A functional central limit theorem for branching random walks, with applications to Quicksort asymptotics

Rudolf Grübel
Leibniz Universität Hannover

based on joint work with

Zakhar Kabluchko
Westfälische Wilhelms Universität Münster

Strobl, AofA 2015

Background and history

Let X_n be the number of comparisons needed by $\textsc{Quicksort}$ to sort the first n in a sequence $(U_m)_{m\in\mathbb{N}}$ of independent uniforms, using the respective first element of the list as a pivot.

- Régnier (1989) 'found the martingale',

$$Y_n := (X_n - EX_n)/(n+1) \to Y_\infty \text{ almost surely, as } n \to \infty,$$

- Rösler (1989) characterized $\mathcal{L}(Y_\infty)$ as a fixed point of a contraction,
- Neininger (2014) obtained an associated second order result,

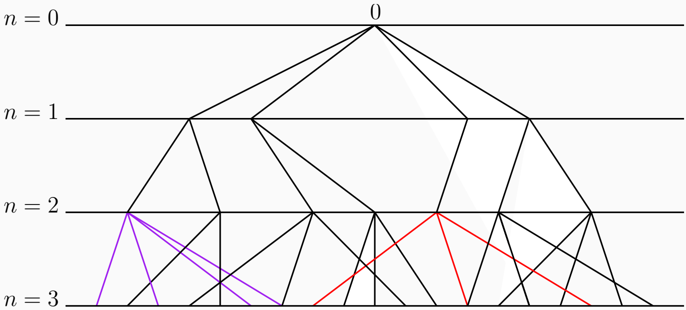
$$\sqrt{n/(2\log n)}\big(Y_n - Y_\infty\big) \to N(0,1) \;\text{ in distribution,}$$

  using the contraction method. (Fuchs (2015): Method of moments)

We will use the well-established link to branching random walks (Biggins, Chauvin, Devroye, Marckert, Roualt and others) to obtain

- yet another proof, indeed: of a stronger result,
- similar results for other (tree) models.

BRW: Basics



Combines splitting (branching) and shifting (random walk).

Basic parameter: (distribution of) a point process

$$\zeta = \sum_{i=1}^{N} \delta_{z_j}, \quad \text{where } \begin{cases} N & : \text{(random) no. of children,} \\ z_1, \dots, z_N & : \text{(random) location shifts,} \end{cases}$$

with i.i.d. copies of ζ for the separate individuals.

BRW: The Biggins martingale

- $(Z_n)_{n\in\mathbb{N}_0}$: a BRW with parameter $\mathcal{L}(\eta)$,
- ν with $\nu(A) := \mathbb{E}\zeta(A)$ is the intensity measure of ζ,
- $\nu_n(A) := \mathbb{E}Z_n(A)$ is the intensity measure of Z_n,
- by construction, $Z_1 = \eta$ and $\nu_1 = \nu$.

Let '^' denote the resp. moment generating function,

$$\hat{\zeta}(\theta) = \int e^{\theta x}\zeta(dx), \quad \hat{Z}_n(\theta) = \int e^{\theta x} Z_n(dx), \quad \hat{\nu}_n(\theta) = \int e^{\theta x}\nu_n(dx).$$

Fundamental observation:

$$\nu_n = \nu^{\star n}, \ \hat{\nu}_n(\theta) = \nu(\theta)^n.$$

As a consequence,

$$W_n(\theta) := \hat{\nu}(\theta)^{-n}\hat{Z}_n(\theta), \ n \in \mathbb{N},$$

is a martingale.

BRW: Assumptions

(A) $P\big(\zeta(\mathbb{R}) = 0\big) = 0,\ P\big(\zeta(\mathbb{R}) < \infty\big) = 1,\ P\big(\zeta(\mathbb{R}) = 1\big) < 1.$

(B) For some $p_0 > 2$, $\theta_0 > 0$,

$$\mathbb{E}\big(\hat{\zeta}(\theta)\big)^{p_0} < \infty \ \text{ for all } \theta \in (-\theta_0, \theta_0).$$

Let $N_n := Z_n(\mathbb{R})$ be the (random) number of particles in generation n. (A) implies $m := \mathbb{E}\zeta(\mathbb{R}) > 1$, hence $(N_n)_{n\in\mathbb{N}_0}$ is supercritical, so that

$$m^{-n} N_n \to \ N_\infty \ \text{ almost surely.}$$

(A) and (B) also imply that $W_n(\theta) \to W_\infty(\theta)$ almost surely, which we regard as a strong law of large numbers for the Biggins martingale.

We will need

$$\sigma^2 := \mathrm{var}(N_\infty), \quad \tau^2 := (\log \hat{\nu})''(0).$$

BRW: A functional central limit theorem

Fix $R > 0$ and let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq R\}$. Regard

$$D_n(u) := m^{n/2}\left(W_\infty\left(\frac{u}{\sqrt{n}}\right) - W_n\left(\frac{u}{\sqrt{n}}\right)\right)$$

as a random element of the space $\mathbb{A}$ of continuous function $f : \mathbb{D} \to \mathbb{C}$ that are analytic on $\mathbb{D}^\circ$. Endow $\mathbb{A}$ with uniform convergence.

Theorem

As $n \to \infty$, D_n converges in distribution to a limit process D_∞. The distribution of D_∞ is the same as the distribution of

$$\mathbb{D} \ni u \mapsto \sigma\sqrt{N_\infty}\,\xi(\tau u),$$

with N_∞, ξ independent and ξ the random analytic function

$$\xi(u) := \sum_{k=1}^{\infty} \xi_k \frac{u^k}{\sqrt{k!}}, \quad (\xi_k)_{k\in\mathbb{N}} \text{ i.i.d. } N(0,1).$$

From BRW to Quicksort

- Switch to continuous time: particles have independent and exponentially distributed lifetimes.
- Generalize the FCLT from fixed times $n = 1, 2, \ldots$ to stopping times $\tau_1, \tau_2, \ldots$ with $\tau_n \uparrow \infty$.
- Use the continuous time BRW with $\eta := 2\,\delta_1$.
- With τ_n the time of birth of the nth particle, Z_{τ_n} is the profile of the Quicksort tree of sublists.
- The punchline:
  - the quantity of interest is the mean of the profile,
  - on the transform side, mean means taking the derivative,
  - the functional $\mathbb{A} \ni \phi \mapsto \phi'(0)$ is continuous.

Now use the FCLT and the Continuous Mapping Theorem.

The strengthening

- We may regard Neininger's result as a martingale CLT.
- It is known that these often hold in the stronger sense of stable convergence (Rényi).

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the martingale filtration.
Basic idea:

Instead of $\mathcal{L}(D_n)$ consider $\mathcal{L}[D_n|\mathcal{F}_n]$.

- These conditional distributions are random variables with values in a space probability measures.
- The space of probability measures on the topological space $\mathbb{A}$ of analytic functions is itself a topological space $\mathcal{M}$ if endowed with weak convergence.

Theorem

In $\mathcal{M}$, $\mathcal{L}[D_n|\mathcal{F}_n]$ converges almost surely to $\mathcal{L}(D_\infty)$.

Back to basics: the Pólya urn

- Start with one black and one white ball. In each step, select one ball u.a.r., and add one ball of the same colour. Let X_n be the number of white balls added after n steps, $X = (X_n)_{n \in \mathbb{N}}$.
- $Y_n := X_n/(n+2)$, $n \in \mathbb{N}$, is a bd. martingale, hence $Y_n \to Y_\infty$ a.s..
- Boundary theory (or direct calculation) says that X, given $Y_\infty = \theta$, is a simple random walk on $\mathbb{Z}$, with θ the probability for a move to the right.
- Let $W_n := \sqrt{n}(Y_\infty - Y_n)$. Standard CLT gives

$$\mathcal{L}[W_n | Y_\infty = \theta] \to N\big(0, \theta(1-\theta)\big) \text{ in distribution.}$$

- We may rewrite this as a.s. convergence of the random p.m. $\mathcal{L}[W_n | Y_\infty]$ to the random (!) p.m. $N\big(0, Y_\infty(1 - Y_\infty)\big)$.

Theorem

$\mathcal{L}[W_n | X_1, \ldots, X_n]$ *converges a.s. to* $N\big(0, Y_\infty(1 - Y_\infty)\big)$.

(Proof uses some beta-gamma algebra.)