begin{array}{cc} W & B \\ \left(\begin{array}{c} a \\ c \end{array}\right. & \left.\begin{array}{c} b \\ d \end{array}\right) \end{array}.$$

Pólya-Eggenberger urns: examples

Pólya urn: $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

Beta limit law; Normal limit for martingale tail sum
Hall and Heyde; **Grübel**.

Triangular urns: $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

with $a, d, \in \mathbb{N}$ and $b \in \mathbb{N}_0$.

Generalized Mittag-Leffler limit law
Janson; **Goldschmidt; Mahmoud**.

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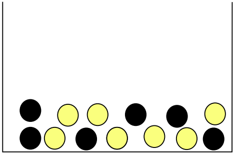
Pólya-Eggenberger urns: examples

Ball replacement matrix $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & -1 \end{smallmatrix}\right)$

Initial configuration: $n = 7$ yellow (white), $m = 6$ black balls



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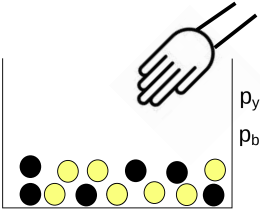

$p_{\text{yellow}} = 7/13$

$p_{\text{black}} = 6/13$

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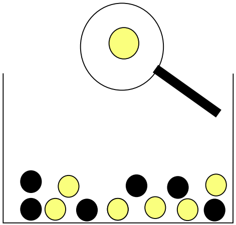

Inspected color:
yellow

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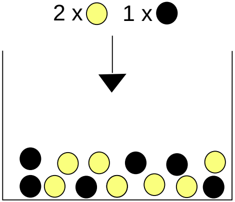

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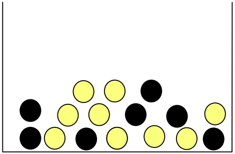

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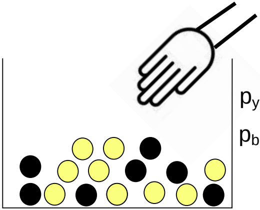

$p_{\text{yellow}} = 9/16$

$p_{\text{black}} = 7/16$

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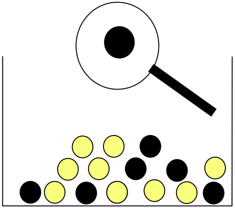

Inspected color:
black

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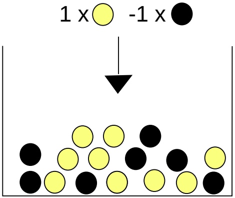

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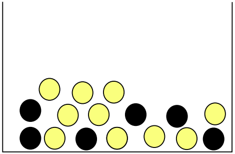

Pólya-Eggenberger urns

We make the following assumptions:

1. **Tenable urns**: process of drawing and adding/removing balls can be continued **ad infinitum**.
2. **Balance condition**:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   with

$$\sigma = a + b = c + d \geqslant 1.$$

   Consequently: total number of balls T_n is **non-random**: $T_n = W_n + B_n = W_0 + T_0 + n \cdot \sigma$.

3. **Two colors**: white and black balls.

Pólya-Eggenberger urns

Main question: Given $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and start with $W_0 \in \mathbb{N}_0$ white and $B_0 \in \mathbb{N}_0$ black balls: configuration W_n (B_n) after n draws?

Main task: Analyse

$$W_n \overset{(d)}{=} W_{n-1} + a \cdot \mathbb{I}_n(\{W\}) + c \cdot \mathbb{I}_n(\{B\}), \quad n \geqslant 1,$$

where

$$\mathbb{P}\{\mathbb{I}_n(\{W\}) = 1 | \mathcal{F}_{n-1}\} = \frac{W_{n-1}}{W_{n-1} + B_{n-1}} = \frac{W_{n-1}}{T_{n-1}}$$

and

$$\mathbb{P}\{\mathbb{I}_n(\{B\}) = 1 | \mathcal{F}_{n-1}\} = \frac{B_{n-1}}{W_{n-1} + B_{n-1}} = \frac{B_{n-1}}{T_{n-1}}.$$

Pólya-Eggenberger urns: existing works

Analytic combinatorics/Symbolic methods: **generating functions, method of moments, etc.**

FLAJOLET, GABARRÓ AND PEKARI; BRENNAN AND PRODINGER; STADJE; DUMAS, FLAJOLET AND PUYHAUBERT; HWANG, K. AND PANHOLZER; MORCRETTE; MAHMOUD AND MORCRETTE; . . .

Probabilistic methods: **stochastic processes, martingales, contraction method, etc.**

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Balanced urn models: partial differential equations; History counting approach

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unbalanced r-color urns: general results; also covers 2-color triangular urn models.

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contraction method for balanced urns; smoothing equations equations for large urns.

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Fundamental result

Theorem (Trichotomy of limit laws - Janson)

For a balanced two-color urn model $M = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix}$ *let* $\Lambda = \frac{\lambda_1}{\lambda_2}$ *denote the ratio of the two eigenvalues of* M.

1. *for* $\Lambda \leqslant \frac{1}{2}$: $\frac{W_n - \mathbb{E}(W_n)}{\sqrt{\mathbb{V}(W_n)}} \to \mathcal{N}(0,1)$.
2. *for* $\Lambda > \frac{1}{2}$: $\frac{W_n - \mathbb{E}(W_n)}{n^{\Lambda}} \to L$.
3. *for* $b_0 \cdot a_1 = 0$ *the urn is triangular and* $\frac{W_n}{n^{\Lambda}} \to T$.

Pólya-Eggenberger urns

. . . New problems.

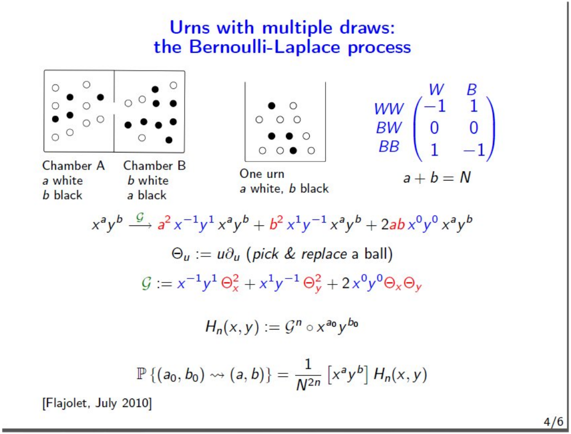
SVANTE JANSON Functional limit theorems for multitype branching processes and generalized Pólya urns. Stoch. Proc. Appl. 110 (2004)

Remark 4.5: "*Mahmoud has initiated the study of urn models where several, say 2, balls are drawn at the same time, and balls are added de- pending on the drawn combination of types. It may be possible to study such models too by the methods of this paper, first considering the corresponding continuous time model, but we have not pursued this. (This case is substantially more complicated than the standard case treated here; for example, the continuous time model will explode in finite time.")*

Pólya-Eggenberger urns

Remark

PHILIPPE FLAJOLET, *Talk by Basile Morcrette and Nicolas Pouyanne, 2011 Conference - Philippe Flajolet and Analytic Combinatorics, slides;*



Urns with multiple draws: the Bernoulli-Laplace process

Chamber A
a white
b black

Chamber B
b white
a black

One urn
a white, b black

$$\begin{array}{ccc} & W & B \\ WW & -1 & 1 \\ BW & 0 & 0 \\ BB & 1 & -1 \end{array}$$

$$a + b = N$$

$$x^a y^b \xrightarrow{\mathcal{G}} a^2 x^{-1} y^1 x^a y^b + b^2 x^1 y^{-1} x^a y^b + 2ab\, x^0 y^0 x^a y^b$$

$\Theta_u := u\partial_u$ (*pick & replace* a ball)

$$\mathcal{G} := x^{-1} y^1 \Theta_x^2 + x^1 y^{-1} \Theta_y^2 + 2\, x^0 y^0 \Theta_x \Theta_y$$

$$H_n(x,y) := \mathcal{G}^n \circ x^{a_0} y^{b_0}$$

$$\mathbb{P}\left\{(a_0, b_0) \rightsquigarrow (a, b)\right\} = \frac{1}{N^{2n}} \left[x^a y^b\right] H_n(x,y)$$

[Flajolet, July 2010]

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Pólya-Eggenberger urns

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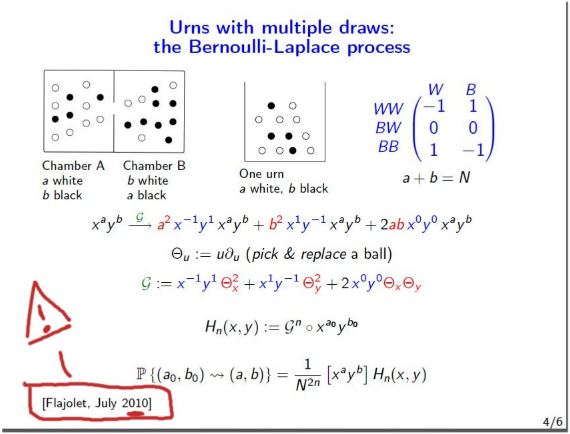

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[Flajolet, July 2010]

Urn models

Multiple drawings
- linear affine class

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Multiple drawings
- linear affine class

Urn models with multiple drawings

Previously: Urn contains w white and b black balls. $M = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix}$

Draw at random a **single** ball: $p_{\text{white}} = \frac{w}{w+b}$, $p_{\text{black}} = \frac{b}{w+b}$.

Multiple drawings: We draw $m \geqslant 1$ balls in order to obtain our sample. Depending on the drawn multiset of white/black balls we add/remove balls according to the $(m+1) \times 2$ ball replacement matrix

$$M = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \dots & \dots \\ a_{m-1} & b_{m-1} \\ a_m & b_m \end{pmatrix}$$

Balance: $\sigma = a_0 + b_0 = \dots = a_m + b_m$.

Urn models with multiple drawings: Example

We sample $m = 2$ balls.

$$M = \begin{array}{c} \\ \{WW\} \\ \{WB\} \\ \{BB\} \end{array} \begin{array}{c} \begin{array}{cc} W & B \end{array} \\ \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \end{array}.$$

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Sampling schemes - unordered samples:

Urn contains w white and b black balls.

Sampling without replacement

$$p_{\{k \text{ times white, } (m-k) \text{ times black }\}} = \frac{1}{(b+w)^{\underline{m}}} \binom{m}{k} w^{\underline{k}} b^{\underline{m-k}},$$

with $x^{\underline{s}} = x(x-1)\dots(x-s+1)$.

Sampling with replacement,

$$p_{\{k \text{ times white, } (m-k) \text{ times black }\}} = \frac{1}{(b+w)^{m}} \binom{m}{k} w^{k} b^{m-k}.$$

Urn models with multiple drawings

Distributions: We consider balanced urn models and study the number of white balls W_n:

$$W_n \overset{(d)}{=} W_{n-1} + \sum_{k=0}^{m} a_{m-k} \cdot \mathbb{I}_n(W^k B^{m-k}), \quad n \geqslant 1.$$

$\mathbb{I}_n(W^k B^{m-k})$, indicator variables k white and $m-k$ black balls.

$\mathcal{F}_{n-1}$ sigma-field generated by first $n-1$ draws

For sampling without replacement $\mathbb{P}\{\mathbb{I}_n(W^k B^{m-k}) = 1 | \mathcal{F}_{n-1}\}$ is given by

$$\frac{\binom{W_{n-1}}{k}\binom{B_{n-1}}{m-k}}{\binom{T_{n-1}}{m}} = \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}}$$

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Urn models with multiple drawings: examples

Pólya urn: $M = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, c \in \mathbb{N}.$

Generalization: If we draw $\{W^k S^{m-k}\}$ we add $k \cdot c$ white and $(m-k) \cdot c$ black balls, with $c \in \mathbb{N}$. $(m+1) \times 2$-matrix

$$M = \begin{pmatrix} mc & 0 \\ (m-1)c & c \\ \dots & \dots \\ c & (m-1)c \\ 0 & mc \end{pmatrix}$$

Urn is balanced: $\sigma = mc$, such that $T_n = W_n + B_n = nmc + T_0$.

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Urn models with multiple drawings: linear affine models.

Given

$$M = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \ldots & \ldots \\ a_{m-1} & b_{m-1} \\ a_m & b_m \end{pmatrix}$$

one cannot easily obtain the expected value.

Difficulty:

$$\mathbb{E}(W_n) = \sum_{k=0}^{m} f_{n,k} \mathbb{E}(W_{n-1}^k),$$

for certain sequences $f_{n,k}$ depending only on n and m (and k).

When is it possible to **calculate** the expected value of W_n (in a **simple** way)?

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Urn models with multiple drawings.

Proposition (Mahmoud and K.)

Given matrix M*, the numbers* a_{m-1}*,* a_m*, and the balance factor* $\sigma = a_k + b_k \geqslant 0$*.* W_n *satisfies a* ***linear affine relation*** *of the form*

$$\mathbb{E}\left[W_n | \mathcal{F}_{n-1}\right] = \alpha_n W_{n-1} + \beta_n, \qquad n \geqslant 1,$$

if and only if, for $0 \leqslant k \leqslant m$*, the numbers* a_k *satisfy the condition*

$$a_k = (m-k)a_{m-1} - (m-k-1)a_m.$$

(equivalently, $a_k = a_0 + hk$*, with* h *an integer guaranteeing tenability.) Then*

$$\alpha_n = \frac{T_{n-1} + m(a_{m-1} - a_m)}{T_{n-1}}, \qquad \textit{and} \qquad \beta_n = a_m, \qquad n \geqslant 1.$$

Remark

For $m = 1$ we reobtain the ordinary balanced urns.

Urn models with multiple drawings.

Proof (sketch). For sampling with replacement:

$$\mathbb{E}\left[W_n|\mathcal{F}_{n-1}\right] = W_{n-1} + \sum_{k=0}^{m} a_{m-k}\mathbb{E}[\mathbb{I}_n(W^k B^{m-k}) = 1|\mathcal{F}_{n-1}]$$

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Taking the expectation gives the ”if-part” of the stated result.

Urn models with multiple drawings.

Although only three parameters a_{m-1}, a_m and σ, we **unify earlier treated** models:

Example

- Case $m = 2$: MAHMOUD's condition $a_0 = 2a_1 - a_2$
- Case $a_m = mc$, $a_{m-1} = (m-1)c$ and $\sigma = mc$: generalized Friedman urn model
- Case $a_m = 0$, $a_{m-1} = c$ and $\sigma = mc$: generalized Pólya urn model
- Case $a_m = 1$ and $a_{m-1} = 0$ and $\sigma = 1$ we obtain $a_k = -(m-k) + 1$, urn model for logical circuits.

Urn models

- A few results

Urn models

- A few results

Pólya-Eggenberger urns

Proposition (Mahmoud and K.)

For balanced affine urn schemes the expected value is given by

$$\mathbb{E}[W_n] = \frac{a_m(n + \frac{T_0}{\sigma})}{1-\Lambda} + \left(W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda}\right) \frac{\binom{n-1+\frac{T_0}{\sigma}+\Lambda}{n}}{\binom{n-1+\frac{T_0}{\sigma}}{n}},$$

as well as the asymptotic expansion

$$\mathbb{E}[W_n] = \frac{a_m}{1-\Lambda} n + \left(W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda}\right) \frac{\Gamma(\frac{T_0}{\sigma})}{\Gamma(\frac{T_0}{\sigma}+\Lambda)} n^{\Lambda} + \mathcal{O}(1),$$

Moreover, for $\Lambda = 1$ *we obtain* $\mathbb{E}[W_n] = W_0 \frac{n\sigma + T_0}{T_0}$.

Pólya-Eggenberger urns

Theorem (Mahmoud and K.)

For balanced affine urn schemes, the variance satisfies the following expansions.
Small-index urns, the case $\Lambda < \frac{1}{2}$*:*

$$\mathbb{V}[W_n] = \frac{a_m b_0 \Lambda^2}{m(1-2\Lambda)(1-\Lambda)^2} n + o(n).$$

Critical-index urns, the case $\Lambda = \frac{1}{2}$*:*

$$\mathbb{V}[W_n] = \frac{a_m b_0}{m} n \log n + \mathcal{O}(n),$$

Large-index urns, the case $\Lambda > \frac{1}{2}$*:*

$$\mathbb{V}[W_n] = C n^{2\Lambda} + \mathcal{O}(n),$$

with the constant C *being model-dependent given by an infinite sum*

Pólya-Eggenberger urns

Theorem (continued)

Constant C*:*

$$C = \frac{W_0^2}{\psi_0} + \sum_{j=1}^{\infty} \frac{\beta_j \mathbb{E}[W_{j-1}] + a_m^2 - \psi_j 2a_m j^{-\Lambda}\Big(\frac{a_m}{1-\Lambda} j^{1-\Lambda} + (W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda})\frac{\Gamma(\frac{T_0}{\sigma})}{\Gamma(\frac{T_0}{\sigma}+\Lambda)}\Big)}{\psi_j}$$

$$+ \frac{2a_m^2}{1-\Lambda}\zeta(2\Lambda-1) + 2a_m\Big(W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda}\Big)\frac{\Gamma(\frac{T_0}{\sigma})}{\Gamma(\frac{T_0}{\sigma}+\Lambda)}\zeta(\Lambda) - \Big(W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda}\Big)^2 \frac{\Gamma^2(\frac{T_0}{\sigma})}{\Gamma^2(\frac{T_0}{\sigma}+\Lambda)},$$

with $\zeta(z)$ *denoting the Riemann zeta function and* β_j, ψ_j *certain sequences:*

$$\psi_n = \begin{cases} \dfrac{\Gamma(n+\lambda_1)\Gamma(n+\lambda_2)}{\Gamma(n+\frac{T_0}{\sigma})\,\Gamma(n+\frac{T_0-1}{\sigma})}, & \textit{without replacement} \\ \dfrac{\Gamma(n+\mu_1)\Gamma(n+\mu_2)}{\Gamma(n+\frac{T_0}{\sigma})^2} & \textit{with replacement,} \end{cases}$$

and $\lambda_{1,2} = \dfrac{m(a_{m-1}-a_m)+T_0-\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4m(a_{m-1}-a_m)(a_{m-1}-a_m+1)}}{\sigma}$,

$\beta_n = (a_{m-1}-a_m)^2\Big(\frac{m}{T_{n-1}} - \frac{m^2}{T_{n-1}^2}\Big) + \frac{2ma_m(a_{m-1}-a_m)}{T_{n-1}} + 2a_m$. *for sampling without replacement,*

or $\mu_{1,2} = \dfrac{m(a_{m-1}-a_m)+T_0\pm(a_{m-1}-a_m)\sqrt{m}}{\sigma}$, *and*

$\beta_n = \dfrac{(a_{m-1}-a_m)^2 m}{T_{n-1}} + \dfrac{2ma_m(a_{m-1}-a_m)}{T_{n-1}} + 2a_m$ *for sampling with replacement;*

Pólya-Eggenberger urns

Theorem (Trichotomy of limit laws for affine linear models - Mahmoud and K.)

For a balanced linear affine two-color urn model $\mathcal{M}$ *with sample size* $m \geqslant 1$ *let* $\Lambda = \frac{\lambda_1}{\lambda_2}$ *denote the ratio of the two eigenvalues of* $\mathcal{M}$.

1. *for* $\Lambda \leqslant \frac{1}{2}$: $\frac{W_n - \mathbb{E}(W_n)}{\sqrt{\mathbb{V}(W_n)}} \to \mathcal{N}(0,1)$.
2. *for* $\Lambda > \frac{1}{2}$: $\frac{W_n - \mathbb{E}(W_n)}{n^\Lambda} \to L$.
3. *for* $b_0 \cdot a_m = 0$ *the urn is triangular and* $\frac{W_n}{n^\Lambda} \to T$.

There exists explicit expressions for all moments of $\mathbb{E}(L^s)$ and $\mathbb{E}(T^s)$ as nested infinite sums; almost sure convergence for large-index urns and triangular urns.

Urn models

- Multiple drawings

Urn models

- Multiple drawings
Analytic combinatorics

Analysis using Analytic combinatorics

Analytic combinatorial approach à la **Flajolet**:

- Symbolic description of the problem, replacement of balls modelled by differential operator
- Generating functions
- Translate description to higher order linear partial differential equation
- Use analytic machinery to analyze parameters of interest

Analysis using Analytic combinatorics

Using the idea DUMAS, FLAJOLET AND PUYHAUBERT we study **urn histories** using differential operators:

∂_z: differential operator with respect to z: $\partial_z(z^n) = n \cdot z^{n-1}$
$\Theta_z = z \cdot \partial_z$, such that $\Theta_z(z^n) = n \cdot z^n$

Assume we have w white, b black balls $\Longrightarrow$ encoded by $x^w y^b$:

$$\frac{\binom{m}{k}}{(b+w)^{\underline{m}}} y^{m-k} \partial_y^{m-k} x^k \partial_x^k (x^w y^b) = \frac{\binom{m}{k} w^{\underline{k}} b^{\underline{m-k}}}{(b+w)^{\underline{m}}} x^w y^b,$$

$$\frac{\binom{m}{k}}{(b+w)^m} \Theta_x^k \Theta_y^{m-k} (x^w y^b) = \frac{\binom{m}{k} w^k b^{m-k}}{(b+w)^m} x^w y^b.$$

Analysis using Analytic combinatorics

Let $H_n(x,y) = \mathbb{E}(x^{W_n} y^{B_n}) H_n$ and $H(x,y,z) = \sum_{n\geqslant 0} H_n(x,y) \frac{z^n}{(n!)^m}$ denote the complete history generating function

Proposition (Morcrette and K.)

Starting with W_0 *white and* B_0 *black balls the generating function of all urn histories* $H(x,y;z)$ *satisfies*

$$\mathcal{D} * H(x,y,z) = \frac{1}{z} \cdot \Theta_z^m * H(x,y,z),$$

with

$$\mathcal{D}_M = \sum_{k=0}^{m} \binom{m}{k} x^{a_{m-k}+k} y^{b_{m-k}+m-k} \partial_x^k \partial_y^{m-k},$$

and

$$\mathcal{D}_R = \sum_{k=0}^{m} \binom{m}{k} x^{a_{m-k}} y^{b_{m-k}} \Theta_x^k \Theta_y^{m-k}.$$

Analysis using Analytic combinatorics

Proposition (Linear affine model - AC version)

The history generating function $H_x(1,1,z)$ *of the expected value* $\mathbb{E}(W_n)$ *satisfies an ordinary differential equation if the coefficients satisfy an affinity condition:* $a_k = a_0 + hk$, *with* h *an integer guaranteeing tenability. Then,*

$$E_xE_y\partial_x\mathcal{D} = \begin{cases} \big(m(1-h)\cdot\mathcal{V}^{\underline{m-1}} + \mathcal{V}^{\underline{m}}\big)E_xE_y\partial_x + (a_0+hm)\mathcal{V}^{\underline{m}}E_xE_y, \\ \big(m(1-h)\cdot\mathcal{V}^{m-1} + \mathcal{V}^{m}\big)E_xE_y\partial_x + (a_0+hm)\mathcal{V}^{m}E_xE_y, \end{cases}$$

with $\mathcal{V} = \sigma\Theta_z + T_0$. *Thus,*

$$\big(m(1-h)\cdot\mathcal{V}^{\underline{m-1}} + \mathcal{V}^{\underline{m}} - \frac{1}{z}\cdot\Theta_z^m\big)H_x(1,1,z) = -(a_0+hm)\mathcal{V}^{\underline{m}}H(1,1,z)$$

for sampling without replacement and

$$\big(m(1-h)\cdot\mathcal{V}^{m-1} + \mathcal{V}^{m} - \frac{1}{z}\cdot\Theta_z^m\big)H_x(1,1,z) = -(a_0+hm)\mathcal{V}^{m}H(1,1,z)$$

for sampling with replacement (with $H(1,1,z)$ *explicitly given and* $E_xf(x) = f(1)$*).*

Urn models

- Outlook

Urn models

- Outlook

Outlook and conclusion

Conclusion

- Characterization of linear affine urns
- Extension of the trichotomy - small, large and triangular urns - from $m = 1$ to $m \geqslant 1$ for linear affine urns.
- Analytic combinatorics - PDEs and an alternative derivation of the linear affine urns.

Outlook

We obtain higher order PDEs using Morcrette's approach also for

- r-colors
- unbalanced models
- ordered samples

Linear affine models: We can characterize and analyze r-colors models (joint work with Hosam Mahmoud).

Thanks for your **attention**!

Danke für Ihre **Aufmerksamkeit!**

Merci de votre **attention!**