# Symbolic evaluation of determinants and rhombus tilings of holey hexagons

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# Beginning of the Story

Inventiones math. 53, 193 – 225 (1979)

*Inventiones mathematicae*



**Plane Partitions (III): The Weak Macdonald Conjecture**

George E. Andrews*

The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Dedicated to the memory of Alfred Young and F.J.W. Whipple

Determinant that counts descending plane partitions:

$$D_{0,0}(n) := \det_{1 \leqslant i,j \leqslant n} \left( \delta_{i,j} + \binom{\mu + i + j - 2}{j - 1} \right),$$

where $\delta_{i,j}$ denotes the Kronecker delta function.

## Andrews's Result

**Theorem.** We have

$$D_{0,0}(n) = 2 \prod_{i=1}^{n-1} R_{0,0}(i),$$

in other words $R_{0,0}(n) = D_{0,0}(n+1)/D_{0,0}(n)$, where

$$R_{0,0}(2n) = \frac{\left(\mu + 2n\right)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{\left(n\right)_n \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}},$$

$$R_{0,0}(2n-1) = \frac{\left(\mu + 2n - 2\right)_{n-1} \left(\frac{\mu}{2} + 2n - \frac{1}{2}\right)_n}{\left(n\right)_n \left(\frac{\mu}{2} + n - \frac{1}{2}\right)_{n-1}},$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n := a \cdot (a+1) \cdots (a+n-1).$$

Another question is the possibility of other general determinants of this nature. At first glance

$$E_m(\mu) = \det\left(\delta_{ij} + \binom{\mu+i+j}{i+1}\right)_{0\le i,j\le m-1}$$

looks interesting. Indeed it turns out that

$$E_1(\mu) = \mu + 1,$$

$$E_2(\mu) = (\mu+2)(\mu+1),$$

$$E_3(\mu) = \frac{(\mu+14)(\mu+3)(\mu+2)(\mu+1)}{12},$$

$$E_4(\mu) = \frac{(\mu+14)(\mu+9)(\mu+4)(\mu+3)(\mu+2)(\mu+1)}{72},$$

$$E_5(\mu) = \frac{(\mu+9)(\mu+5)(\mu+4)(\mu+3)(\mu+2)(\mu+1)(\mu^3+45\mu^2+722\mu+3432)}{8640}.$$

Emprically it seems reasonable to guess that

$$\frac{E_{2m}(\mu)}{E_{2m-1}(\mu)} = f_{2m,2m}(\mu-2),$$

George Andrews (1980): Macdonald's conjecture and descending plane partitions

# Andrews's Conjecture (1980)

Let $D_{1,1}(n)$ denote Andrews's interesting-looking determinant:

$$D_{1,1}(n) := \det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j} + \binom{\mu+i+j-2}{j}\right).$$

**Conjecture.** The following holds:

$$\begin{aligned}\frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} &= (-1)^{(n-1)(n-2)/2}\, 2^n \frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(-\frac{\mu}{2}-2n+\frac{3}{2}\right)_{\lfloor (n-1)/2\rfloor}}\\ &= 2^n \frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(\frac{\mu}{2}+\left\lfloor\frac{3n}{2}\right\rfloor+\frac{1}{2}\right)_{\lfloor (n-1)/2\rfloor}}\\ &= \frac{\left(\mu+2n\right)_n\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{\left(n\right)_n\left(\frac{\mu}{2}+n+\frac{1}{2}\right)_{n-1}}.\end{aligned}$$

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**Theorem.** The following holds:

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$\longrightarrow$ Proven by us in 2013.

$$
\begin{aligned}
D_{1,1}(1) &= \mu+1 \\
D_{1,1}(2) &= (\mu+1)(\mu+2) \\
D_{1,1}(3) &= \tfrac{1}{12}(\mu+1)(\mu+2)(\mu+3)(\mu+14) \\
D_{1,1}(4) &= \tfrac{1}{72}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+9)(\mu+14) \\
D_{1,1}(5) &= \tfrac{1}{8640}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+9) \\
&\quad \times(\mu^3+45\mu^2+722\mu+3432) \\
D_{1,1}(6) &= \tfrac{1}{518400}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6) \\
&\quad \times(\mu+8)(\mu+13)(\mu+15)(\mu^3+45\mu^2+722\mu+3432) \\
D_{1,1}(7) &= \tfrac{1}{870912000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_{1,1}(8) &= \tfrac{1}{731566080000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_{1,1}(9) &= \tfrac{1}{221225582592000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+21)^2 \\
&\quad \times(\mu^6+142\mu^5+8505\mu^4+277100\mu^3+5253404\mu^2+52937808\mu \\
D_{1,1}(10) &= \tfrac{1}{334493080879104000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+25)(\mu+27) \\
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$$C(n) = \frac{(-1)^n + 3}{2} \prod_{i=1}^{n} \frac{\left\lfloor \frac{i}{2} \right\rfloor !}{i!},$$

$$E(n) = (\mu+1)_n \left( \prod_{i=1}^{\left\lfloor \frac{3}{2} \left\lfloor \frac{1}{2}(n-1) \right\rfloor - 2 \right\rfloor} \Big(\mu + 2i + 6\Big)^{2\left\lfloor \frac{1}{3}(i+2) \right\rfloor} \right)$$

$$\times \left( \prod_{i=1}^{\left\lfloor \frac{3}{2} \left\lfloor \frac{n}{2} \right\rfloor - 2 \right\rfloor} \Big(\mu + 2i + 2\left\lfloor \frac{3}{2}\left\lfloor \frac{n}{2}+1 \right\rfloor \right\rfloor - 1\Big)^{2\left\lfloor \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{3}(i-1) \right\rfloor - 1} \right),$$

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$$F_m(n) = \left( \prod_{i=1}^{\left\lfloor \frac{1}{4}(n-1) \right\rfloor} (\mu + 2i + n + m)^{1-2i-m} \right)$$

$$\times \left( \prod_{i=1}^{\left\lfloor \frac{n}{4} - 1 \right\rfloor} (\mu - 2i + 2n - 2m + 1)^{1-2i-m} \right),$$

# Our Conjecture

...further let ...

$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even,} \\ E(n)F_1(n) \displaystyle\prod_{i=1}^{\frac{1}{2}(n-5)} (\mu + 2i + 2n - 1), & \text{if } n \text{ is odd,} \end{cases}$$

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$$\begin{aligned} T(k) = {} & 55296k^6 + 41472(\mu-1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 \\ & + 96(\mu-1)(15\mu^2 - 42\mu + 61)k^3 \\ & + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 \\ & + (\mu-1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ & + 2(\mu-3)(\mu-2)(\mu-1)(\mu+1), \end{aligned}$$

# Our Conjecture

…and let …

$$
\begin{aligned}
S_1(n) = \sum_{k=1}^{n-1} \Big( & 2^{6k}(\mu+8k-1)\left(\tfrac{1}{2}\right)_{2k-1}^{2}\left(\tfrac{1}{2}(\mu+5)\right)_{2k-3} \\
& \times \left(\tfrac{1}{2}(\mu+4k+2)\right)_{k-2}\left(\tfrac{1}{2}(\mu+4k+2)\right)_{2n-2k-2} T(k)\Big) \\
& \Big/ \Big( (2k)!\left(\tfrac{1}{2}(\mu+6k-3)\right)_{3k+4} \Big), \\
S_2(n) = \sum_{k=1}^{n-1} \Big( & 2^{6k}(\mu+8k+3)\left(\tfrac{1}{2}\right)_{2k}^{2}\left(\tfrac{1}{2}(\mu+5)\right)_{2k-2} \\
& \times \left(\tfrac{1}{2}(\mu+4k+4)\right)_{k-2}\left(\tfrac{1}{2}(\mu+4k+4)\right)_{2n-2k-2} T\big(k+\tfrac{1}{2}\big)\Big) \\
& \Big/ \Big( (2k+1)!\left(\tfrac{1}{2}(\mu+6k+1)\right)_{3k+5} \Big),
\end{aligned}
$$

# Our Conjecture

$$P_1(n) = 2^{3n-1} \frac{\left(\frac{1}{2}(\mu+6n-3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-3}} \times \left( \frac{\left(\frac{1}{2}(\mu+2)\right)_{2n-2}}{(\mu+3)^2} + \frac{\mu(\mu-1)}{2^{13}} S_1(n) \right),$$

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$$G(n) = \begin{cases} P_1\left(\frac{1}{2}(n+1)\right), & \text{if } n \text{ is odd,} \\ P_2\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Then for every positive integer $n$ we have

$$D_{1,1}(n) = C(n)\, F(n)\, G\left(\left\lfloor \tfrac{1}{2}(n+1) \right\rfloor\right).$$

**Definition:** For $n, s, t \in \mathbb{Z}$, $n \geqslant 1$, and $\mu$ an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$-determinant:

$$\begin{aligned} D_{s,t}(n) &:= \det_{\substack{s \leqslant i < s+n \\ t \leqslant j < t+n}} \left( \delta_{i,j} + \binom{\mu + i + j - 2}{j} \right) \\ &= \det_{1 \leqslant i,j \leqslant n} \left( \delta_{i+s-1,j+t-1} + \binom{\mu + i + j + s + t - 4}{j + t - 1} \right) \end{aligned}$$

# Generalization

**Definition:** For $n, s, t \in \mathbb{Z}$, $n \geqslant 1$, and $\mu$ an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$-determinant:

$$D_{s,t}(n) := \det_{\substack{s \leqslant i < s+n \\ t \leqslant j < t+n}} \left( \delta_{i,j} + \binom{\mu + i + j - 2}{j} \right)$$
$$= \det_{1 \leqslant i,j \leqslant n} \left( \delta_{i+s-1,j+t-1} + \binom{\mu + i + j + s + t - 4}{j + t - 1} \right)$$

**Known special cases:**

- closed form for $D_{0,0}(n)$ (Andrews 1979)
- closed form for $D_{1,1}(2n)/D_{1,1}(2n-1)$ (Andrews 1980)
- monstrous conjecture for $D_{1,1}(n)$ (K-T 2013)

# Desnanot-Jacobi-Carroll Identity (DJC)

**Theorem.** Let $\left(m_{i,j}\right)_{i,j\in\mathbb{Z}}$ be an infinite sequence and denote by $M_{s,t}(n)$ the determinant of the $(n \times n)$-matrix whose upper left entry is $m_{s,t}$, more precisely the matrix $\left(m_{i,j}\right)_{s\leqslant i<s+n, t\leqslant j<t+n}$. Then:

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

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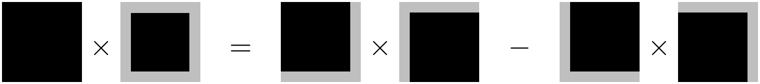
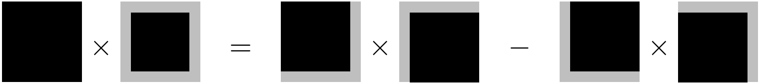
$$M_{s,t}(n)M_{s+1,t+1}(n-2) =$$
$$M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

**Schematically:**



# DJC for $D_{1,1}(n)$



By (DJC) we obtain a recurrence equation for $D_{1,1}(n)$:

$$D_{0,0}(n+1)D_{1,1}(n-1) = D_{0,0}(n)D_{1,1}(n) - D_{1,0}(n)D_{0,1}(n).$$

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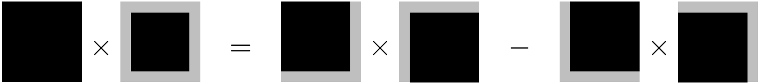

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We rewrite it slightly:

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{= R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

$\longrightarrow$ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

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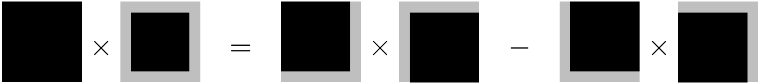

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$\longrightarrow$ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

**Question:** What is the combinatorial interpretation of the general determinant $D_{s,t}(n)$?

# Lindström-Gessel-Viennot Lemma

Let $G$ be a directed acyclic graph and consider base vertices $A = \{a_1, \ldots, a_n\}$ and destination vertices $B = \{b_1, \ldots, b_n\}$.

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$$e(a,b) = \sum_{P: a \to b} \omega(P) \qquad \text{and}$$

$$M = \begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & \cdots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & \cdots & e(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n,b_1) & e(a_n,b_2) & \cdots & e(a_n,b_n) \end{pmatrix}.$$

# Lindström-Gessel-Viennot Lemma

Let $G$ be a directed acyclic graph and consider base vertices $A = \{a_1, \ldots, a_n\}$ and destination vertices $B = \{b_1, \ldots, b_n\}$. For each path $P$, let $\omega(P)$ be the product of its edge weights. Let

$$e(a, b) = \sum_{P: a \to b} \omega(P) \qquad \text{and}$$

$$M = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & \cdots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & \cdots & e(a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n, b_1) & e(a_n, b_2) & \cdots & e(a_n, b_n) \end{pmatrix}.$$

Then the determinant of $M$ is the signed sum over all $n$-tuples $P = (P_1, \ldots, P_n)$ of non-intersecting paths from $A$ to $B$:

$$\det(M) = \sum_{(P_1, \ldots, P_n):\ A \to B} \operatorname{sign}(\sigma(P)) \prod_{i=1}^{n} \omega(P_i).$$

where $\sigma$ denotes a permutation that is applied to $B$.

# Lindström-Gessel-Viennot Lemma

**Application:** In our context, the lemma implies the following.

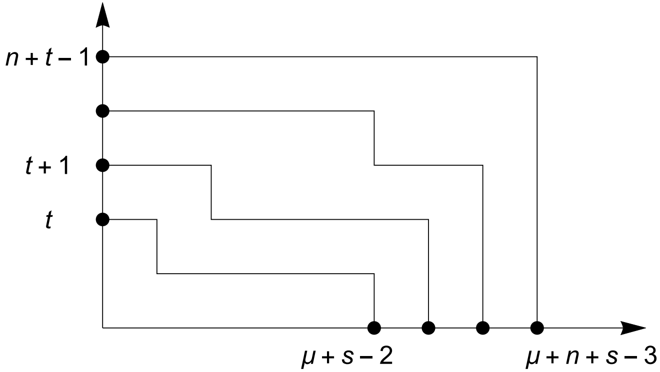
Look at the determinant without the Kronecker-Delta:

$$\det_{1\leqslant i,j\leqslant n} \binom{\mu+i+j+s+t-4}{j+t-1}.$$

It counts $n$-tuples of non-intersecting paths in the lattice $\mathbb{N}^2$:

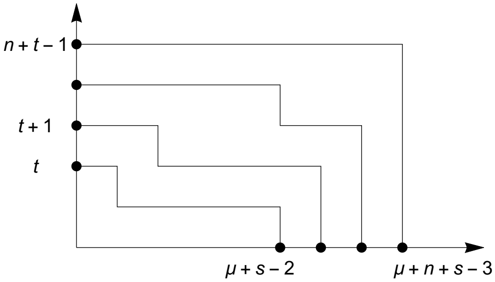
- The starting points are $(0,t),(0,t+1),\ldots,(0,t+n-1)$.
- The end points are $(\mu+s-2,0),\ldots,(\mu+s+n-3,0)$.
- The allowed steps are $(1,0)$ and $(0,-1)$.

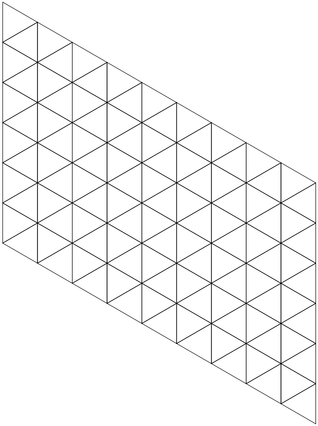
# Non-intersecting Lattice Paths




For $1 \leqslant i, j \leqslant n$ the number of paths from $(0, t+j-1)$ to $(\mu+s+i-3, 0)$ is given by $\binom{\mu+i+j+s+t-4}{j+t-1}$, which is precisely the $(i,j)$-entry of our matrix.

# Lattice Paths — Rhombus Tilings



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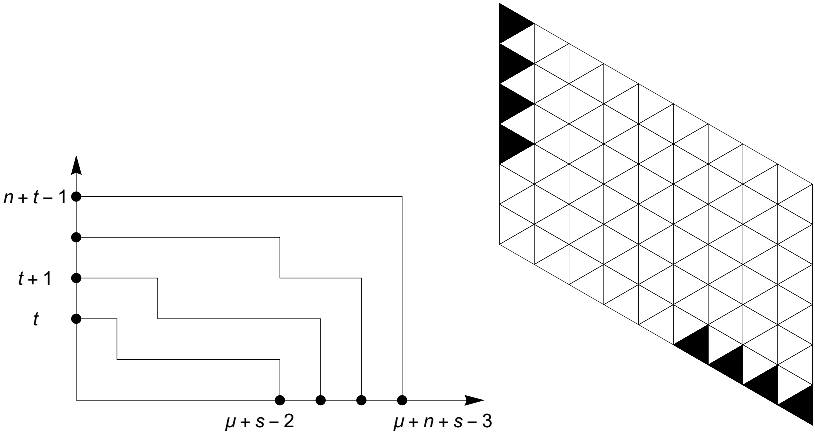

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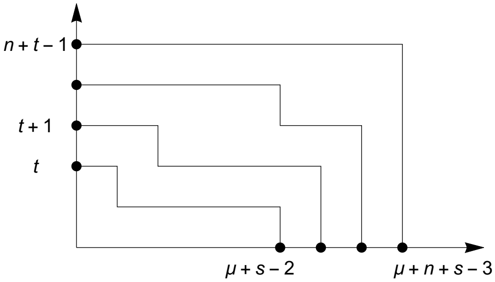


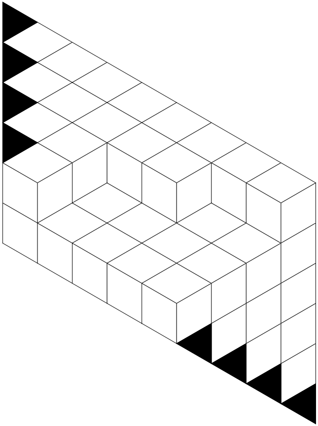

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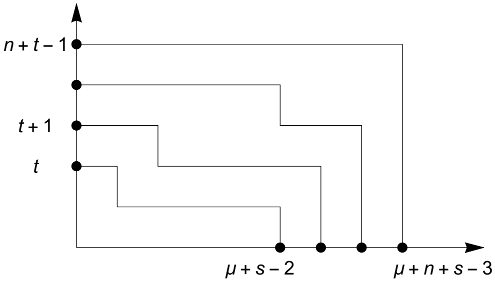


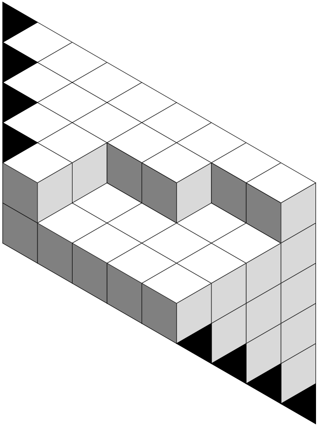

## Determinant with Kronecker-Delta

From the Laplace expansion one immediately sees that

$$\begin{vmatrix} \cdots & b_{1,j}+1 & b_{1,j+1} & \cdots \\ \cdots & b_{2,j} & b_{2,j+1}+1 & \cdots \\ & \vdots & \vdots & \end{vmatrix} =$$

$$\begin{vmatrix} \cdots & b_{1,j} & b_{1,j+1} & \cdots \\ \cdots & b_{2,j} & b_{2,j+1}+1 & \cdots \\ & \vdots & \vdots & \end{vmatrix} \pm \begin{vmatrix} \cdots & b_{2,j-1} & b_{2,j+1}+1 & \cdots \\ & \vdots & \vdots & \end{vmatrix}$$

By applying this procedure recursively, one obtains

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\left(M^{I}_{I+s-t}\right) \qquad (s \geqslant t),$$

where $M^I_J$ denotes the matrix that is obtained by deleting all rows with indices in $I$ and all columns with indices in $J$ from the matrix

$$\left(\binom{\mu+i+j+s+t-4}{j+t-1}\right)_{1 \leqslant i,j \leqslant n}.$$

# Kronecker-Deltas on the Main Diagonal

**General formula:**

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\left(M^{I}_{I+s-t}\right) \qquad (s \geqslant t)$$

**Special case:** If $s = t$ we obtain

$$D_{s,s}(n) = \sum_{I \subseteq \{1,\ldots,n\}} \det\left(M^{I}_{I}\right),$$

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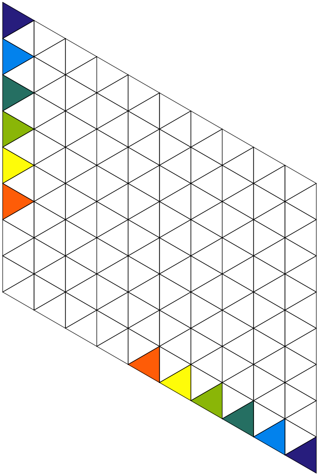
**Hence:** $D_{s,s}(n)$ counts all $k$-tuples of non-intersecting lattice paths, where $k$ runs from $0$ to $n$, and where the start and end points are given by the same $k$-subset.

# Kronecker-Deltas on the Main Diagonal

$s = 2$
$t = 2$
$n = 6$
$\mu = 4$



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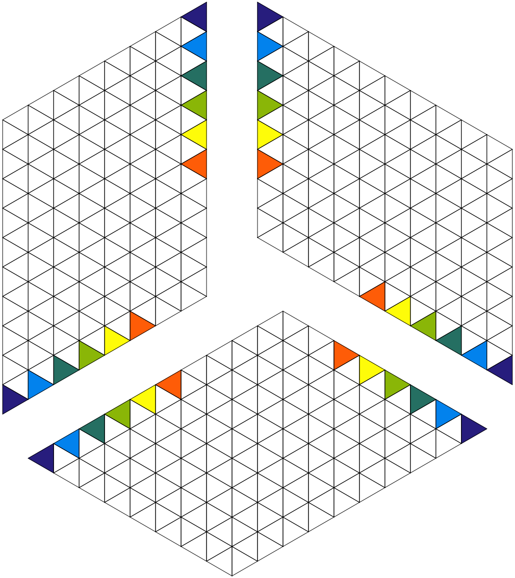

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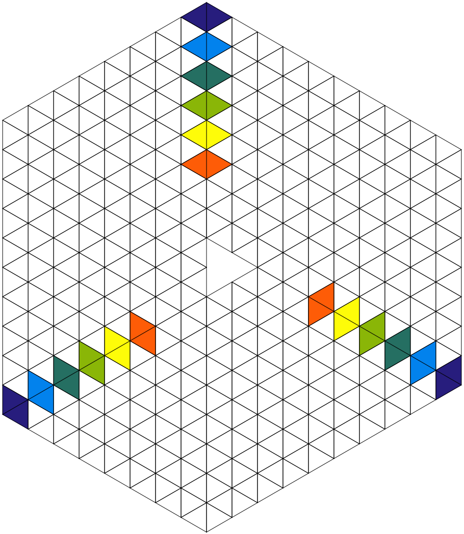

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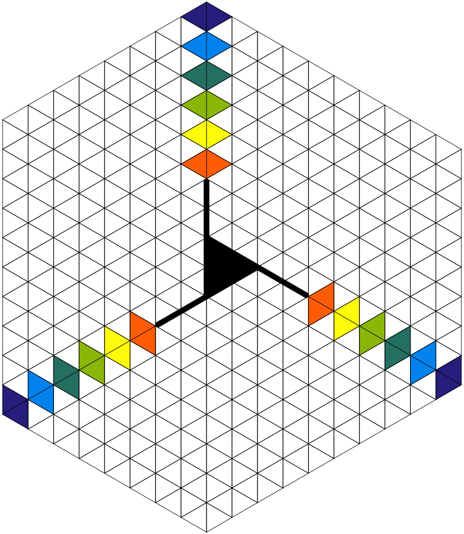

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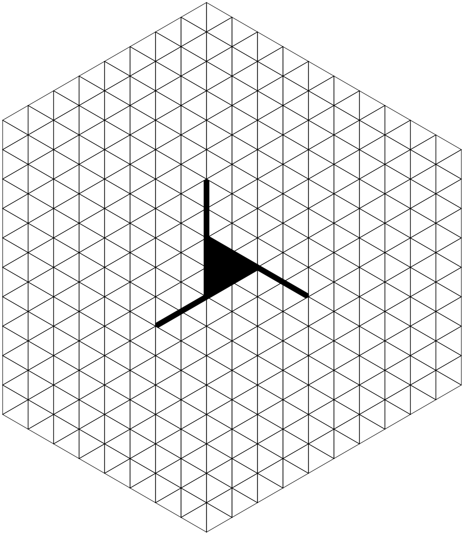

# Rhombus Tilings

**Finding:** The determinant $D_{s,s}(n)$ counts

- rhombus tilings
- of a hexagon with a funny-shaped hole ("holey hexagon")
- that are cyclically symmetric.
- The hole has the shape of a triangle (of size $\mu - 2$) with "boundary lines" (of length $s$) sticking out of its corners.

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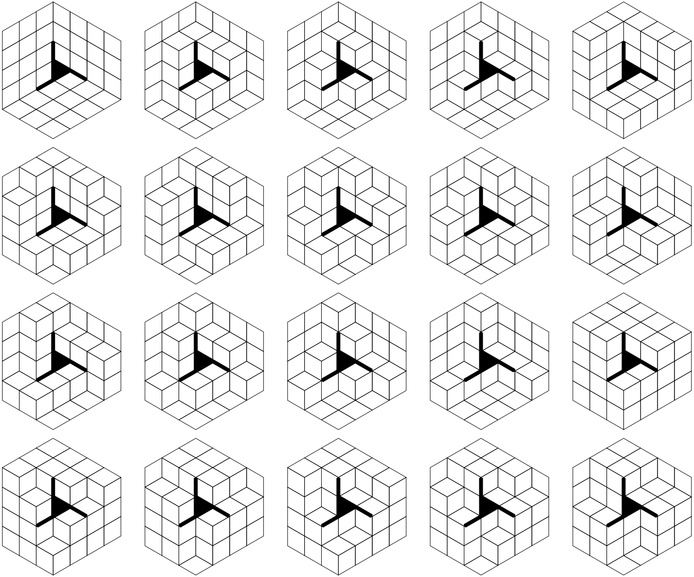
**Example:** For $s = t = 1$, $n = 2$, and $\mu = 3$ we obtain

$$D_{1,1}(2)\big|_{\mu\to 3} = \begin{vmatrix} 4 & 6 \\ 4 & 11 \end{vmatrix} = 20.$$

# Cyclically Symmetric Rhombus Tilings of a Holey Hexagon



# Off-Diagonal Kronecker-Deltas

Now let's look at the situation $s \neq t$.

**General formula:**

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n+s-t\}} (-1)^{(s-t)\cdot|I|} \ \det\bigl(M_I^{I+t-s}\bigr) \qquad (t \geqslant s)$$

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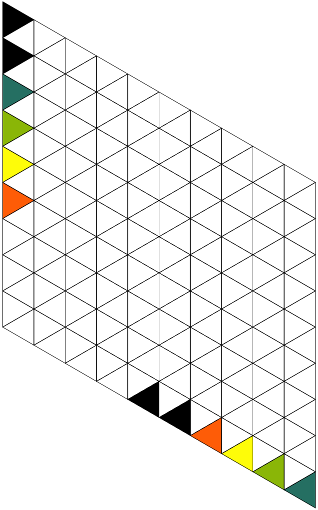
**Remark:** If $s-t$ is odd, we perform a weighted count with weights $+1$ and $-1$, according to the length of the tuples of paths.

# Off-Diagonal Kronecker-Deltas

$s = 1$
$t = 3$
$n = 6$
$\mu = 5$



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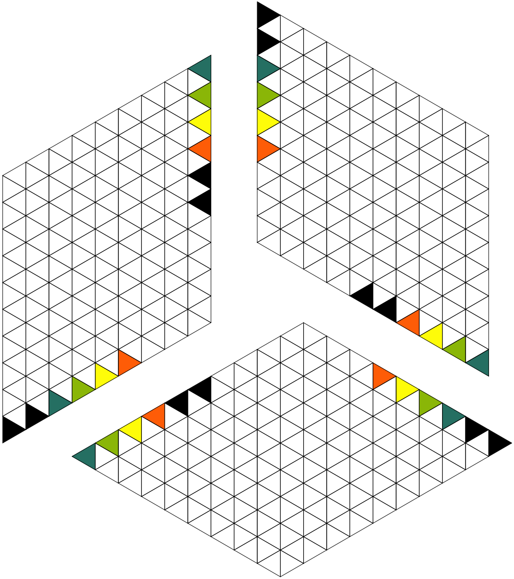

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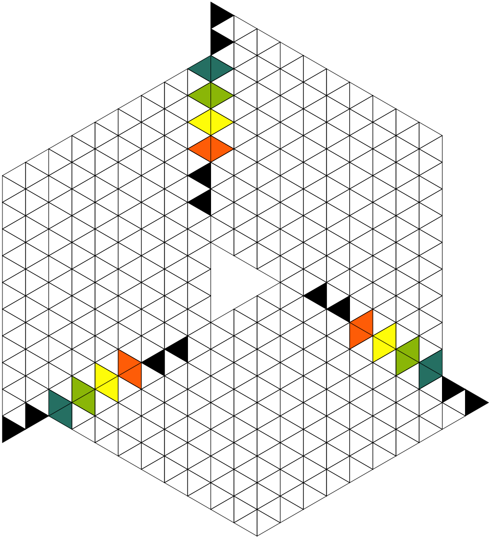

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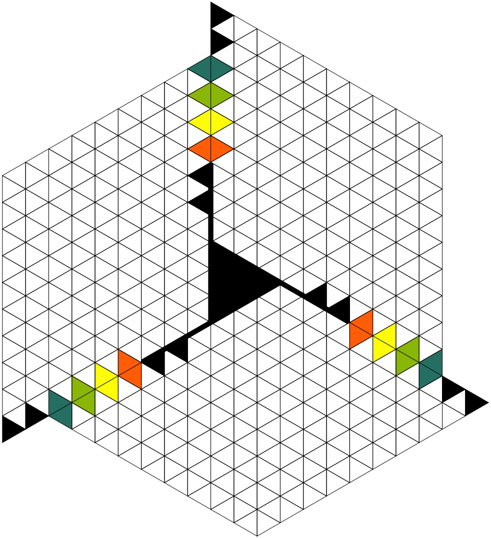

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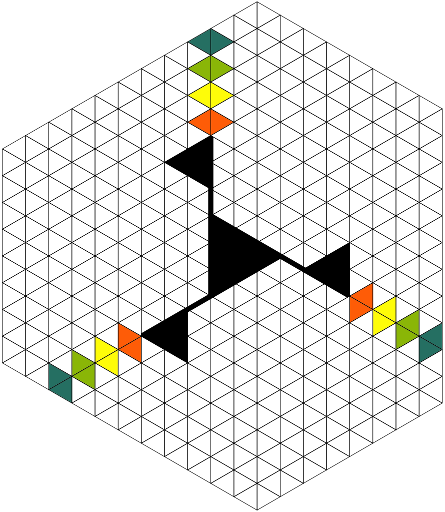

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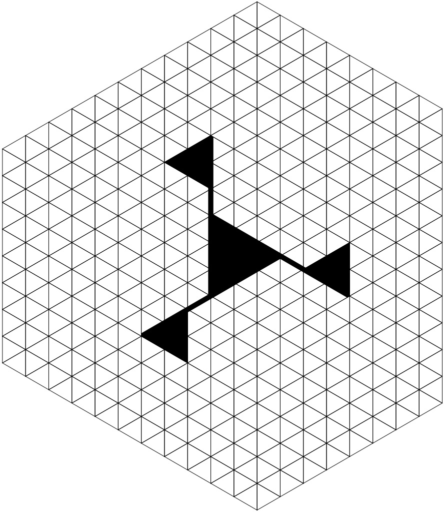
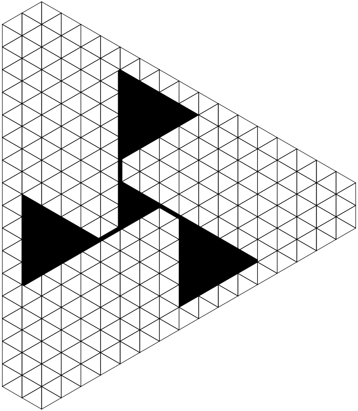
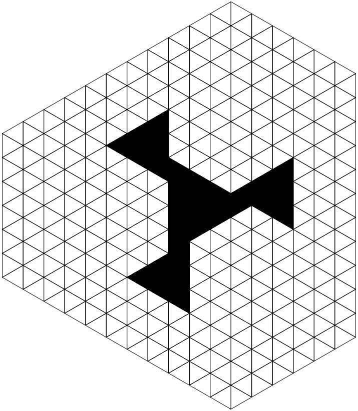


# Off-Diagonal Kronecker-Deltas

**Example:** Shapes for different choices of the parameters



$s = 5,\ t = 1,\ n = 5,\ \mu = 4$



$s = -1,\ t = 2,\ n = 6,\ \mu = 6$

# Back to $D_{1,1}(n)$

**Recall:** We wanted to evaluate $D_{1,1}(n)$ using (DJC):

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{= R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n) D_{0,1}(n)}{D_{0,0}(n)}.$$

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For example, we can show that $D_{1,0}(2n) = 0 = D_{0,1}(2n)$ for all $n$.

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- Normalize each generator $c_n$ (last component $= 1$).
- "Guess" recurrence equations for the bivariate sequence $c_{n,j}$.
- Use the holonomic systems approach (Zeilberger) to prove that $D_{0,1}(2n) \cdot c_n = 0$ for all $n$.

## The HOLONOMIC ANSATZ II. Automatic DISCOVERY(!) and PROOF(!!) of Holonomic Determinant Evaluations

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- $b_n \neq 0$ for all $n \geqslant 1$.

## Expansion Formula

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

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- $A_n^{(i,j)}$: matrix $A_n$ with row $i$ and column $j$ deleted

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- We obtain $\sum_{j=1}^{n} a_{i,j} c_{n,j} = \delta_{i,n} (\det A_n)/(\det A_{n-1})$

# Determinant Evaluation: Proof by Induction

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Now use $c_{n,j}$ to do Laplace expansion of $A_n$ w.r.t. the last row:

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The induction step is completed by proving that this is equal to $b_n$.

# Recipe for the Holonomic Ansatz

**Problem:** Given $a_{i,j}$ and $b_n \neq 0$. Show that $\det\left(a_{i,j}\right)_{1\leqslant i,j\leqslant n} = b_n$.

**Method:** "Pull out of the hat" a function $c_{n,j}$ and prove

$$
\begin{aligned}
c_{n,n} &= 1 && (n \geqslant 1),\\
\sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leqslant i < n),\\
\sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{b_n}{b_{n-1}} && (n \geqslant 1).
\end{aligned}
$$

Then $\det\left(a_{i,j}\right)_{1\leqslant i,j\leqslant n} = b_n$ holds.

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**Example:** For $D_{0,0}(2n)$ we obtain the following holonomic system of recurrence relations for $c_{n,j}$.

$$
\begin{aligned}
&\{(j+\mu+2n-3)(2\mu j^6+8nj^6-2j^6+3\mu^2j^5-48n^2j^5-12\mu j^5-24nj^5+9j^5+\\
&\mu^3j^4+48n^3j^4-11\mu^2j^4-84\mu n^2j^4+204n^2j^4+21\mu j^4-20\mu^2nj^4+38\mu nj^4-\\
&10nj^4-11j^4+216n^4j^3-2\mu^3j^3+312\mu n^3j^3-408n^3j^3+7\mu^2j^3+28\mu^2n^2j^3+\\
&122\mu n^2j^3-198n^2j^3-2\mu j^3-9\mu^3nj^3+68\mu^2nj^3-113\mu nj^3+78nj^3-3j^3-\\
&864n^5j^2-756\mu n^4j^2+432n^4j^2-\mu^3j^2-112\mu^2n^3j^2-308\mu n^3j^2+600n^3j^2+\\
&11\mu^2j^2-3\mu^3n^2j^2-66\mu^2n^2j^2+189\mu n^2j^2-168n^2j^2-23\mu j^2-2\mu^4nj^2+\\
&15\mu^3nj^2-28\mu^2nj^2+33\mu nj^2-34nj^2+13j^2+864n^6j+432\mu n^5j+432n^5j-\\
&144\mu^2n^4j+1116\mu n^4j-1104n^4j+2\mu^3j-88\mu^3n^3j+384\mu^2n^3j-392\mu n^3j-\\
&36n^3j-10\mu^2j-14\mu^4n^2j+45\mu^3n^2j+40\mu^2n^2j-317\mu n^2j+270n^2j+14\mu j-\\
&\mu^5nj+3\mu^4nj+17\mu^3nj-89\mu^2nj+112\mu nj-42nj-6j+432\mu n^6-864n^6+\\
&432\mu^2n^5-1080\mu n^5+432n^5+144\mu^3n^4-324\mu^2n^4-156\mu n^4+456n^4+20\mu^4n^3-\\
&18\mu^3n^3-220\mu^2n^3+470\mu n^3-204n^3+\mu^5n^2+3\mu^4n^2-37\mu^3n^2+57\mu^2n^2+\\
&36\mu n^2-60n^2+2\mu^4n-18\mu^3n+54\mu^2n-62\mu n+24n)c_{n,j}-(j+\mu-3)(2j+\mu-\\
&3)(j-2n+1)(\mu+4n-1)(j^4+2\mu j^3-6j^3+\mu^2j^2-12n^2j^2-9\mu j^2-6\mu nj^2+\\
&6nj^2+13j^2-3\mu^2j-12\mu n^2j+36n^2j+13\mu j-6\mu^2nj+24\mu nj-18nj-12j+\\
&2\mu^2-2\mu^2n^2+20\mu n^2-24n^2-6\mu-\mu^3n+11\mu^2n-22\mu n+12n+4)c_{n,j+1}+\\
&2(2j+\mu-2)n(2n+1)(-j+2n+1)(-j+2n+2)(j+\mu+2n-1)(\mu+4n-3)(\mu+\\
&4n-1)c_{n+1,j},-(j+1)(2j+\mu)(j-2n)(j+\mu+2n-3)c_{n,j}+(4j^4+8\mu j^3-8j^3+\\
&5\mu^2j^2-8n^2j^2-5\mu j^2-4\mu nj^2+12nj^2-8j^2+\mu^3j+2\mu^2j-8\mu n^2j+8n^2j-15\mu j-\\
&4\mu^2nj+16\mu nj-12nj+12j+\mu^3-3\mu^2-2\mu^2n^2+16n^2-2\mu-\mu^3n+3\mu^2n+\\
&8\mu n-24n+8)c_{n,j+1}-(j+\mu-2)(2j+\mu-2)(j-2n+2)(j+\mu+2n-1)c_{n,j+2}\}
\end{aligned}
$$

$$\{(j+\mu+2n-3)(2\mu j^6+8nj^6-2j^6+3\mu^2j^5-48n^2j^5-12\mu j^5-24nj^5+9j^5+\mu^3j^4+48n^3j^4-11\mu^2j^4-84\mu n^2j^4+204n^2j^4+21\mu j^4-20\mu^2nj^4+38\mu nj^4-10nj^4-11j^4+216n^4j^3-2\mu^3j^3+312\mu n^3j^3-408n^3j^3+7\mu^2j^3+28\mu^2n^2j^3+122\mu n^2j^3-198n^2j^3-2\mu j^3-9\mu^3nj^3+68\mu^2nj^3-113\mu nj^3+78nj^3-3j^3-864n^5j^2-756\mu n^4j^2+432n^4j^2-\mu^3j^2-112\mu^2n^3j^2-308\mu n^3j^2+600n^3j^2+11\mu^2j^2-3\mu^3n^2j^2-66\mu^2n^2j^2+189\mu n^2j^2-168n^2j^2-23\mu j^2-2\mu^4nj^2+15\mu^3nj^2-28\mu^2nj^2+33\mu nj^2-34nj^2+13j^2+864n^6j+432\mu n^5j+432n^5j-144\mu^2n^4j+1116\mu n^4j-1104n^4j+2\mu^3j-88\mu^3n^3j+384\mu^2n^3j-392\mu n^3j-36n^3j-10\mu^2j-14\mu^4n^2j+45\mu^3n^2j+40\mu^2n^2j-317\mu n^2j+270n^2j+14\mu j-\mu^5nj+3\mu^4nj+17\mu^3nj-89\mu^2nj+112\mu nj-42nj-6j+432\mu n^6-864n^6+432\mu^2n^5-1080\mu n^5+432n^5+144\mu^3n^4-324\mu^2n^4-156\mu n^4+456n^4+20\mu^4n^3-18\mu^3n^3-220\mu^2n^3+470\mu n^3-204n^3+\mu^5n^2+3\mu^4n^2-37\mu^3n^2+57\mu^2n^2+36\mu n^2-60n^2+2\mu^4n-18\mu^3n+54\mu^2n-62\mu n+24n)c_{n,j}-(j+\mu-3)(2j+\mu-3)(j-2n+1)(\mu+4n-1)(j^4+2\mu j^3-6j^3+\mu^2j^2-12n^2j^2-9\mu j^2-6\mu nj^2+6nj^2+13j^2-3\mu^2j-12\mu n^2j+36n^2j+13\mu j-6\mu^2nj+24\mu nj-18nj-12j+2\mu^2-2\mu^2n^2+20\mu n^2-24n^2-6\mu-\mu^3n+11\mu^2n-22\mu n+12n+4)c_{n,j+1}+2(2j+\mu-2)n(2n+1)(-j+2n+1)(-j+2n+2)(j+\mu+2n-1)(\mu+4n-3)(\mu+4n-1)c_{n+1,j},-(j+1)(2j+\mu)(j-2n)(j+\mu+2n-3)c_{n,j}+(4j^4+8\mu j^3-8j^3+5\mu^2j^2-8n^2j^2-5\mu j^2-4\mu nj^2+12nj^2-8j^2+\mu^3j+2\mu^2j-8\mu n^2j+8n^2j-15\mu j-4\mu^2nj+16\mu nj-12nj+12j+\mu^3-3\mu^2-2\mu^2n^2+16n^2-2\mu-\mu^3n+3\mu^2n+8\mu n-24n+8)c_{n,j+1}-(j+\mu-2)(2j+\mu-2)(j-2n+2)(j+\mu+2n-1)c_{n,j+2}\}$$

## Back to $D_{1,1}(n)$

Using Zeilberger's method, we obtain product formulas for the missing determinants.

$$D_{1,1}(n) = R_{0,0}(n)D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

Since $D_{0,1}(n) = D_{1,0}(n) = 0$ for even $n$, the recurrence simplifies:

$$D_{1,1}(n) = R_{0,0}(n)D_{1,1}(n-1) \qquad (n \text{ even}).$$

For odd $n$ we obtain $D_{1,1}(n) =$

$$= R_{0,0}(n)D_{1,1}(n-1) + (\mu-1)\frac{\Big(\prod_{j=1}^{\frac{n-1}{2}} R_{1,0}(j)\Big)\Big(\prod_{j=1}^{\frac{n-1}{2}} R_{0,1}(j)\Big)}{2\prod_{j=1}^{n-1} R_{0,0}(j)}$$

$$= R_{0,0}(n)D_{1,1}(n-1) + \frac{(\mu-1)}{2}\prod_{j=1}^{(n-1)/2} \frac{R_{1,0}(j)R_{0,1}(j)}{R_{0,0}(2j-1)R_{0,0}(2j)}.$$

# Main Result

**Theorem.** Let $\mu$ be an indeterminate and let $\rho_k$ be defined as $\rho_0(a,b) = a$ and $\rho_k(a,b) = b$ for $k > 0$. If $n$ is an odd positive integer then

$$D_{1,1}(n) = \sum_{k=0}^{(n+1)/2} \rho_k\left(4(\mu-2), \frac{1}{(2k-1)!}\right) \frac{(\mu-1)_{3k-2}}{2\left(\frac{\mu}{2}+k-\frac{1}{2}\right)_{k-1}}$$
$$\times \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_{j-1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)^2$$
$$\times \left(\prod_{j=k}^{(n-1)/2} \frac{(\mu+2j)_j^2\left(\frac{\mu}{2}+2j-\frac{1}{2}\right)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j+1}}{(j)_j\,(j+1)_{j+1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_j^2}\right)$$

If $n$ is an even positive integer then... [similar formula]

# More Results

We can give closed-form evaluations of some infinite one-dimensional families of $D_{s,t}(n)$.

| $t \setminus s$ | $\cdots$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vdots$ | | | | $\vdots$ | $\vdots$ | $\vdots$ | | | | | | |
| $6$ | | | | D | A | C | | | | | | |
| $5$ | | | | F | B | E | | | | | | |
| $4$ | | | | D | A | C | | | | | | |
| $3$ | | | | F | B | E | | | | | | |
| $2$ | | | | D | A | C | | | | | | |
| $1$ | | | | F | B | E | C | E | C | E | C | $\cdots$ |
| $0$ | | | | D | A | B | A | B | A | B | A | $\cdots$ |
| $-1$ | | | | A′ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |
| $-2$ | | | A′ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |
| $-3$ | | A′ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |
| $\vdots$ | $\iddots$ | $\iddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

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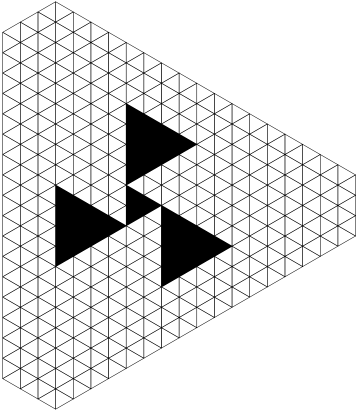

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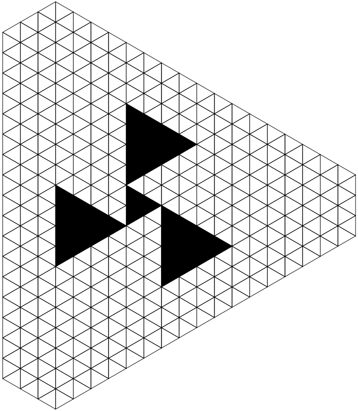

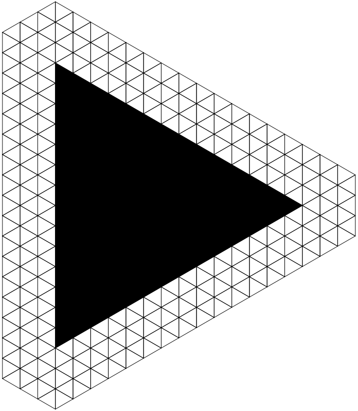

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**Family B:** If $n \geqslant 2r$ is an even number, then $D_{2r-1,0}(n) = 0$.

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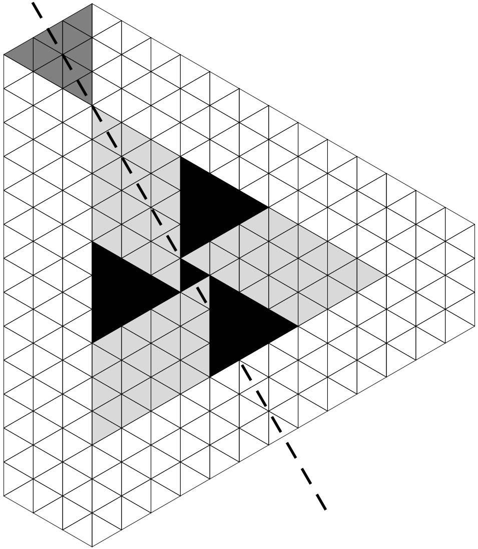

# Reference

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