Multi-drawing, multi-colour Pólya urns

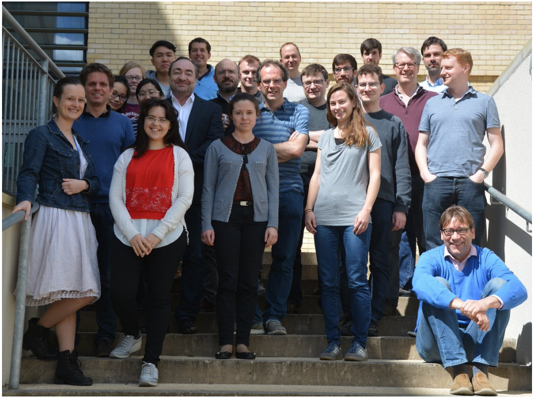
– Cécile Mailler –

`ArXiV:1611.09090`
joint work with Nabil Lassmar and Olfa Selmi (Monastir, Tunisia)

October 11th, 2017

Happy birthday, Henning!!

Professorship @ Bath!!



Deadline for applications: 01/01/2018.

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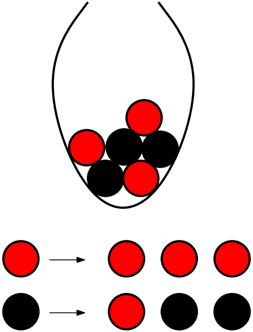
UNIVERSITY OF BATH

The "classical" Pólya urn model



Two parameters:

- the replacement matrix

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- and the initial composition

$$U_0 = \begin{pmatrix} U_{0,1} \\ U_{0,2} \end{pmatrix}$$

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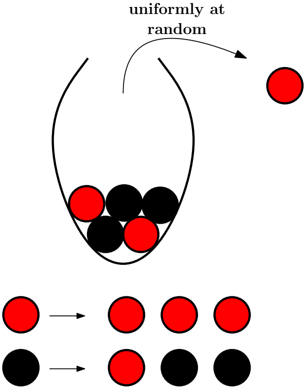

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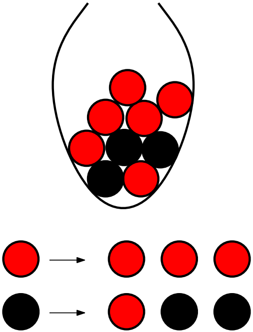

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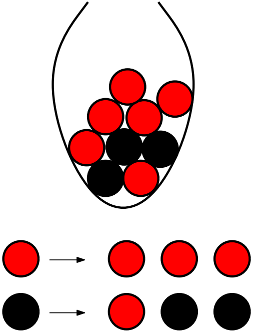

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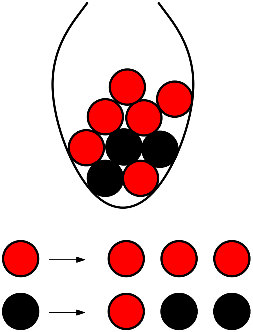

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Questions:

How does U_n behave when n is large?
How does this asymptotic behaviour depend on R and U_0?

Asymptotic theorems

- **Perron-Frobenius:** If R is irreducible, then its spectral radius λ_1 is positive, and a simple eigenvalue of R. And there exists an eigenvector u_1 with positive coordinates such that ${}^tRu_1 = \lambda_1 u_1$.
- λ_2 is the eigenvalue of R with the second largest real part, and $\sigma = {}^{\mathrm{Re}\lambda_2}/_{\lambda_1}$.

Theorem (see, e.g. [Athreya & Karlin '68] [Janson '04]):

Assume that R is irreducible and $\sum_{i=1}^{d} U_{0,i} > 0$, then,

- $U_n/n \to u_1$ $(n \to \infty)$ almost surely;
- furthermore, when $n \to \infty$,
  - if $\sigma < {}^1/_2$, then $n^{-1/2}\big(U_n - nu_1\big) \to \mathcal{N}(0, \Sigma^2)$ in distribution;
  - if $\sigma = {}^1/_2$, then $(n \log n)^{-1/2}\big(U_n - nu_1\big) \to \mathcal{N}(0, \Theta^2)$ in distribution;
  - if $\sigma > {}^1/_2$, then $n^{-\sigma}\big(U_n - nu_1\big)$ cv. a.s. to a finite random variable.

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- Both Σ and Θ don't depend on the initial composition.

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- Both Σ and Θ don't depend on the initial composition.
- It actually applies to a largest class of urns: R can be reducible as long as there is a Perron-Frobenius-like eigenvalue.
- The non-Perron-Frobenius-like cases are much less understood (see, e.g. [Janson '05]).

Multi-drawing d-colour Pólya urns

Three parameters: an integer $m \geq 1$, the initial composition U_0, and the replacement rule $R : \Sigma_m^{(d)} \to \mathbb{N}^d$, where

$$\Sigma_m^{(d)} = \{v \in \mathbb{N}^d : v_1 + \ldots + v_d = m\}.$$

Start with $U_{0,i}$ balls of colour i in the urn ($\forall 1 \leq i \leq d$). At step n,

- pick m balls in the urn (with or without replacement), denote by $\xi_{n+1} \in \Sigma_m^{(d)}$ the composition of the set drawn;
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With replacement:

For all $v \in \Sigma_m^{(d)}$,
$\mathbb{P}_n(\xi_{n+1} = v) = \binom{m}{v_1 \ldots v_d} \prod_{i=1}^d Z_{n,i}^{v_i}$.

Without replacement:

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The method

~~Embed the urn into continuous-time onto a multi-type branching processes.~~
[Athreya & Karlin '68, Janson '04]

Restrict to the "affine" case and use martingales.
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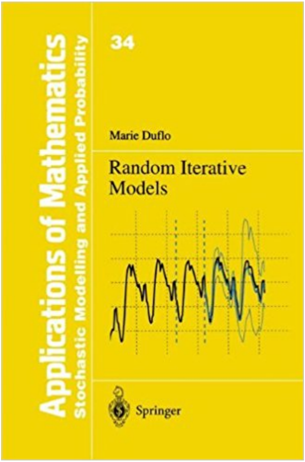
Use stochastic approximation!



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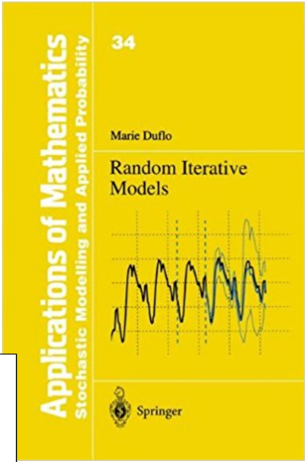
MICHEL BENAÏM

Dynamics of stochastic approximation algorithms

Séminaire de probabilités (Strasbourg), tome 33 (1999), p. 1-68.



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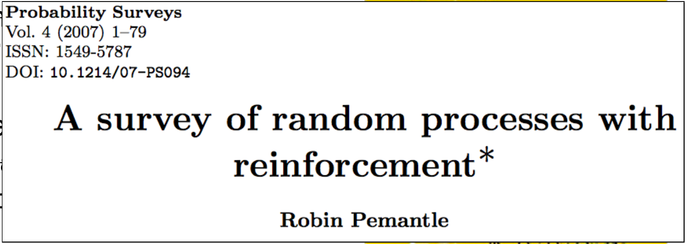

Probability Surveys
Vol. 4 (2007) 1–79
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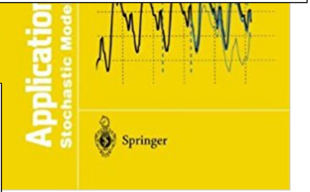
A survey of random processes with reinforcement[*]

Robin Pemantle



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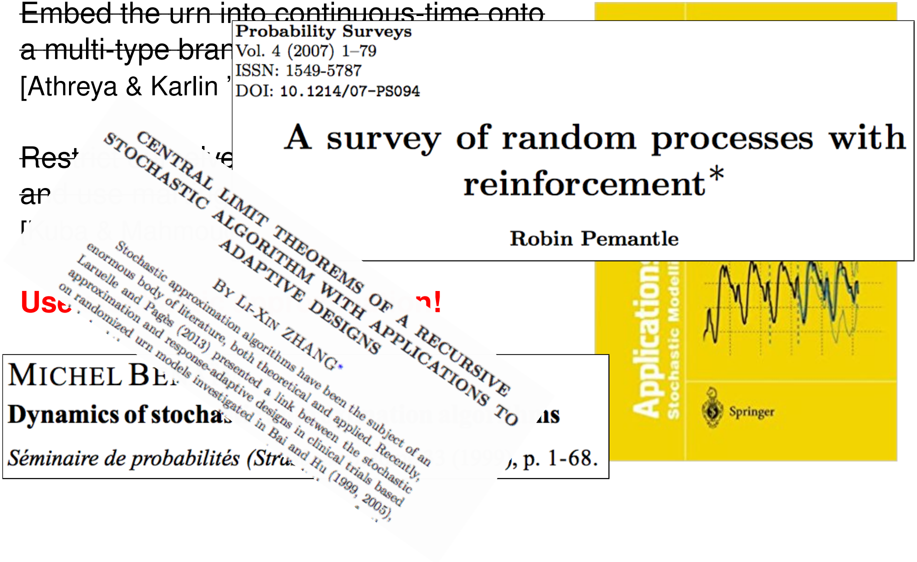

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CENTRAL LIMIT THEOREMS OF A RECURSIVE STOCHASTIC ALGORITHM WITH APPLICATIONS TO ADAPTIVE DESIGNS

BY LI-XIN ZHANG

Stochastic approximation algorithms have been the subject of an enormous body of literature, both theoretical and applied. Recently, Laruelle and Pagès (2013) presented a link between the stochastic approximation and response-adaptive designs in clinical trials based on randomized urn models investigated in Bai and Hu (1999, 2005),

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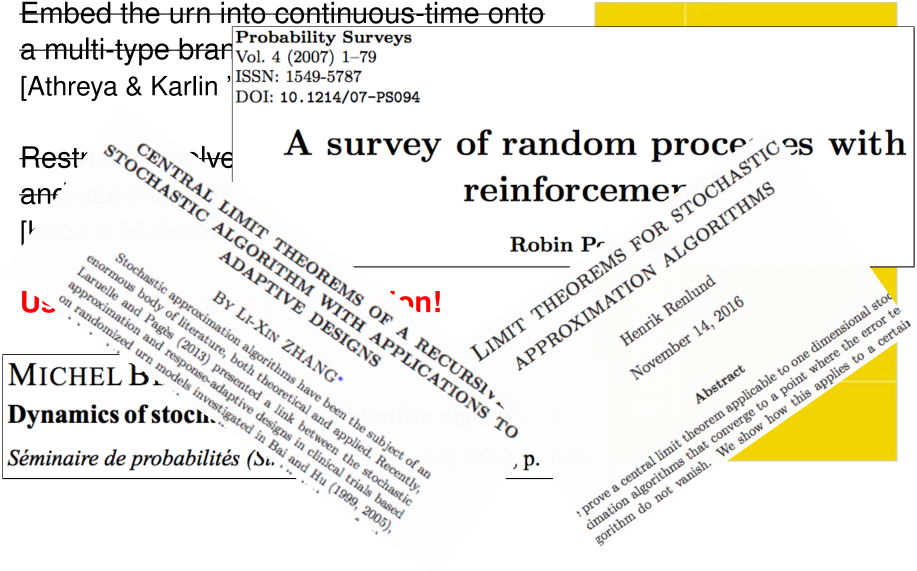

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LIMIT THEOREMS FOR STOCHASTIC
APPROXIMATION ALGORITHMS

Henrik Renlund

November 14, 2016

Abstract

prove a central limit theorem applicable to one dimensional sto
imation algorithms that converge to a point where the error te
orithm do not vanish. We show how this applies to a certai

Stochastic approximations

A sequence $(Z_n)_{n\geq 0}$ is a stochastic approximation if it satisfies

$$Z_{n+1} = Z_n + \frac{1}{\gamma_n}\Big(h(Z_n) + \Delta M_{n+1} + r_{n+1}\Big),$$

where

- h is a Lipschitz function,
- ΔM_{n+1} is a martingale increment, i.e. $\mathbb{E}_n\big[\Delta M_{n+1}\big] = 0$,
- $r_n \to 0$ a.s. is a remainder term,
- $(\gamma_n)_{n\geq 0}$ satisfies $\sum \frac{1}{\gamma_n} = +\infty$ and $\sum \frac{1}{\gamma_n^2} < +\infty$.

[Robbins-Monro '51]

Our urn is a stochastic approximation

Notations:

$U_{n,i}$ = number of balls of colour i in the urn at time n
$Z_{n,i}$ = proportion of balls of colour i in the urn at time n
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$\bar{R}(v) = \sum_{i=1}^{d} R_i(v)$ = total # of balls added when the sample drawn is v.

$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}} \left(R(\xi_{n+1}) - \bar{R}(\xi_{n+1}) Z_n \right)$$

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$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}} \underbrace{\left(R(\xi_{n+1}) - \bar{R}(\xi_{n+1}) Z_n\right)}_{Y_{n+1}}$$

Let $Y_{n+1} = R(\xi_{n+1}) - \bar{R}(\xi_{n+1})Z_n$, then

$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}} Y_{n+1} = Z_n + \frac{1}{T_{n+1}}\Big(\mathbb{E}_n Y_{n+1} + \underbrace{Y_{n+1} - \mathbb{E}_n Y_{n+1}}_{\text{martingale increment}}\Big)$$

$$\begin{aligned}\mathbb{E}_n Y_{n+1} &= \sum_{v \in \Sigma_m^{(d)}} \mathbb{P}_n(\xi_{n+1} = v)\big(R(v) - \bar{R}(v)Z_n\big)\\ &= \sum_{v \in \Sigma_m^{(d)}} \binom{m}{v_1, \dots, v_d}\left(\prod_{i=1}^{d} Z_{n,i}^{v_i}\right)\big(R(v) - \bar{R}(v)Z_n\big) =: h(Z_n)\end{aligned}$$

A stochastic approximation!

$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}}\big(h(Z_n) + \Delta M_{n+1}\big)$$

Stochastic approximation: the heuristic

Let $Z_n = U_n/T_n$ renormalised composition vector.
$Z_n \in \Sigma^{(d)} = \{(x_1,\dots,x_d) \in [0,1]^d : \sum_{i=1}^d x_i = 1\}$.

A stochastic approximation!

$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}}\big(h(Z_n) + \Delta M_{n+1}\big)$$

where ΔM_{n+1} is a martingale increment, and

$$h(x) = \sum_{v\in\Sigma_m^{(d)}} \binom{m}{v_1,\dots,v_d}\left(\prod_{i=1}^d x_i^{v_i}\right)\big(R(v) - \bar{R}(v)x\big), \text{ with } \bar{R}(v) = \sum_{i=1}^d R_i(v).$$

NB: $h : \Sigma^{(d)} \to \{(y_1,\dots,y_d) : \sum_{i=1}^d y_i = 0\}$

Theorem [Benaim '99]:

If $T_n = \Theta(n)$, then, the linear interpolation of the trajectory $(Z_n)_{n\geq 1}$ "asymptotically follows the flow of $\dot{y} = h(y)$" in $\Sigma^{(d)}$.

Main result: the “law of large numbers”

Balance assumption: $\bar{R}(v) = S$ for all $v \in \Sigma_m^{(d)}$.

Theorem: Diagonal balanced case

If $h \equiv 0$, then $(Z_n)_{n\geq 0}$ is a positive martingale and thus $Z_n \to Z_\infty$ a.s.

Limit set of $(Z_n)_{n\geq 0} := \bigcap_{n\geq 0} \overline{\bigcup_{m\geq n} Z_m}$.

Theorem [LMS++]:

For all d-colour m-drawing balanced Pólya urn scheme,

- the limit set of $(Z_n)_{n\geq 0}$ is almost surely a compact connected set of $\Sigma^{(d)}$ stable by the flow of the differential equation $\dot{x} = h(x)$;
- if there exists $\theta \in \Sigma^{(d)}$ such that $h(\theta) = 0$ and, for all $n \geq 0$, $\langle h(Z_n), Z_n - \theta \rangle < 0$, then Z_n converges almost surely to θ.

A bit disappointing?

- Favourable case: h has only one zero θ on $\Sigma^{(d)}$, and $\langle h(x), x-\theta\rangle < 0$ for all x in $\Sigma^{(d)}$ (true on "most" examples). Such a θ must verify that all eigenvalues of $Dh(\theta)$ are non-positive.
- The $m=1$ Perron-Frobenius-like cases are favourable: the only zero of $h(x) = ({}^tR - S\mathrm{Id})x$ (R =replacement matrix) on $\Sigma^{(d)}$ is the left eigenvector u_1 associated to S. `#AthreyaKarlin`
- Non-favourable cases $\Leftrightarrow$ ($m=1$)-non-Perron-Frobenius-like cases. Not surprising that they are much harder to analyse (see [Janson '05])
- "Affine" case of Kuba and Mahmoud $\Leftrightarrow$ $h(x) = Ax + b$.
- h has polynomial components of degree at most m. Thus, given a replacement rule, one can easily check if it is a favourable case, using ~~Maple~~Sage, for example.

The good news...

θ is a stable zero of h iff all eigenvalues of $Dh(\theta)$ are negative.

Theorem [LMS++]: For all balanced d-colour, m-drawing urn:

Assume that there exists a stable zero θ of h such that $Z_n \to \theta$ a.s. Let Λ be the eigenvalue of $-Dh(\theta)$ with the smallest real part. Then,

- if $\mathrm{Re}(\Lambda) > S/2$, then $\sqrt{n}(Z_n - \theta) \Rightarrow \mathcal{N}(0, \Sigma)$ when $n \to \infty$.

Assume additionally that all Jordan blocks of $Dh(\theta)$ associated to Λ are of size 1. Then,

- if $\mathrm{Re}(\Lambda) = S/2$, then $\sqrt{n/\log n}(Z_n - \theta) \Rightarrow \mathcal{N}(0, \Theta)$ when $n \to \infty$.
- if $\mathrm{Re}(\Lambda) < S/2$, then $n^{\mathrm{Re}(\Lambda)/S}(Z_n - \theta)$ converges almost surely to a finite random variable.

see [Zhang ’17]

- We have explicit formulas for Σ and Θ, they don’t depend on the initial condition.
- Generalisation of the $m = 1$ case and the “affine” case.

Two-colour examples

The replacement rule can be expressed by a matrix:

$$R = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix}$$

If the set we drew at random contains k red balls, we add a_{m-k} red balls and b_{m-k} black balls in the urn.

[Kuba Mahmoud '16]

We have $h(x, 1-x) = \begin{pmatrix} h_1(x, 1-x) \\ -h_1(x, 1-x) \end{pmatrix}$. Let $g(x) := h_1(x, 1-x)$:

Corollary [LMS++]:

Let $g(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^k a_{m-k} - Sx$, then

- either $g \equiv 0$ and then $Z_n \to Z_\infty$ a.s. (diagonal case),
- or g has isolated zeros, and $Z_n \to (\theta, 1-\theta)$ where $g(\theta) = 0$, and $g'(\theta) \leq 0$.

Second order depending on the relative order of $-g'(\theta)/S$ and $1/2$ (if $g'(\theta) < 0$).

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Example 1:

$$R = \begin{pmatrix} 4 & 0 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}$$

- $g(x) = (1-x)(1-3x)$, $g'(1) = 2$, $g'(1/3) = -2$
- thus $Z_n \to (1/3, 2/3)$ a.s.;
- $-g'(1/3)/S = 1/2$, and thus:

$$\sqrt{n/\log n}\left(Z_{n,1} - 1/3\right) \Rightarrow \mathcal{N}(0, 1/18)$$

NB: the urn is not "affine" since g has degree 2.

If $m = 2$, there is at most one stable zero, but when $m \geq 3$:

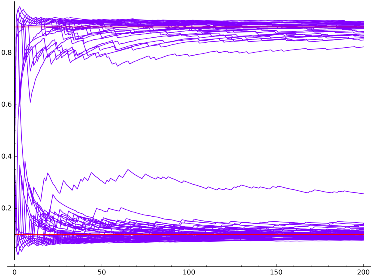
Example 2:

$$R = \begin{pmatrix} 82 & 9 \\ 91 & 0 \\ 0 & 91 \\ 9 & 82 \end{pmatrix}$$

- $g(x) = -200(x - {}^{1}\!/_{10})(x - {}^{1}\!/_{2})(x - {}^{9}\!/_{10})$
- $g'({}^{1}\!/_{2}) > 0$, $g'({}^{1}\!/_{10}) = g'({}^{9}\!/_{10}) = -64$
- $-{}^{64}\!/_{91} > {}^{1}\!/_{2}$, thus

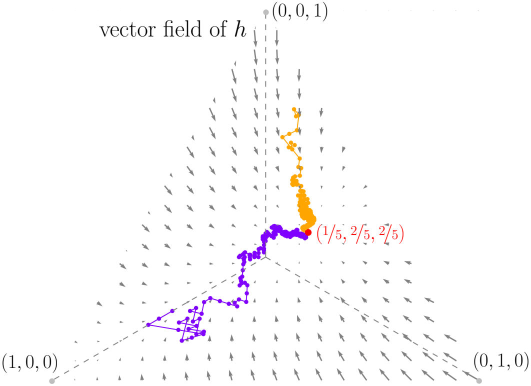
$$Z_{n,1} \to X_\infty \in \{{}^{1}\!/_{10}, {}^{9}\!/_{10}\} \text{ and } \sqrt{n}\big(Z_{n,1} - X_\infty\big) \Rightarrow \mathcal{N}(0, {}^{4131}\!/_{67340}).$$

We have simulated 100 trajectories (200 steps each) of this urn starting at $({}^{2}\!/_{5}, {}^{3}\!/_{5})$:



Some three-colour examples ($m = 2$)

$$
\begin{aligned}
R : (2,0,0) &\mapsto (2,0,0)\\
(0,2,0) &\mapsto (1,0,1)\\
(0,0,2) &\mapsto (1,1,0)\\
(1,1,0) &\mapsto (0,0,2)\\
(1,0,1) &\mapsto (0,2,0)\\
(0,1,1) &\mapsto (0,1,1)
\end{aligned}
$$



We have simulated two 200-step trajectories starting from $(6,3,3)$ and $(2,6,20)$:

$$\sqrt{n}\Big(Z_n - (^1/_5, {}^2/_5, {}^2/_5)\Big) \Rightarrow \mathcal{N}(0, \Sigma)$$

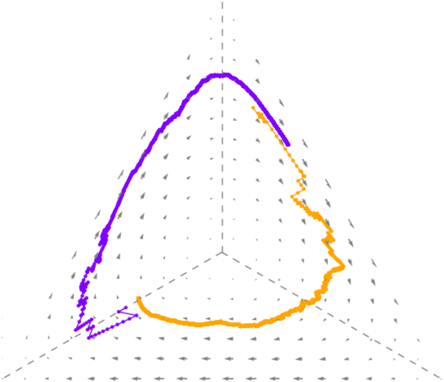
$$\Sigma = \frac{1}{25}\begin{pmatrix} 2 & -1 & -1 \\ -1 & ^{19}/_{13} & -^6/_{13} \\ -1 & -^6/_{13} & ^{19}/_{13} \end{pmatrix}$$

NB: $\Sigma \cdot (1,1,1)^t = (0,0,0)^t$.

A non-favourable case: “rock, scissor, paper”

$$\begin{aligned} R: (2,0,0) &\mapsto (1,0,0) \\ (0,2,0) &\mapsto (0,1,0) \\ (0,0,2) &\mapsto (0,0,1) \\ (1,1,0) &\mapsto (1,0,0) \\ (1,0,1) &\mapsto (0,0,1) \\ (0,1,1) &\mapsto (0,1,0) \end{aligned}$$

h has four zeros: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1/3, 1/3, 1/3)$, but all of them are “repulsive”.



Theorem [Laslier & Laslier ++]:

The trajectory of Z_n accumulates on a cycle stable by the flow of $\dot{y} = h(y)$.

In a nutshell

We have

- a theorem that gives, in the "favourable" cases, convergence almost sure to some θ ($h(\theta) = 0$);
- conditionally on $Z_n \to \theta$, an easy-to-apply theorem that gives the speed of convergence in terms of a "central limit theorem".

Flaws:

- there seems to be no "easy criterion" that says which replacement rule R leads to a favourable case (other than calculating h);
- the second order results only apply if all eigenvalues of $Dh(\theta)$ on $\Sigma^{(d)}$ are negative.

I believe that this is the best we can do in full generality.

Future work

Remove the balance assumption.

- for 2-colour urns, we can prove $Z_n \to \theta$ where $h(\theta) = 0$ a.s., and partial result for the central limit theorem;
- but there is a lack of stochastic approximation results for d-dimensional, with random increment $1/T_n$: [Renlund '16]

$$Z_{n+1} = Z_n + \frac{1}{T_{n+1}}\big(h(Z_n) + \Delta M_{n+1}\big).$$

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Thank you!!