The Node Profile of Symmetric Digital Search Trees

(joint with M. Drmota, H.-K. Hwang and R. Neininger)

Michael Fuchs

Department of Applied Mathematics
National Chiao Tung University

June 8th, 2015

Node Profile of (Rooted) Trees

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$I_{n,k}$ = number of internal nodes at level k.

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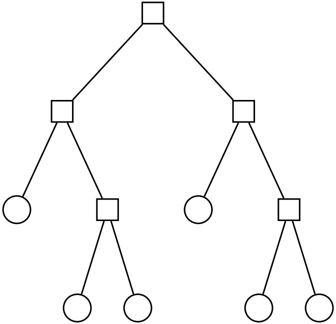

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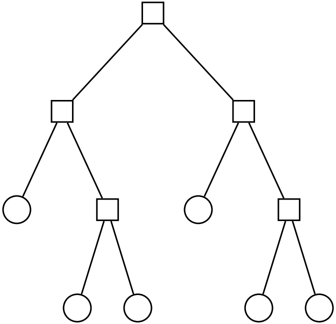

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$B_{5,2} = 2,$

$B_{5,3} = 4,$

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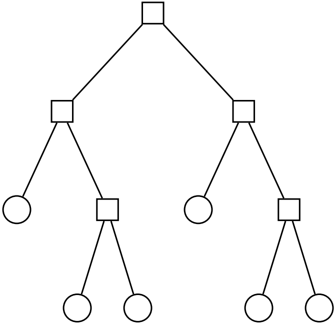

$B_{5,0} = 0,$ $I_{5,0} = 1;$

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$B_{5,3} = 4,$ $I_{5,3} = 0.$

Relations to Other Shape Parameters

Many shape parameters can by analyzed through the profile.

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- Depth: $P(D_n = k) = B_{n,k}/(n+1)$;
- Width: $\max\{B_{n,k} \ : \ k \geq 0\}$;
- Total Path Length: $\sum_k kB_{n,k}$;
- Height: $\max\{k \ : \ B_{n,k} > 0\}$;
- Shortest Path: $\min\{k \ : \ B_{n,k} > 0\}$;
- Fill-up Level: $\max\{k \ : \ I_{n,k} = 2^k\}$;
- Etc.

Profile of Random Trees

- $\sqrt{n}$-Trees:

  Aldous (1991); Drmota and Gittenberger (1997); Kersting (1998); Pitman (1999); etc.

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- $\log n$-Trees:
  - Binary Search Trees: Chauvin, Drmota, Jabbour-Hattab (2001); Drmota and Hwang (2005); F., Hwang, Neininger (2006).
  - Recursive Trees: Drmota and Hwang (2005); F., Hwang, Neininger (2006).
  - Plane-oriented Recursive Trees: Hwang (2007).
  - m-ary Seach Trees: Drmota, Janson, Neininger (2008).

Tries

René de la Briandais (1959)

Name from data retrieval (suggested by Fredkin).

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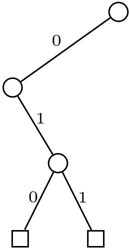

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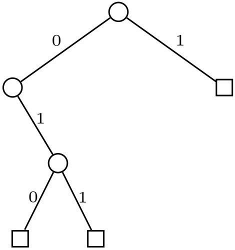

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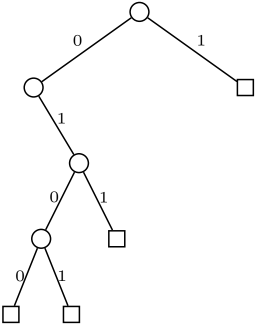

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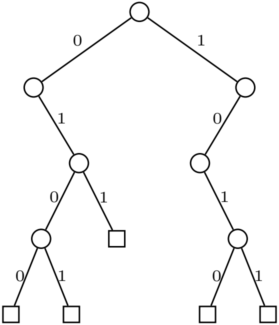

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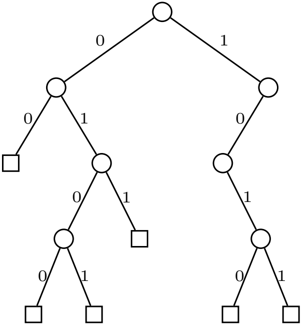

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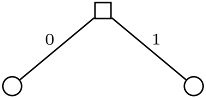

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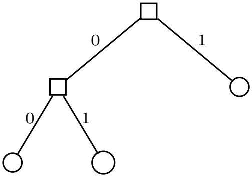

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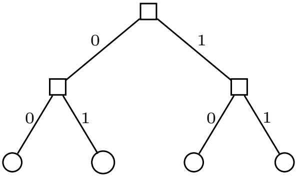

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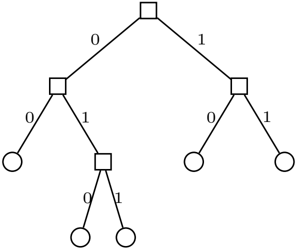

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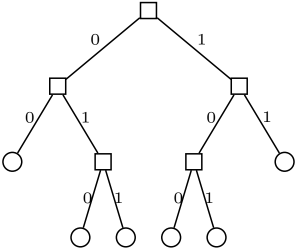

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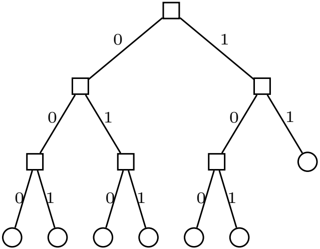

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In this talk, we are interested in mean, variance and limit laws of the profile for symmetric DSTs.

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  Variance & limit laws: Drmota, F., Hwang, Neininger ($\rightarrow$ this talk).

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Profile of Tries

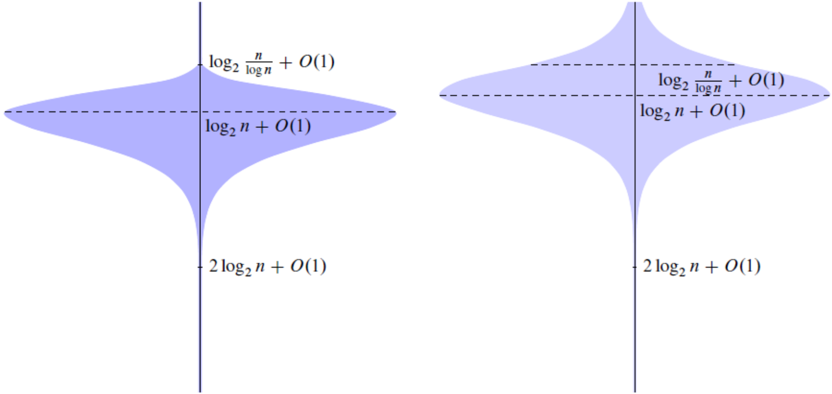
G. Park[1], H.-K Hwang[2], P. Nicodème[3], and W. Szpankowski[4]

[1] Department of Computer Science, State University of New York at Geneseo, Geneseo, 14554, USA
park@geneseo.edu
[2] Institute of Statistical Science, Academia Sinica, 11529 Taipei, Taiwan
hkhwang@stat.sinica.edu.tw
[3] Laboratory LIX, École polytechnique, 91128 Palaiseau Cedex, France
nicodeme@lix.polytechnique.fr
[4] Department of Computer Sciences, Purdue University, 250 N. University Street, West Lafayette, Indiana, 47907-2066, USA
spa@cs.purdue.edu

Abstract. The *profile* of a trie, the most popular data structures on words, is a parameter that represents the number of nodes (either internal or external) with the same distance to the root. Several, if not

Plot of Mean Profile of Symmetric Tries

Hwang, Nicodéme, Park, Szpankowski (2009):



Symmetric Tries: Mean

We have,

$$\mu_{n,k} := \mathbb{E}(B_{n,k}) \sim \begin{cases} n(1-2^{-k})^{n-1}, & \text{if } 2^{-k}n \to \infty; \\ \tilde{M}_{k,1}(n), & \text{if } 4^{-k}n \to 0, \end{cases}$$

where

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$$\tilde{M}_{k,1}(n) \sim \begin{cases} ne^{-n/2^k}, & \text{if } 2^{-k}n \to \infty; \\ \Theta(n), & \text{if } 2^{-k}n = \Theta(1); \\ 2^{-k}n^2, & \text{if } 2^{-k}n \to 0. \end{cases}$$

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Thus, the profile has maximum of order n (asymmetric tries: $n/\sqrt{\log n}$)

Symmetric Tries: Variance

We have,

$$\sigma_{n,k}^2 := \mathrm{Var}(B_{n,k}) \sim \begin{cases} n(1-2^{-k})^{n-1}, & \text{if } 2^{-k}n \to \infty; \\ \tilde{V}_k(n), & \text{if } 4^{-k}n \to 0, \end{cases}$$

where

$$\tilde{V}_k(z) = z(e^{-z/2^k} - e^{-z/2^{k-1}}) + 2^{-k}z^2 e^{-z/2^{k-1}} - 2^{1-k}z^2(e^{-z/2^k} - e^{-z/2^{k-1}})^2.$$

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In particular,

$$\tilde{V}_k(n) \sim \begin{cases} ne^{-n/2^k} \sim \tilde{M}_k(n), & \text{if } 2^{-k}n \to \infty; \\ \Theta(n), & \text{if } 2^{-k}n = \Theta(1); \\ 2^{1-k}n^2 \sim 2\tilde{M}_k(n), & \text{if } 2^{-k}n \to 0. \end{cases}$$

Poissonization and Depoissonization

Poisson Model: Build digital tree from Poisson-distributed number of records.

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Poisson moments:

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$$\tilde{M}_{k,\ell}(z) \text{ sufficiently smooth} \quad \Longrightarrow \quad \mathbb{E}(B^{\ell}_{n,k}) \approx \tilde{M}_{k,\ell}(n).$$

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Poisson heuristic made precise by the **Theory of Analytic Depoissonization** (Jacquet & Szpankowski; 1998).

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With this choice:

$$\mathrm{Var}(B_{n,k}) \sim \tilde{V}_k(n)$$

when $4^{-k}n \to 0$.

Symmetric DSTs: Mean

Let

$$Q(z) = \prod_{\ell=1}^{\infty} \left(1 - z2^{-\ell}\right), \qquad Q_n = \prod_{\ell=1}^{n} \left(1 - 2^{-\ell}\right) = \frac{Q(2^{-n})}{Q(1)}.$$

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Theorem

We have,

$$\mu_{n,k} \begin{cases} \sim \dfrac{2^k}{Q_k}\left(1 - 2^{-k}\right)^n, & \text{if } 2^{-k}n \to \infty; \\ = 2^k F(n/2^k) + \mathcal{O}(1), & \text{if } 4^{-k}n \to 0, \end{cases}$$

where $F(x)$ is the positive function

$$F(x) = \sum_{j\geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q(1)} e^{-2^j x}.$$

$F(x)$ (i)

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and as $x \to 0$,

$$F(x) \sim \frac{X^{1/\log 2}}{\sqrt{2\pi x}} \exp\left(-\frac{(\log X \log X)^2}{\log 2} - \sum_{j\in\mathbb{Z}} c_j (X \log X)^{-\chi_j}\right),$$

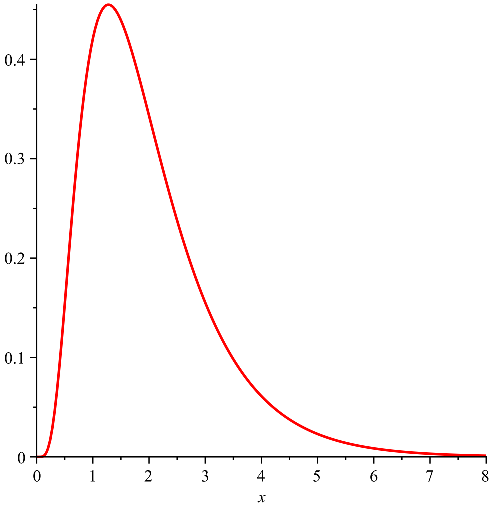
where $X = 1/(x \log 2)$, $\chi_j = 2j\pi i/\log 2$,

$$c_0 = \frac{\log 2}{12} + \frac{\pi^2}{6 \log 2}$$

and

$$c_j = \frac{1}{2j \sinh(2j\pi/\log 2)}, \qquad (j \neq 0).$$

$F(x)$ (ii)



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$$\tilde{M}_{k,1}(z) = 2^k \sum_{0\le j\le k} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q_{k-j}} e^{-z/2^{k-j}}.$$

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From this,

$$\mu_{n,k} = 2^k \sum_{0\le j\le k} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q_{k-j}} \left(1 - 2^{j-k}\right)^n.$$

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This is useful if $n2^{-k} \to \infty$.

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Lemma

We have,

$$\tilde{M}_{k,1}(z) = 2^k \sum_{r\geq 0} \frac{2^{-\binom{r+1}{2}-kr}}{Q_r} F^{(r)}\left(\frac{z}{2^k}\right).$$

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Lemma

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This gives,

$$\tilde{M}_{k,1}(z) = 2^k F\left(\frac{z}{2^k}\right) + \mathcal{O}(1).$$

Result follows from depoissonization.

Symmetric DSTs: Variance

Theorem (Drmota, F., Hwang, Neininger)

We have,

$$\sigma_{n,k}^2 \begin{cases} \sim \dfrac{2^k}{Q_k}\left(1-2^{-k}\right)^n, & \text{if } 2^{-k}n \to \infty; \\ = 2^k H(n/2^k) + \mathcal{O}(1), & \text{if } 4^{-k}n \to 0, \end{cases}$$

where $H(x)$ is a function with

$$H(x) = \frac{e^{-x}}{Q(1)} + \mathcal{O}(xe^{-2x}), \qquad (x \to \infty)$$

and

$$H(x) \sim 2F(x), \qquad (x \to 0).$$

$H(x)$ (i)

We have,

$$H(x) = \sum_{j,r=0}^{\infty} \sum_{0\le h,\ell\le j} \frac{2^{-j}(-1)^{r+h+\ell}2^{-\binom{r}{2}-\binom{h}{2}-\binom{\ell}{2}+2h+2\ell}}{Q_r Q(1) Q_h Q_{j-h} Q_\ell Q_{j-\ell}} \varphi(2^{r+j}, 2^h + 2^\ell; x),$$

where

$$\varphi(u,v;x) = \begin{cases} \dfrac{e^{-ux} - ((v-u)x+1)e^{-vx}}{(v-u)^2}, & \text{if } u \neq v; \\ x^2 e^{-ux}/2, & \text{if } u = v. \end{cases}$$

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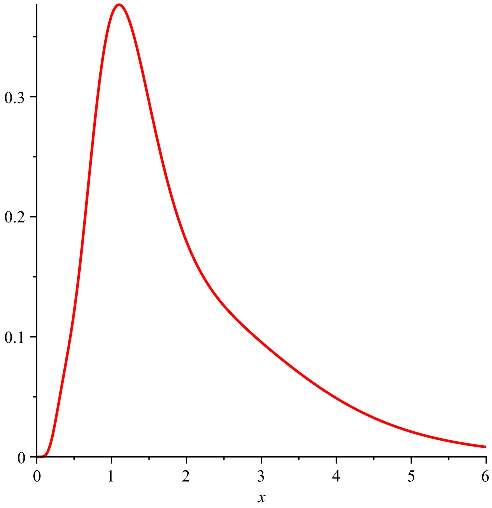
Proposition (Drmota, F., Hwang, Neininger)

$H(x)$ is a positive function on $(0,\infty)$.

$H(x)$ (ii)



Some Details of the Proof (i)

We have,

$$\tilde{V}_k(z) + \tilde{V}'_k(z) = 2\tilde{V}_{k-1}(z/2) + z\tilde{M}''_{k,2}(z)^2.$$

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By Laplace transform and its inverse,

$$\tilde{V}_k(z) = \sum_{(j,r,h,\ell)\in\mathscr{V}} \frac{2^{k-j}(-1)^{r+h+\ell}2^{-\binom{r}{2}-\binom{h}{2}-\binom{\ell}{2}+2h+2\ell}}{Q_rQ_{k-j-r}Q_hQ_{j-h}Q_\ell Q_{j-\ell}}\varphi\left(2^{r+j}, 2^h+2^\ell, \frac{z}{2^k}\right)$$

with

$$\mathscr{V} = \{(j,r,h,\ell)\ :\ 0 \le j \le k, 0 \le r \le k-j, 0 \le h, \ell \le j\}$$

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Lemma

We have,

$$\tilde{V}_k(z) = 2^k \sum_{m\geq 0} \frac{2^{-\binom{m+1}{2}-km}}{Q_m} H^{(m)}\left(\frac{z}{2^k}\right).$$

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The Laplace transform of $H(z)$:

$$\mathscr{L}[H(z); s] = \sum_{j\ge 0} 4^{-j} \frac{\tilde{g}_j^*(2^{-j}s)}{Q(-2^{1-j}s)}$$

where

$$\tilde{g}_j^*(s) = \sum_{0\le k,\ell\le j} \frac{(-1)^{h+\ell} 2^{-\binom{h}{2}-\binom{\ell}{2}+2h+2\ell}}{Q_k Q_{j-k} Q_\ell Q_{j-\ell}} \frac{1}{(2^j s + 2^h + 2^\ell)^2}.$$

Some Details of the Proof (iii)

Lemma

We have, as $s \to \infty$,

$$\frac{\tilde{g}_0^*(s)}{Q(-2s)} \sim \frac{1}{s^2 Q(-2s)}, \qquad 4^{-1}\frac{\tilde{g}_1^*(2^{-1}s)}{Q(-s)} \sim \frac{9}{sQ(-2s)}$$

and, for $j \geq 2$,

$$4^{-j}\frac{\tilde{g}_j^*(2^{-j}s)}{Q(-2^{1-j}s)} \sim \frac{(2j-3)!}{((j-2)!)^2}\frac{2^{\binom{j}{2}}}{s^{j-2}Q(-2s)}.$$

Thus,

$$\mathscr{L}[H(z);s] \sim \frac{2}{Q(-2s)}$$

and hence, $H(x) \sim 2F(x)$ *as* $x \to 0$.

Symmetric DSTs: Limit Laws

Corollary (Drmota, F., Hwang, Neininger)

We have,

$$\mu_{n,k} \longrightarrow \infty \quad \textit{iff} \quad \sigma_{n,k}^2 \longrightarrow \infty.$$

Theorem (Drmota, F., Hwang, Neininger)

Assume that $\mu_{n,k} \longrightarrow \infty$. Then,

$$\frac{B_{n,k} - \mu_{n,k}}{\sigma_{n,k}} \stackrel{d}{\longrightarrow} N(0,1),$$

where $N(0,1)$ denotes a standard normal distribution.

Application to the Height

H_n =height of a symmetric DST of size n.

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Theorem (Drmota, F., Hwang, Neininger)

Set

$$k_n = \min\{k \geq \log_2 n \ : \ 2^k F(n/2^k) \leq 1\}.$$

Then,

$$k_n = \log_2 n + \sqrt{2 \log_2 n} - \log_2 \left(\sqrt{\log_2 n}\right) + \mathcal{O}(1).$$

Moreover,

$$P(H_n = k_n - 2 \text{ or } H_n = k_n - 1) \to 1.$$

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This solves an open problem of Aldous & Shields.

Summary of Results for Symmetric DSTs

- Mean profile tends to infinity when k is roughly in the range

$$\log_2 n - \log_2 \log n \leq k \leq \log_2 n + \sqrt{2 \log_2 n};$$

otherwise it is bounded.

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- Our results have many applications, e.g., they allow us to solve a problem of Aldous & Shields.