Best Simultaneous Diophantine Approximations
under a Constraint on the Denominator

by Iskander ALIEV*

Technische Universität Wien

Institut für Diskrete Mathematik und Geometrie
Forschungsgruppe Konvexe und Diskrete Geometrie
Technische Universität Wien
Wiedner Hauptstrasse 8-10 / 1046
A-1040 Wien, Austria
ialiev@osiris.tuwien.ac.at
Tel.: +43 1 58801 10461
Fax: +43 1 58801 10496

*The author was supported by FWF Austrian Science Fund, project M672.
Abstract. This paper studies the problem of best Diophantine approximations under a constraint on the denominator proposed by W. B. Jurkat in [9]. We give new lower estimates for optimal approximation constants in terms of critical determinants of suitable convex bodies. To obtain these results we study in details Diophantine approximations of rationals by rationals with smaller denominators. Finally, we apply results on such approximations to the problem of decomposition of integer vectors.

Keywords: Convex body, critical determinant, integer vector, modular lattice, polar lattice, successive minima

2000 MS Classification: 11J13, 11H06
1 Introduction

The general theory of simultaneous Diophantine approximations with constraints was presented by W. B. Jurkat in [9]. W. Kratz [10], [11] considered the following particular problem. Let \( x \in \mathbb{R}^k, k \geq 2 \) and \( g(y) = \|y\|_2 \). For \( Q > 0 \) we introduce the successive minima \( \lambda_i = \lambda_i(x, Q), i \in \{1, \ldots, k+1\} \) as in [10], i.e. \( \lambda_i \) is the minimum of all non-negative numbers \( \lambda \), such that there exist \( i \) linearly independent vectors \( p_1, \ldots, p_i \in \mathbb{Z}^{k+1} \) with

\[
g(p_{jk+1}x - (p_j1, \ldots, p_jk)) \leq \lambda, \quad |p_{jk+1}| \leq Q, \quad j \in \{1, \ldots, i\}.
\]

We are interested in an optimal constant \( c = c(k, \| \cdot \|_2) \) such that

\[
\lambda_1 \cdots \lambda_k < c Q.
\]

W. Kratz proved in [11] that \( c(2, \| \cdot \|_2) = 2/\sqrt{3} \).

Let now \( g(y) \) be the distance function of a centrally symmetric convex body \( K \) in \( \mathbb{R}^k \). In this paper we consider the above problem with respect to the function \( g(y) \) and show (Theorem 3 below) that

\[
c(k, g) \geq (\Delta(K))^{-1},
\]

where \( \Delta(K) \) is the critical determinant of \( K \). Let \( \gamma_k \) denote the Hermite constant. Since the critical determinant of the \( k \)-dimensional unit ball equals \( \gamma_k^{-k/2} \), we conclude that \( c(k, \| \cdot \|_2) \geq \gamma_k^{k/2} \).

To obtain these results we will study in details simultaneous Diophantine approximations of rationals by rationals with smaller denominators. Namely, let \( n = (n_1, \ldots, n_k, n_{k+1}), k \geq 2 \) be an integer vector. Assume that \( 0 < n_1 \leq \ldots \leq n_{k+1} \) and that \( \gcd(n_1, \ldots, n_{k+1}) = 1 \). We consider the problem of approximation of the vector \((n_1/n_{k+1}, \ldots, n_k/n_{k+1})\) by vectors \((m_1/m_{k+1}, \ldots, m_k/m_{k+1})\) with \( m_i \in \mathbb{Z}, i \in \{1, \ldots, k+1\} \) and \( 0 \leq m_{k+1} < n_{k+1} \). More precisely, we study the behavior of the points

\[
\left(\frac{m_1 - m_{k+1} n_1}{n_{k+1}}, \ldots, \frac{m_k - m_{k+1} n_k}{n_{k+1}}\right),
\]

when \( m = (m_1, \ldots, m_k, m_{k+1}) \) runs through all vectors from \( \mathbb{Z}^{k+1} \). Since the points (1) form a \( k \)-dimensional lattice \( \Lambda(n) \) (see Section 2 for detail), the main tools of this research belong to geometry of numbers. Using a different approach, J. Lagarias and J. Hastad [12] study the number \( N(n, \Delta) \) of vectors \( v \in \Lambda(n) \) such that

\[
\|v\|_\infty \leq \frac{\Delta}{n_{k+1}}.
\]
M. Lempel and A. Paz [13] and G. Rote [15] consider an algorithmic problem concerning \( \Lambda(n) \) for \( n \in \mathbb{Z}^3 \).

We shall use the following notation. By \( \alpha(K) \) we denote the anomaly of a set \( K \), and if \( \Lambda \) is a lattice then \( \Lambda^* \) will denote its polar lattice (also called the reciprocal lattice or dual lattice). For detailed information on these objects and geometry of numbers in general, see e. g. [7].

The first result of this paper presents a connection between \( \Lambda(n) \) and the lattice \( \Lambda^\perp(n) \) of integer vectors orthogonal to the vector \( n \). Namely, let \( \Lambda^\perp_{k+1}(n) \) be the \( k \)–dimensional lattice obtained by omitting the \( (k+1) \)–th coordinate in \( \Lambda^\perp(n) \), then the following lemma holds.

**Lemma 1.** The lattice \( \Lambda^\perp_{k+1}(n) \) is the polar lattice for \( \Lambda(n) \),

\[
\Lambda^\perp_{k+1}(n) = (\Lambda(n))^*.
\]

This result, together with a technique introduced by A. Schinzel in [17], provide us with a way to study the lattice \( \Lambda(n) \). Namely, the next theorem shows that, roughly speaking, given an arbitrary lattice \( \Lambda \subset \mathbb{Q}^k \), we can construct a sequence of integer vectors \( n(t) \) such that the sequence of corresponding lattices \( \Lambda(n(t)) \) after an appropriate normalization tends to \( \Lambda \).

**Theorem 1.** For any rational lattice \( \Lambda \) with basis \( b_1, \ldots, b_k \in \mathbb{Q}^k \) and for all rational numbers \( \alpha_1, \ldots, \alpha_k \) with \( 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1 \), there exist an arithmetic progression \( P \) and a sequence \( n(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t)) \in \mathbb{Z}^{k+1}, t \in P, \) such that

\[
gcd(n_1(t), \ldots, n_k(t), n_{k+1}(t)) = 1
\]

and \( \Lambda(n(t)) \) has a basis \( a_1(t), \ldots, a_k(t) \) with

\[
a_{ij}(t) = \frac{b_{ij}}{dt} + O\left(\frac{1}{t^2}\right), \quad i, j \in \{1, \ldots, k\}, \tag{2}
\]

where \( d \in \mathbb{N} \) such that \( db_{ij}, d\alpha_j b_{ij} \in \mathbb{Z} \) for all \( i, j \in \{1, \ldots, k\} \). Moreover,

\[
||n(t)||_\infty = \frac{d^{k+1}}{\det \Lambda} + O(t^{k-1}) \tag{3}
\]

and

\[
\alpha_i(t) := \frac{n_i(t)}{n_{k+1}(t)} = \alpha_i + O\left(\frac{1}{t}\right). \tag{4}
\]
Let $\lambda_i(K, \Lambda)$ denote the $i$–th successive minimum of the lattice $\Lambda$ w. r. t. the set $K$. The following theorem presents the main result of this paper on Diophantine approximations of rationals.

**Theorem 2.** Let $K$ be a centrally symmetric convex body in $\mathbb{R}^k$ and 
$$U^{k+1} = \{ x = (x_1, \ldots, x_{k+1}) \in \mathbb{Z}^{k+1} : 0 < x_1 \leq \cdots \leq x_{k+1}, \gcd(x_1, \ldots, x_{k+1}) = 1 \}.$$ 

Then 
$$C(K) := \sup_{n \in U^{k+1}} \frac{\lambda_1(K, \Lambda(n)) \cdots \lambda_k(K, \Lambda(n))}{\det \Lambda(n)} = \frac{\alpha(K)}{\Delta(K)}.$$ 

Moreover, for all $\alpha_1, \ldots, \alpha_k \in \mathbb{Q}$, $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1$ there exists an infinite sequence of integer vectors $n(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t)) \in U^{k+1}$, $t \in T = \{t_1, t_2, \ldots \}$, such that 
$$\lim_{t \to \infty, t \in T} \frac{\lambda_1(K, \Lambda(n(t))) \cdots \lambda_k(K, \Lambda(n(t)))}{\det \Lambda(n(t))} = C(K),$$ 

$$\lim_{t \to \infty, t \in T} \frac{n_i(t)}{n_{k+1}(t)} = \alpha_i, \quad i \in \{1, \ldots, k\}.$$ 

and 
$$\lim_{t \to \infty, t \in T} n_{k+1}(t) = \infty.$$ 

The proof of Theorem 2 is based on the following lemma of independent interest.

**Lemma A.** If $\Lambda_t$ is a sequence of lattices in $\mathbb{R}^k$ convergent to a full lattice $\Lambda$ and $K$ is a centrally symmetric convex body then for each $i \leq k$ 
$$\lim_{t \to \infty} \lambda_i(K, \Lambda_t) = \lambda_i(K, \Lambda).$$ 

This result was recently proved in the joint paper with A. Schinzel and W. M. Schmidt [3].
Theorem 3. Let $g(y)$ be the distance function of a centrally symmetric convex body $K$ in $\mathbb{R}^k$. Then
\[ c(k, g) \geq (\Delta(K))^{-1}. \]

In Section 7 we apply Theorem 1 to the problem of decomposition of integer vectors. We consider the problem with respect to sup norm. For recent results on this problem with respect to $|| \cdot ||_2$–norm see [3]. By a tradition we denote the sup norm of a vector $a$ by $h(a)$.

Given $m$ linearly independent vectors $n_1, \ldots, n_m$ in $\mathbb{Z}^{k+1}$ let $H(n_1, \ldots, n_m)$ denote the maximum of the absolute values of $m \times m$–minors of the matrix $(n_1^t, \ldots, n_m^t)$ and $D(n_1, \ldots, n_m)$ the greatest common divisor of these minors. Then $h(n) = H(n)$ for $n \neq 0$. Let for $k + 1 > l > m > 0$
\[ c_0(k + 1, l, m) = \sup \inf \left( \frac{D(n_1, \ldots, n_m)}{H(n_1, \ldots, n_m)} \right) \prod_{i=1}^{l} h(p_i), \quad (8) \]
where the supremum is taken over all sets of linearly independent vectors $n_1, \ldots, n_m$ in $\mathbb{Z}^{k+1}$ and the infimum is taken over all sets of linearly independent vectors $p_1, \ldots, p_l$ in $\mathbb{Z}^{k+1}$ such that for all $i \leq m$
\[ n_i = \sum_{j=1}^{l} u_{ij} p_j, \quad u_{ij} \in \mathbb{Q}. \]
It has been proved in [16] that for fixed $l, m$
\[ \limsup_{k \to \infty} c_0(k + 1, l, m) < \infty, \quad (9) \]
in [2] that
\[ c_0(k + 1, 2, 1) \leq \frac{2}{(k + 1)^{1/k}}, \]
and in [6] that $c_0(3, 2, 1) = 2/\sqrt{3}$. Note that
\[ c_0(k + 1, 2, 1) = \sup_{n \in \mathbb{Z}^{k+1} \setminus \{0\}} \inf_{\text{dim}(p,q)=2} \frac{h(p)h(q)}{h(n)^{1-1/k}}. \]
In this paper we continue to study the behavior of $c_0(k + 1, 2, 1)$ and prove the following theorem.
Theorem 4. For \( k \geq 3 \)

\[
\limsup_{n \in \mathbb{Z}^{k+1}} \inf_{h(n) \to \infty} \frac{h(p)h(q)}{h(n)^{1-1/k}} \geq \frac{1}{(k+1)^{1/k}}.
\]

This result improves an estimate of S. Chaładus [5] concerning a more general situation, which implies a weaker inequality with \( 1/2 \) on the right hand side.

2 Lattice \( \Lambda(n) \), rational Weyl sequences and systems of linear congruences

In this section we construct a special lattice such that its points correspond to points (1). Let us extend the vector \( n \) by vectors \( v_1, \ldots, v_k \) to a basis of the lattice \( \mathbb{Z}^{k+1} \). Consider \( k \)-dimensional vectors

\[
v'_i = (v_{i1} - v_{ik+1}n_1/n_{k+1}, \ldots, v_{ik} - v_{ik+1}n_k/n_{k+1}), \quad i \in \{1, \ldots, k\}.
\]

Since the equality

\[
A_1v'_1 + \ldots + A_kv'_k = 0
\]

implies

\[
n_{k+1}A_1v_1 + \ldots + n_{k+1}A_kv_k + A_{k+1}n = 0
\]

with \( A_{k+1} = -A_1v_{1k+1} - \ldots - A_kv_{kk+1} \), the vectors \( v'_1, \ldots, v'_k \) are linearly independent. Denote by \( \Lambda(n) \) the \( k \)-dimensional lattice with basis \( v'_1, \ldots, v'_k \). Since

\[
1 = \det \begin{pmatrix}
v_{11} & \ldots & v_{k1} & n_1 \\
\vdots & \ddots & \vdots & \vdots \\
v_{1k} & \ldots & v_{kk} & n_k \\
v_{1k+1} & \ldots & v_{kk+1} & n_{k+1}
\end{pmatrix}
\]

\[
= n_{k+1} \det \begin{pmatrix}
v_{11} - v_{1k+1}n_1/n_{k+1} & \ldots & v_{k1} - v_{kk+1}n_1/n_{k+1} & n_1/n_{k+1} \\
\vdots & \ddots & \vdots & \vdots \\
v_{1k} - v_{1k+1}n_k/n_{k+1} & \ldots & v_{kk} - v_{kk+1}n_k/n_{k+1} & n_k/n_{k+1} \\
0 & \ldots & 0 & 1
\end{pmatrix},
\]
we have det $\Lambda(n) = 1/n_{k+1}$. Further, as is easily seen, for every non–zero vector $v \in \Lambda(n)$ there exists a vector $m \in \mathbb{Z}^{k+1}$ such that

$$v = (m_1 - m_{k+1}n_1/n_{k+1}, \ldots, m_k - m_{k+1}n_k/n_{k+1})$$

and this vector can be uniquely chosen under condition $0 \leq m_{k+1} < n_{k+1}$. Therefore, there is 1–1 correspondence between non–zero points of $\Lambda(n)$ and non–zero integer vectors with $0 \leq m_{k+1} < n_{k+1}$. Note also that since $v \neq 0$ the vectors $m$ and $n$ are linearly independent.

The lattice $\Lambda(n)$ appears in some problems of number theory. Let $\theta_1, \ldots, \theta_k$, $k \geq 2$ be real numbers and $W_k$ be the sequence of $k$–dimensional vectors

$$(i\theta_1 \mod 1, \ldots, i\theta_k \mod 1), \quad i = 0, 1, 2 \ldots$$

$–k$–dimensional Weyl sequence. We shall consider the case when

$$\theta_1 = \frac{n_1}{n_{k+1}}, \ldots, \theta_k = \frac{n_k}{n_{k+1}}.$$ 

Then $W_k$ is $n_{k+1}$–periodic and the set

$$\Lambda(W_k) = \{x + y : x \in \mathbb{Z}^k, y \in W_k\}$$

is a $k$–dimensional lattice. It can be easily shown that

$$\Lambda(W_k) = \Lambda(n).$$

Let us consider the lattice $n_{k+1}\Lambda(n) = n_{k+1}\Lambda(W_k) \subset \mathbb{Z}^k$. The points (10) multiplied by $n_{k+1}$ can be written as

$$(in_1 \mod n_{k+1}, \ldots, in_k \mod n_{k+1}), \quad i = 0, 1, 2 \ldots$$

Therefore, any point $(x_1, \ldots, x_k) \in n_{k+1}\Lambda(n)$ is a solution of the system

$$\begin{cases} x_1 + rn_1 \equiv 0 \pmod{n_{k+1}} \\ \vdots \\ x_k + rn_k \equiv 0 \pmod{n_{k+1}} \end{cases}$$

for an integer $r$ which corresponds to $m_{k+1}$. Hence Theorems 1, 2 can be considered as results on rational Weyl sequences and solutions of the system (11).
3 Proof of Lemma 1

Let \( \mathbf{v} \) be a primitive non–zero vector of \( \Lambda(\mathbf{n}) \) and \( \mathbf{V} = n_{k+1} \mathbf{v} \). Let us choose a vector \( \mathbf{m} \in \mathbb{Z}^{k+1} \) such that

\[
\mathbf{v} = (m_1 - m_{k+1}n_1/n_{k+1}, \ldots, m_k - m_{k+1}n_k/n_{k+1}).
\]

Let \( \Lambda(\mathbf{m}, \mathbf{n}) \) denote the lattice with basis \( \mathbf{m}, \mathbf{n} \). By primitivity of \( \mathbf{v} \) we have

\[
\Lambda(\mathbf{m}, \mathbf{n}) = S(\mathbf{m}, \mathbf{n}) \cap \mathbb{Z}^{k+1},
\]

where \( S(\mathbf{m}, \mathbf{n}) \) denotes the subspace of \( \mathbb{Q}^{k+1} \) spanned by the vectors \( \mathbf{m}, \mathbf{n} \).

Consider the lattice \( \Lambda^\perp(\mathbf{m}, \mathbf{n}) \) of integer vectors orthogonal to \( S(\mathbf{m}, \mathbf{n}) \) and choose a basis

\[
\begin{align*}
\mathbf{a}_1' &= (a_{11}, \ldots, a_{1k}, a_{1k+1}), \\
\vdots \\
\mathbf{a}_{k-1}' &= (a_{k-11}, \ldots, a_{k-1k}, a_{k-1k+1}), \\
\mathbf{a}_k' &= (a_{k1}, \ldots, a_{kk}, a_{kk+1})
\end{align*}
\]

(12)

of the lattice \( \Lambda^\perp(\mathbf{n}) \) such that the first \( k-1 \) vectors \( \mathbf{a}_1' \ldots \mathbf{a}_{k-1}' \) form a basis for \( \Lambda^\perp(\mathbf{m}, \mathbf{n}) \). As is easily seen, vectors

\[
\begin{align*}
\mathbf{a}_1 &= (a_{11}, \ldots, a_{1k}), \\
\vdots \\
\mathbf{a}_k &= (a_{k1}, \ldots, a_{kk})
\end{align*}
\]

form a basis of \( \Lambda^\perp_{k+1}(\mathbf{n}) \). Consider the matrix

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1k} & a_{1k+1} \\
\vdots & & & \vdots \\
a_{k-11} & \cdots & a_{k-1k} & a_{k-1k+1}
\end{pmatrix}
\]

and denote by \( A_{ij} \) the minor obtained by omitting the \( i \)th and \( j \)th columns in \( A \). Let

\[
\mathbf{V}_i' = \mathbf{m}, \mathbf{n} - n_i \mathbf{m}
\]

and let \( \mathbf{V}_i \) be the vector obtained by omitting the \( i \)th coordinate in \( \mathbf{V}_i' \) (this coordinate is obviously equal to 0). Omitting the coordinate we assume that the numeration of remaining coordinates is preserved. For example, we consider \( \mathbf{V}_3 \) as a vector from the \( k \)–dimensional space with coordinates \( x_1, x_2, x_4, \ldots, x_{k+1} \). In particular, \( \mathbf{V}_{k+1} = \mathbf{V} \).

Also allow \( \Lambda^\perp_{k+1}(\mathbf{m}, \mathbf{n}) \) to denote the lattice obtained by omitting the \( i \)th coordinate in \( \Lambda^\perp(\mathbf{m}, \mathbf{n}) \) with the same rule for the numeration of remaining coordinates. Denote by \( V_{ij} \) the \( j \)th coordinate of \( \mathbf{V}_i \). The following result holds.

**Lemma 2.** \( V_{ij} = \epsilon_{ij} A_{ij} \), where \( \epsilon_{ij} = \pm 1 \) and \( \epsilon_{k+1i}^1 \epsilon_{k+1j} = (-1)^{i-j} \).
Proof. Since $V_i' \in \Lambda(m, n)$, we obtain $V_i' \perp \Lambda^+(m, n)$ and therefore $V_i \perp \Lambda^+_i(m, n)$. Hence $V_i$ can be represented in the form

$$V_i = s_i(\text{external product of the vectors of a basis of } \Lambda^+_i(m, n)), \quad s_i \in \mathbb{R}.$$  \hfill (13)

Therefore,

$$V_{ij} = \epsilon_{ij}t_iA_{ij}, \quad \epsilon_{ij} = \pm 1, \quad t_i > 0$$

and obviously $\epsilon_{k+1,i}\epsilon_{k+1,j} = (-1)^{i-j}$. In order to see this, it is enough to put the basis $a'_1 \ldots a'_{k-1}$ of $\Lambda^+_i(m, n))$ obtained from (12) as a basis on the right hand side of (13). Further, the equation $V_{ij} = -V_{ji}$ implies $t_i = t_j$. Let $t = t_1 = \ldots = t_k$. It is a well known fact (see e.g. [4], pp. 27–28) that

$$\det \Lambda(m, n) = \det \Lambda^+(m, n).$$ \hfill (14)

The first determinant

$$\det \Lambda(m, n) = \begin{pmatrix} mn & mn \\ mn & mn \end{pmatrix} = \frac{1}{2} \sum_{i \neq j} V_{ij}^2 = \frac{t^2}{2} \sum_{i \neq j} A_{ij}^2.$$ 

On the other hand, by the Laplace identity (see e. g. [18], Lemma 6D), the second determinant is

$$\det \Lambda^+(m, n) = \det(a'_i a'_j)_{i,j=1}^{k-1} = \frac{1}{2} \sum_{i \neq j} A_{ij}^2,$$

and by (14) $t = t_1 = \ldots = t_k = 1$. \hfill \square

Since $V = n_{k+1}v$, Lemma 2 implies that the vector $v$ is orthogonal to the vectors $a_1, \ldots, a_{k-1}$ and

$$va_k = \frac{1}{n_{k+1}}Va_k = \frac{1}{n_{k+1}}(V_{k+1}a_{k1} + \ldots + V_{k+1}a_{kk})$$

$$= \pm \frac{1}{n_{k+1}}(A_{k+1}a_{k1} - A_{k+1}a_{k2} + \ldots + (-1)^{k-1}A_{k+1}a_{kk})$$

$$= \pm \frac{1}{n_{k+1}} \det \Lambda^+_{k+1}(n) = \pm 1.$$ 

Taking, if necessary, the vector $-v$ instead of $v$, we can assume that $va_k = 1$. This shows that $v \in (\Lambda^+_{k+1}(n))^*$ and, consequently, $\Lambda(n)$ is a sublattice of $(\Lambda^+_{k+1}(n))^*$. Since

$$\det \Lambda(n) = \det(\Lambda^+_{k+1}(n))^* = \frac{1}{n_{k+1}},$$

these lattices coincide.
4 Proof of Theorem 1

Let \( b_1^*, \ldots, b_k^* \) be a basis of the polar lattice \( \Lambda^* \) such that

\[
    b_i^* b_j = \begin{cases} 
    1, & i = j, \\
    0, & \text{otherwise}.
\end{cases}
\]

We shall apply Theorem 1 of [17], taking \( m = 1, F = 1 \), and taking for \( F_1\nu, \nu \in \{1, \ldots, k + 1\} \) all minors of order \( k \) of the matrix

\[
    M = M(T, T_1, \ldots, T_k)
\]

\[
    = \begin{pmatrix}
    db_{11}^* T + T_1 & db_{12}^* T & \cdots & db_{1k}^* T & d \sum_{i=1}^{k} \alpha_i b_{i1}^* T \\
    db_{21}^* T & db_{22}^* T + T_2 & \cdots & db_{2k}^* T & d \sum_{i=1}^{k} \alpha_i b_{i2}^* T \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    db_{k1}^* T & db_{k2}^* T & \cdots & db_{kk}^* T + T_k & d \sum_{i=1}^{k} \alpha_i b_{ki}^* T
    \end{pmatrix},
\]

where \( T, T_1, \ldots, T_k \) are variables. Let \( M_i = M_i(T, T_1, \ldots, T_k) \) and \( B_i^* \) be the minor obtained by omitting the \( i \)th column in \( M \) or in the matrix

\[
    \begin{pmatrix}
    b_{11}^* & b_{12}^* & \cdots & b_{1k}^* \sum_{i=1}^{k} \alpha_i b_{i1}^* \\
    b_{21}^* & b_{22}^* & \cdots & b_{2k}^* \sum_{i=1}^{k} \alpha_i b_{i2}^* \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{k1}^* & b_{k2}^* & \cdots & b_{kk}^* \sum_{i=1}^{k} \alpha_i b_{ki}^*
    \end{pmatrix},
\]

respectively. As in the proof of Theorem 2 in [17] we note that

\[
    |B_{k+1}^*| = |\det(b_{ij}^*)| \neq 0, \quad (15)
\]

\[
    |B_i^*| = \alpha_i |B_{k+1}^*|, \quad (16)
\]

\[
    M_i = d^k B_i^* T^k + \text{polynomial of degree less than } k \text{ in } T \quad (17)
\]

and \( M_1, \ldots, M_k \) have no common factor. By Theorem 1 of [17] there exist integers \( t_1, \ldots, t_k \) and an arithmetic progression \( P \) such that for \( t \in P \) we have

\[
    \gcd(M_1(t, t_1, \ldots, t_k), \ldots, M_{k+1}(t, t_1, \ldots, t_k)) = 1.
\]

Let

\[
    n(t) = (M_1(t, t_1, \ldots, t_k), \ldots, (-1)^k M_{k+1}(t, t_1, \ldots, t_k)),
\]

then we see that (3) and (4) hold.
To prove the equality (2) let us consider a lattice $\Lambda_{k+1}^\perp(n(t))$, $t \in P$ with basis

$$
a_i^*(t) = \begin{pmatrix} db_{i1}^* T + T_1, & db_{i2}^* T, & \ldots, & db_{ik}^* T \end{pmatrix},
$$

$$
a_2^*(t) = \begin{pmatrix} db_{11}^* T, & db_{12}^* T + T_2, & \ldots, & db_{k2}^* T \end{pmatrix},
$$

\vdots

$$
a_k^*(t) = \begin{pmatrix} db_{k1}^* T, & db_{k2}^* T, & \ldots, & db_{kk}^* T + T_k \end{pmatrix}.
$$

By Lemma 1 $\Lambda(n(t))$ is the polar lattice for $\Lambda_{k+1}^\perp(n(t))$. Let $a_1(t), \ldots, a_k(t)$ be a basis of $\Lambda(n(t))$ such that

$$
a_i^*(t)a_j(t) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases}
$$

Consider the matrices $A^*(t) = (a_{ij}^*(t))_{i,j=1}^k$ and $B^* = (b_{ij}^*)_{i,j=1}^k$. Let $A_{ij}^*(t)$ and $B_{ij}^*$ be the minors obtained by omitting the $i$th row and $j$th column in $A^*(t)$ or $B^*$ respectively. Then, in particular,

$$
A_{ij}^*(t) = d^{k-1}t^{k-1}B_{ij}^* + O(t^{k-2}). \quad (18)
$$

Further

$$
a_i(t) = \lambda^*(A_{i1}^*(t), -A_{i2}^*(t), \ldots, (-1)^{k-1}A_{ik}^*(t)),
$$

where $\lambda^* = \det \Lambda(n(t)) = (\det \Lambda_{k+1}^\perp(n(t)))^{-1}$. To check it just note that

$$
\det \Lambda_{k+1}^\perp(n(t)) = a_i^*(t)(A_{i1}^*(t), -A_{i2}^*(t), \ldots, (-1)^{k-1}A_{ik}^*(t)).
$$

Analogously,

$$
b_i = \lambda(B_{i1}^*, -B_{i2}^*, \ldots, (-1)^{k-1}B_{ik}^*),
$$

where $\lambda = (B_{k+1}^*)^{-1} = (\det B^*)^{-1}$, since obviously

$$
\det B^* = b_i^*(B_{i1}^*, -B_{i2}^*, \ldots, (-1)^{k-1}B_{ik}^*).
$$

By (17)

$$
\lambda^* = (d^kt^k\lambda^{-1} + O(t^{k-1}))^{-1}.
$$

Therefore, by (18)

$$
a_{ij}(t) = (-1)^{j-1}\frac{d^{k-1}t^{k-1}B_{ij}^* + O(t^{k-2})}{d^kt^k\lambda^{-1} + O(t^{k-1})} = (-1)^{j-1}\frac{d^{k-1}t^{k-1}B_{ij}^*}{d^kt^k\lambda^{-1}(1 + O(1/t))} + O\left(\frac{1}{t^2}\right)
$$

\begin{align*}
&= (-1)^{j-1}\frac{\lambda B_{ij}^*}{dt} + O\left(\frac{1}{t^2}\right) = \frac{b_{ij}}{dt} + O\left(\frac{1}{t^2}\right).
\end{align*}
5 Proof of Theorem 2

The inequality
\[ C(K) = \sup_{n \in U^{k+1}} \frac{\lambda_1(K, \Lambda(n)) \cdots \lambda_k(K, \Lambda(n))}{\det \Lambda(n)} \leq \frac{\alpha(K)}{\Delta(K)} \]
holds by the definition of anomaly (see [7], pp. 191, 192). Let us show that
\[ \sup_{n \in U^{k+1}} \frac{\lambda_1(K, \Lambda(n)) \cdots \lambda_k(K, \Lambda(n))}{\det \Lambda(n)} \geq \frac{\alpha(K)}{\Delta(K)} \]  
(19)

Let \( \Lambda_0 = \Lambda_0(K) \) be a lattice such that
\[ \lambda_1(K, \Lambda_0) \cdots \lambda_k(K, \Lambda_0) = \frac{\alpha(K)}{\Delta(K)} \det \Lambda_0 \]  
(20)

The existence of such lattices for bounded star bodies in \( \mathbb{R}^2 \) was proved in [14] and for all dimensions in [8] (see also [19]). Let \( \mathbf{r}_1, \ldots, \mathbf{r}_k \) be a basis of \( \Lambda_0 \). Take a positive \( \delta < 1 \) and choose linearly independent vectors \( \mathbf{b}_1(\delta), \ldots, \mathbf{b}_k(\delta) \) in \( \mathbb{Q}^k \) such that
\[ ||\mathbf{b}_j(\delta) - \mathbf{r}_j||_\infty < \delta, \quad j \in \{1, \ldots, k\}, \]
\[ |\det(\mathbf{b}_1^T(\delta), \ldots, \mathbf{b}_k^T(\delta)) - \det \Lambda_0| < \delta \det \Lambda_0 \]  
(21)

Let us apply Theorem 1 to the lattice \( \Lambda \) with basis \( \mathbf{b}_1(\delta), \ldots, \mathbf{b}_k(\delta) \) and arbitrarily chosen rational numbers \( \alpha_1, \ldots, \alpha_k \) with \( 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1 \), then we obtain an arithmetic progression \( \mathcal{P} \) and a sequence \( \mathbf{n}(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t)) \in \mathbb{Z}^{k+1}, \]
\( t \in \mathcal{P} \), such that \( \Lambda(\mathbf{n}(t)) \) has a basis \( \mathbf{a}_1(t), \ldots, \mathbf{a}_k(t) \) with
\[ dta_{ij}(t) = b_{ij}(\delta) + O \left( \frac{1}{t} \right), \quad i, j \in \{1, \ldots, k\}, \]
where \( d = d(\delta) \in \mathbb{N} \) such that \( db_{ij}(\delta), d\alpha_i b_{ij}(\delta) \in \mathbb{Z} \) for all \( i, j \in \{1, \ldots, k\} \). Let us choose any \( t_0 = t_0(\delta) \in \mathcal{P} \) such that
\[ ||d\alpha_i a_{ij}(\delta) - r_{ij}||_\infty < \delta, \quad j \in \{1, \ldots, k\} \]  
(22)

and \( t_0 > 1/\delta \). Put \( \Lambda_\delta = d\alpha_0 \Lambda(\mathbf{n}(t_0)) \). For \( \delta \to 0 \) we obtain an infinite sequence of lattices \( \{\Lambda_\delta\} \) and by (22) \( \Lambda_\delta \to \Lambda_0 \). In view of Lemma A
\[ \lambda_1(K, \Lambda_\delta) \cdots \lambda_k(K, \Lambda_\delta) \to \frac{\alpha(K)}{\Delta(K)} \det \Lambda_0, \quad \text{as} \ \delta \to 0. \]
We have
\[
\lambda_1(K, \Lambda(n(t_0))) \cdots \lambda_k(K, \Lambda(n(t_0))) = \frac{\lambda_1(K, \Lambda_0) \cdots \lambda_k(K, \Lambda_0)}{(d(\delta)t_0(\delta))^k},
\]
and by (3) and (21)
\[
(d(\delta)t_0(\delta))^k = \frac{\det \Lambda}{\det \Lambda(n(t_0))} + O(t_0^{k-1}) < \frac{(1 + \delta) \det \Lambda_0}{\det \Lambda(n(t_0))} (1 + O(\delta)).
\]
Therefore, for every \(\epsilon > 0\) and for sufficiently small \(\delta > 0\) there exists an integer vector \(n = n(t_0(\delta))\) such that
\[
\lambda_1(K, \Lambda(n)) \cdots \lambda_k(K, \Lambda(n)) > \frac{(1 - \epsilon)\alpha(K)}{\Delta(K)} \det \Lambda(n).
\]
This implies (19) and shows that (5) holds for the sequence \(\{n(t_0(\delta))\}\). Finally, we take this sequence as a sequence from the statement of the theorem. Then the equality (6) holds by (4) and (7) holds by (3).

6 Proof of Theorem 3

We shall show that for every \(\epsilon > 0\) there exist a vector \(x \in \mathbb{R}^k\) and a real number \(Q > 0\) such that
\[
\{\lambda_1(x, Q)\}^k > \frac{1 - \epsilon}{\Delta(K)Q}.
\]
(23)

Let
\[
C_1(K) := \limsup_{n \in U^{k+1}} \frac{\{\lambda_1(K, \Lambda(n))\}^k}{\det \Lambda(n)}.\]

The proof of Theorem 2 can be easily modified to prove that
\[
C_1(K) = \frac{1}{\Delta(K)}.\]
(24)

Namely, we have to consider any critical lattice of \(K\) as the lattice \(\Lambda_0 = \Lambda_0(K)\) and to replace (20) by the equality
\[
\{\lambda_1(K, \Lambda_0)\}^k = \frac{\det \Lambda_0}{\Delta(K)}.
\]
By (24) there exists a sequence \( \{n(t)\} \), such that \( ||n(t)||_\infty \to \infty \) and for sufficiently large \( t \) holds
\[
\{\lambda_1(K, \Lambda(n(t)))\}^k > \frac{(1 - \epsilon) \det \Lambda(n(t))}{\Delta(K)(1 - 1/n_{k+1}(t))}
\]
\[
= \frac{1 - \epsilon}{\Delta(K)(n_{k+1}(t) - 1)}.
\]

Now it is enough to put \( x = (n_1(t)/n_{k+1}(t), \ldots, n_k(t)/n_{k+1}(t)) \), \( Q = n_{k+1}(t) - 1 \) and to note that \( \lambda_1(K, \Lambda(n(t))) = \lambda_1(x, Q) \).

**Remark.** For any \( \epsilon > 0 \) in the proof of Theorem 3 we obtain not only rational solutions \( x \) of the inequality (23). In fact, all vectors sufficiently close to a vector \( x \) satisfying (23) satisfy (23) as well. Moreover, since we apply Theorem 1 with arbitrarily chosen rational numbers \( \alpha_i \), the equality (4) implies that solutions of (23) approximate any rational point \( (\alpha_1, \ldots, \alpha_k) \) with \( 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq 1 \).

## 7 Proof of Theorem 4

For any \( \epsilon > 0 \) we shall find an infinite sequence \( \{n(t)\} \) of integer vectors such that \( h(n(t)) \to \infty \) and for all sufficiently large \( t \) the inequality
\[
\inf_{p, q \in \mathbb{Z}^{k+1}} \frac{h(p)h(q)}{h(n(t))^{1-1/k}} > \frac{1 - \epsilon}{(k + 1)^{1/k}}
\]
(25)
holds.

Let \( n = (n_1, \ldots, n_{k+1}) \), \( 0 < n_1 \leq \ldots \leq n_{k+1} \) be a primitive integer vector, that is \( \gcd(n_1, \ldots, n_{k+1}) = 1 \), and \( m = (m_1, \ldots, m_{k+1}) \) be an integer vector, such that \( m \) and \( n \) are linearly independent. Consider the polygon \( \Pi = \Pi(m, n) \)
\[
\Pi : |m_i y - n_i x| \leq 1, \quad i \in \{1, \ldots, k + 1\}.
\]
(26)

Let
\[
v = v(m) := (m_1 - m_{k+1}n_1/n_{k+1}, \ldots, m_k - m_{k+1}n_k/n_{k+1}) \in \Lambda(n).
\]
(27)

The following lemma is implicit in [2].

**Lemma B.** Assume that \( 0 < n_1 < \ldots < n_{k+1} \), \( \xi > 0 \). There exists a centrally symmetric, convex set \( M_\xi = M_\xi(n) \subseteq \mathbb{R}^k \), such that \( v(m) \in M_\xi \) for an integer vector \( m \) if and only if

\[
\Delta(\Pi(m, n)) \geq \frac{1}{n_{k+1}\xi}.
\]
Moreover,
\[ V_k(\mathcal{M}_\xi) > (k + 1)\xi^k. \] \hspace{1cm} (28)

Indeed, a set \( \mathcal{M}_\xi \) satisfying the equivalence stated in the lemma is described by the formula (6) of [2] and the inequality (28) is proved in Lemma 12 ibid. Let \( f_n(x) \) be the distance function of the set \( \mathcal{M}_1(n) \). By the definition of \( \mathcal{M}_\xi \), for \( v \) as in (27), we have
\[ f_n(v) = (n_{k+1}\Delta(\Pi))^{-1}. \] \hspace{1cm} (29)

Consider a generalized honeycomb \( E_k^1 \) given by the inequalities
\[ E_k^1 = \{ (x_1, \ldots, x_k) \in \mathbb{R}^k : |x_i| \leq 1, |x_i - x_j| \leq 1, i, j \in \{1, \ldots, k\}, i \neq j \}. \]

Observe that
\[ E_k^1 = \bigcap_{p < q} \{ (x_1, \ldots, x_k) \in \mathbb{R}^k : (x_p, x_q) \in E_2^1 \}. \]

Let \( g_k(x) \) be the distance function of \( E_k^1 \). Then obviously
\[ g_k(x) = \max_{1 \leq i < j \leq k} g_2((x_i, x_j)). \]

By Lemma 1 of [2]
\[ V_k(E_k^1) = k + 1, \quad \Delta(E_k^1) = \frac{k + 1}{2^k} \]
and \( E_k^1 \) has a unique critical lattice \( \Lambda(E_k^1) \) with basis
\[
\begin{align*}
b_1 &= (1, 1/2, \ldots, 1/2), \\
b_2 &= (1/2, 1, \ldots, 1/2), \\
& \vdots \\
b_k &= (1/2, 1/2, \ldots, 1).
\end{align*}
\]

**Lemma 3.** For any \( \epsilon > 0 \) there exists a \( \delta = \delta(\epsilon) > 0 \) such that for all integer vectors \( n = (n_1, \ldots, n_k, n_{k+1}) \) with \( 1 - \delta < n_1/n_{k+1} < \ldots < n_k/n_{k+1} < 1 \), for all \( x \in \mathbb{R}^k \setminus \{0\} \)
\[ f_n(x) > (1 - \epsilon/2)g_k(x). \]
Proof. By the formula (6) of [2], the set $M_1(n)$ is the intersection of the sets $G_{pqr}$, where
\[ G_{pqr} = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k : (x_p, x_q) \in B_1 \left( \frac{n_p}{n_{k+1}}, \frac{n_q}{n_{k+1}} \right) \right\} \]
for $p < q < r = k + 1$ and
\[ G_{pqr} = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k : (x_p - \frac{n_q}{n_r} x_r, x_q - \frac{n_q}{n_r} x_r) \in \gamma B_1 \left( \frac{n_p}{n_r}, \frac{n_q}{n_r} \right) \right\} \]
for $p < q < r < k + 1$, $\gamma = n_{k+1}/n_r$. The set $B_1 = B_1(\alpha, \beta)$, $0 < \alpha < \beta < 1$ is defined by the formulae (8)–(13) of [1]. The boundary of $B_1$ consists of two horizontal segments
\[ \pm S_h = \{ \pm (t, 1) : -(1 - \alpha)/(1 + \beta) \leq t \leq (1 + \alpha)/(1 + \beta) \}, \]
two vertical segments
\[ \pm S_v = \{ \pm (1, t) : -(1 - \beta)/(1 + \alpha) \leq t \leq (1 - \beta)/(1 - \alpha) \}, \]
and four curvilinear arcs $\pm L_1, \pm L_2$ with
\[ \pm L_1 = \{ \pm (x(t), tx(t)) : (1 - \beta)/(1 - \alpha) \leq t \leq (1 + \beta)/(1 + \alpha) \}, \]
\[ x(t) = \frac{-t^2(1 + \alpha)^2 + 2t(1 - \alpha + \beta + \alpha \beta) - (1 - \beta)^2}{4t(\beta - \alpha t)} \]
and
\[ \pm L_2 = \{ \pm (X(t), -tX(t)) : (1 - \beta)/(1 + \alpha) \leq t \leq (1 + \beta)/(1 - \alpha) \}, \]
\[ X(t) = \frac{-t^2(1 - \alpha)^2 + 2t(1 + \alpha + \beta - \alpha \beta) - (1 - \beta)^2}{4t(\beta + \alpha t)} \].

By Lemma 1 of [1], $B_1$ is a centrally symmetric convex set.

Assume that there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exist an integer vector $n = (n_1, \ldots, n_k, n_{k+1})$, $1 - \delta < n_1/n_{k+1} < \ldots < n_k/n_{k+1} < 1$ and a point $x \in \mathbb{R}^k \setminus \{0\}$ with
\[ f_n(x) \leq (1 - \epsilon/2)g_k(x). \]
We shall show that this leads to a contradiction. By (30), there is a point $a = (a_1, \ldots, a_k) = \lambda x$, $\lambda > 0$, such that $f_n(a) = 1$ and
\[ g_k(a) = g_2((a_i, a_j)) \geq (1 - \epsilon/2)^{-1} \]
for some $i, j \in \{1, \ldots, k\}$, $i < j$. Let $\alpha = n_i/n_{k+1}$, $\beta = n_j/n_{k+1}$. Since $a \in M_1(n)$, we have $(a_i, a_j) \in B_1(\alpha, \beta)$.

First we consider the case $\alpha, \beta \geq 0$. By Lemma 2 of [1]
\[ B_1(\alpha, \beta) \subset C_1 := \{ x \in \mathbb{R}^2 : ||x||_\infty \leq 1 \} \]

17
and thus
\[
\{(x_i, x_j) \in B_1(\alpha, \beta) : x_i x_j \geq 0\} \subset \{(x_i, x_j) \in E_1^2 : x_i x_j \geq 0\},
\]
which contradicts (31).

Let us now consider the case \(a_i a_j < 0\). Suppose \(a_j = -ta_i\). We may assume without loss of generality that
\[
(1 - \epsilon/2)^{-1} - 1 \leq t \leq ((1 - \epsilon/2)^{-1} - 1)^{-1}.
\]
Otherwise \((a_i, a_j) \not\in C_1\) and we get a contradiction with (32). Since \((1 - \beta)/(1 + \alpha)\) tends to 0 and \((1 + \beta)/(1 - \alpha)\) tends to infinity as \(\delta\) tends to 0, we have
\[
(1 - \beta)/(1 + \alpha) < t < (1 + \beta)/(1 - \alpha)
\]
for \(\delta\) small enough. Then \(\mu(a_i, a_j) \in \pm L_2\) for some \(\mu \geq 1\). Further, for any \(t\) from the interval (33)
\[
X(t) \to \frac{1}{1 + t}, \quad \text{as } \delta \to 0.
\]
Since \(q_2(1/(1 + t), -t/(1 + t)) = 1\), we obtain a contradiction with (31) for sufficiently small \(\delta\).

\[\square\]

**Lemma 4.** For any \(\epsilon > 0\) there exist an arithmetic progression \(P\) and a sequence of primitive integer vectors \(n(t) = (n_1(t), \ldots, n_k(t), n_{k+1}(t)), t \in P\), such that \(h(n(t)) \to \infty\) and for sufficiently large \(t \in P\), for every non–zero vector \(v \in \Lambda(n(t))\)
\[
f_{n(t)}(v) > (1 - \epsilon) \left\{n_{k+1}(t)\Delta(E_1^k)\right\}^{-1/k}.
\]

**Proof.** Let us choose rational numbers \(1 - \delta(\epsilon) < \alpha_1 < \alpha_2 < \cdots < \alpha_k < 1\) and apply Theorem 1 to the lattice \(\Lambda = \Lambda(E_1^k)\), the basis \(b_1, \ldots, b_k\) of \(\Lambda\) and numbers \(\alpha_1, \alpha_2, \ldots, \alpha_k\). Then we obtain an arithmetic progression \(P\) and a sequence of primitive integer vectors \(n(t), t \in P\) such that \(h(n(t)) \to \infty\) and corresponding lattices \(\Lambda(n(t))\) have bases \(a_1(t), \ldots, a_k(t)\) with
\[
a_{ij}(t) = \frac{b_{ij}}{dt} + O\left(\frac{1}{t^2}\right), \quad i, j \in \{1, \ldots, k\}, \tag{34}
\]
where \(d \in \mathbb{N}\) such that \(d b_{ij}, \alpha_j b_{ij} \in \mathbb{Z}\) for all \(i, j \in \{1, \ldots, k\}\). Moreover
\[
\alpha_i(t) := \frac{n_i(t)}{n_{k+1}(t)} = \alpha_i + O\left(\frac{1}{t}\right)
\]
and thus for \(t\) large enough
\[
1 - \delta(\epsilon) < \frac{n_1(t)}{n_{k+1}(t)} < \ldots < \frac{n_k(t)}{n_{k+1}(t)} < 1.
\]

18
Let us show that for sufficiently large \( t \in \mathcal{P} \)
\[
\lambda_1(E_1^k, \Lambda(n(t))) > (1 - \epsilon/2)n_{k+1}(t)\Delta(E_1^k)^{-1/k}.
\] (35)
The equality (34) implies that
\[ dt\Lambda(n(t)) \to \Lambda, \quad \text{as} \quad t \to \infty, \quad t \in \mathcal{P}. \]
Thus, by Lemma A we obtain
\[ \lambda_1(E_1^k, dt\Lambda(n(t))) \to 1, \quad \text{as} \quad t \to \infty, \quad t \in \mathcal{P}. \]
Since
\[ \lambda_1(E_1^k, \Lambda(n(t))) = \lambda_1(E_1^k, dt\Lambda(n(t))) \]
and by (3)
\[ dt = (n_{k+1}(t) \det \Lambda)^{1/k}(1 + O(1/t))^{1/k}, \]
the inequality (35) holds for \( t \) large enough. By Lemma 3 and (35) for sufficiently large \( t \in \mathcal{P} \) for every non–zero vector \( v \in \Lambda(n(t)) \)
\[
\frac{f_{n(t)}(v)}{(1 - \epsilon/2)\lambda_1(E_1^k, \Lambda(n(t)))} > (1 - \epsilon)\{n_{k+1}(t)\Delta(E_1^k)^{-1/k}\}
\] (36)
By (4) for \( t \) large enough \( h(n(t)) = n_{k+1}(t) \). Finally, by (36), (29) and Lemma 4 for sufficiently large \( t \)
\[
\frac{h(p)h(q)}{h(n(t))^{1-1/k}} \geq \frac{1}{2} (n_{k+1}(t))^{1/k}f_{n(t)}(v(m)) > \frac{1 - \epsilon}{k+1}^{1/k}.
\]
Acknowledgement. The author wishes to thank Professor A. Schinzel for many valuable comments and suggestions.

References


