Abstract

We show that $\mathfrak{b} = \mathfrak{c} = \omega_3$ is consistent with the existence of a $\Delta^1_3$-definable wellorder of the reals and a $\Pi^1_2$-definable $\omega$-mad subfamily of $[\omega]^{\omega}$ (resp. $\omega^\omega$).

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1. Introduction

The existence of a projective, in fact $\Delta^1_3$-definable wellorder of the reals in the presence of large continuum, i.e. $\mathfrak{c} \geq \omega_3$, was established by Harrington in [8]. In the present paper, we develop an iteration technique which allows one not only to obtain the consistency of the existence of a $\Delta^1_3$-definable wellorder of the reals with large continuum (see Theorem 1), but in addition the existence of a $\Pi^1_2$-definable $\omega$-mad family with $\mathfrak{b} = \mathfrak{c} = \omega_3$ (see Theorem 2). The method is a natural generalization to models with large continuum of the iteration technique developed in [5]. We expect that an application of Jensen’s coding techniques will lead to the same result with essentially arbitrary values for $\mathfrak{c}$.

For a more detailed introduction to the subject of projective wellorders of the reals and projective mad families, see [5] and [7]. Recall that a family $\mathcal{A}$ of infinite subsets of $\omega$ is almost disjoint if any two of its elements have finite intersection. An infinite almost disjoint family $\mathcal{A}$ is maximal (abbreviated mad family), if for every infinite subset $b$ of $\omega$, there is an element $a \in \mathcal{A}$ such that $|a \cap b| = \omega$. If $\mathcal{A}$ is an almost disjoint family, let $\mathcal{L}(\mathcal{A}) = \{b \in [\omega]^\omega : b$ is not covered by finitely many elements of $\mathcal{A}\}$. A mad family $\mathcal{A}$
is \( \omega \)-mad if for every \( B \in [\mathcal{L}(\mathcal{A})]^{\omega} \), there is \( a \in \mathcal{A} \) such that \( |a \cap b| = \omega \) for all \( b \in B \). For the definition of \( b \), as well as an introduction to the subject of cardinal characteristics of the continuum we refer the reader to [1].

In section 2 we introduce a model in which \( b = \epsilon = \omega_3 \) and there is a \( \Delta^1_2 \)-definable wellorder of the reals. In section 3 we show how to modify the argument to obtain in addition the existence of a \( \Pi^1_2 \)-definable \( \omega \)-mad family. We begin by fixing an appropriate sequence \( \bar{S} = \langle S_\alpha : 1 < \alpha < \omega_3 \rangle \) of stationary subsets of \( \omega_3 \) and explicitly destroying the stationarity of each \( S_\alpha \) by adding a closed unbounded subset of \( \omega_3 \) disjoint from it. The wellorder is produced by introducing reals (see Steps 1 through 3 in section 2) which code this stationary kill for certain stationary sets from \( \bar{S} \). For this purpose, we use almost disjoint coding as well as a modified version of the method of localization (see [4] and [5, Definition 1]).

2. Projective Wellorders with Large Continuum

Throughout the paper we work over the constructible universe \( L \), thus unless otherwise specified \( V = L \). Let \( \langle G_\xi : \xi \in \omega_2 \cap \text{cof}(\omega_1) \rangle \) be a \( \Diamond_{\omega_2}(\text{cof}(\omega_1)) \) sequence which is \( \Sigma_1 \) definable over \( L_{\omega_2} \). For every \( \alpha < \omega_3 \), let \( W_\alpha \) be the \( L \)-least subset of \( \omega_2 \) coding the ordinal \( \alpha \). Let \( \bar{S}' = \langle S_\alpha : 1 < \alpha < \omega_3 \rangle \) be the sequence of stationary subsets of \( \omega_2 \) defined as follows: \( S_\alpha = \{ \xi \in \omega_2 \cap \text{cof}(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset \} \). In particular, the sets \( S_\alpha \) are stationary subsets of \( \text{cof}(\omega_1) \cap \omega_2 \) which are mutually almost disjoint (that is, for all \( 1 < \alpha, \beta < \omega_3, \alpha \neq \beta \), we have that \( S_\alpha \cap S_\beta \) is bounded). Let \( S_{-1} = \{ \xi \in \omega_2 \cap \text{cof}(\omega_1) : G_\xi = 0 \} \). Note that \( S_{-1} \) is a stationary subset of \( \omega_2 \cap \text{cof}(\omega_1) \) disjoint from all \( S_\alpha \)'s.

Say that a transitive \( \text{ZF}^- \) model \( M \) is suitable if \( \omega_3^M \) exists and \( \omega_3^M = \omega_3^L \). From this it follows, of course, that \( \omega_1^M = \omega_1^L \) and \( \omega_2^M = \omega_2^L \).

**Step 0.** For every \( \alpha : \omega_2 \leq \alpha < \omega_3 \) shoot a closed unbounded set \( C_\alpha \) disjoint from \( S_\alpha \) via a poset \( \mathbb{P}_0^\alpha \). The poset \( \mathbb{P}_0^\alpha \) consists of all bounded, closed subsets of \( \omega_2 \), which are disjoint from \( S_\alpha \). The extension relation is end-extension. Note that \( \mathbb{P}_0^\alpha \) is countably closed and \( \text{N}_2 \)-distributive (see [3]). For every \( \alpha \in \omega_2 \) let \( \mathbb{P}_0^\alpha \) be the trivial poset.

Let \( \mathbb{P}^0 = \prod_{\alpha < \omega_3} \mathbb{P}_0^\alpha \) be the direct product of the \( \mathbb{P}_0^\alpha \)'s with supports of size \( \omega_1 \). Then \( \mathbb{P}^0 \) is countably closed and by the \( \Delta \)-system Lemma, also \( \omega_3 \)-c.c. Its \( \omega_2 \)-distributivity is easily established using the stationary set \( S_{-1} \subseteq \omega_2 \cap \text{cof}(\omega_1) \).

**Step 1.** We begin by fixing some notation. Let \( X \) be a set of ordinals. Denote by \( 0(X) \), \( I(X) \), and \( II(X) \) the sets \( \{ \eta : 3\eta \in X \} \), \( \{ \eta : 3\eta + 1 \in X \} \) and \( \{ \eta : 3\eta + 2 \in X \} \), respectively. Let \( \text{Even}(X) \) be the set of even ordinals in \( X \) and \( \text{Odd}(X) \) be the set of odd ordinals in \( X \).
In the following we treat 0 as a limit ordinal. For every $\alpha : \omega_2 \leq \alpha < \omega_3$ let $D_\alpha \subset \omega_2$ be a set coding the tuple $\langle C_\alpha, W_\alpha, W_\gamma \rangle$, where $\gamma$ is the largest limit ordinal $\leq \alpha$. More precisely $D_\alpha$ is such that $0(D_\alpha)$, $I(D_\alpha)$, and $II(D_\alpha)$ equal $C_\alpha$, $W_\alpha$, and $W_\gamma$, respectively. Now let $E_\alpha$ be the club in $\omega_2$ of intersections with $\omega_2$ of elementary submodels of $L_{\alpha + \omega_2 + 1}[D_\alpha]$ which contain $\omega_1 \cup \{D_\alpha\}$ as a subset. (These elementary submodels form an $\omega_2$-chain.) Now choose $Z_\alpha$ to be a subset of $\omega_2$ such that $Even(Z_\alpha) = D_\alpha$, and if $\beta < \omega_2$ is $\omega_2^M$ for some suitable model $M$ such that $Z_\alpha \cap \beta \in M$, then $\beta$ belongs to $E_\alpha$. (This is easily done by placing in $Z_\alpha$ a code for a bijection $\phi : \beta_1 \to \omega_1$ on the interval $(\beta_0, \beta_0 + \omega_1)$ for each adjacent pair $\beta_0 < \beta_1$ from $E_\alpha$.) Then we have:

\[\text{(*)_a: If } \beta < \omega_2 \text{ and } M \text{ is any suitable model such that } \omega_1 \subset M, \omega_2^M = \beta, \text{ and } Z_\alpha \cap \beta \in M, \text{ then } M \models \psi(\omega_2, Z_\alpha \cap \beta), \text{ where } \psi(\omega_2, X) \text{ is the formula } " \text{Even}(X) \text{ codes a tuple } \langle \tilde{C}, \tilde{W}, \tilde{W} \rangle, \text{ where } \tilde{W} \text{ and } \tilde{W} \text{ are the L-least codes of ordinals } \tilde{\alpha}, \tilde{\alpha} < \omega_2 \text{ such that } \tilde{\alpha} \text{ is the largest limit ordinal not exceeding } \alpha, \text{ and } \tilde{C} \text{ is a club in } \omega_2 \text{ disjoint from } S_{\tilde{\alpha}}."\]

Indeed, given a suitable model $M$ with $\omega_2^M = \beta$ and $Z_\alpha \cap \beta \in M$, note that $\beta \in E_\alpha$ by the construction of $Z_\alpha$ and also that $D_\alpha \cap \beta \in M$. Let $N$ be an elementary submodel of $L_{\alpha + \omega_2 + 1}[D_\alpha]$ such that $\omega_1 \cup \{D_\alpha\} \subset N$ and $N \cap \omega_2 = \beta$. Denote by $\tilde{N}$ the transitive collapse of $N$. Then $N = L_\xi[D_\alpha]$ for some $\omega_2 > \xi > \beta$ and $\omega_2^N = \omega_2^M = \beta$. Therefore $\tilde{N} \subset M$. Let $Z_\alpha' \subset \omega_2$ be such that Even($Z_\alpha'$) = Odd($Z_\alpha'$) = $D_\alpha$. By the definition of $D_\alpha$, $L_{\alpha + \omega_2 + 1}[D_\alpha] \models \psi(\omega_2, Z_\alpha')$. By elementarity, $\tilde{N} \models \psi(\omega_2, Z_\alpha' \cap \beta)$. Since the formula $\psi$ is $\Sigma_1$, $\omega_2^\tilde{N} = \omega_2^M$, we conclude that $M \models \psi(\omega_2, Z_\alpha' \cap \beta)$. Since $Z_\alpha \cap \beta \in M$ and $Even(Z_\alpha') = Even(Z_\alpha)$, we have $M \models \psi(\omega_2, Z_\alpha \cap \beta)$, which finishes the proof of (*)$_a$.

Now similarly to $S_\xi$ we can define a sequence $\tilde{\alpha} = \langle \alpha_\xi : \xi < \omega_2 \rangle$ of stationary subsets of $\omega_1$ using the “standard” $\diamondsuit$-sequence. Then in particular this sequence is nicely definable over $L_{\omega_1}$ and almost disjoint. Now we code $Z_\alpha$ by a subset $X_\alpha$ of $\omega_1$ with the forcing $P_\alpha$ consisting of all tuples $\langle s_0, s_1 \rangle \in [\omega_1]^{|\omega_1|} \times [Z_\alpha]^{|\omega_1|}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff $s_0$ is an initial segment of $t_0$, $s_1 \subseteq t_1$ and $t_0 \cap s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$. Then $X_\alpha$ obviously satisfies the following condition:

\[\text{(**)$_a$: If } \omega_1 < \beta < \omega_2 \text{ and } M \text{ is a suitable model such that } \omega_2^M = \beta \text{ and } \{X_\alpha\} \cup \omega_1 \subset M, \text{ then } M \models \phi(\omega_1, \omega_2, X_\alpha), \text{ where } \phi(\omega_1, \omega_2, X) \text{ is the formula } " \text{Using the sequence } \tilde{\alpha}, X \text{ almost disjointly codes a subset } \tilde{Z} \text{ of } \omega_2, \text{ whose even part } Even(\tilde{Z}) \text{ codes a tuple } \langle \tilde{C}, \tilde{W}, \tilde{W} \rangle, \text{ where } \tilde{W} \text{ and } \tilde{W} \text{ are the L-least codes of ordinals } \tilde{\alpha}, \tilde{\alpha} < \omega_3 \text{ such that } \tilde{\alpha} \text{ is the largest limit ordinal not exceeding } \alpha, \text{ and } \tilde{C} \text{ is a club in } \omega_2 \text{ disjoint from } S_{\tilde{\alpha}}."\]
Let $\mathbb{P}^1 = \prod_{\alpha<\omega_1} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for $\alpha \in \omega_2$, be the product of the $\mathbb{P}^1_\alpha$'s with countable support. The poset $\mathbb{P}^1$ is easily seen to be countably closed. Moreover, it has the $\omega_2$-c.c. by a standard $\Delta$-system argument.

**Step 2.** Now we shall force a localization of the $X_\alpha$’s. Fix $\phi$ as in $(**)_\alpha$.

**Definition 1.** Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $M$ with $\omega_1^M = \omega_1^1$ containing $X$ and $X'$, respectively. We denote by $L(X, X')$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of $r$ is a countable limit ordinal such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$
3. if $\gamma \leq |r|$, $M$ is a countable suitable model containing $r \upharpoonright \gamma$ as an element and $\gamma = \omega_1^M$, then $M \models \phi(\omega_1, \omega_2, X \cap \gamma) \land \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension.

Set $\mathbb{P}^2_{\alpha+m} = L(X_{\alpha+m}, X_\alpha)$ for every $\alpha \in Lim(\omega_3) \setminus \omega_2$ and $m \in \omega$. Let $\mathbb{P}^2_{\alpha+m}$ be the trivial poset for every $\alpha \in Lim(\omega_2)$ and $m \in \omega$. Let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. By the $\Delta$-system Lemma in $L^{\mathbb{P}^0 \ast \mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the $\omega_2$-c.c.

Observe that the poset $\mathbb{P}^2_{\alpha+m}$, where $\alpha > 0$, produces a generic function from $\omega_1$ (of $L^{\mathbb{P}^0 \ast \mathbb{P}^1}$) into 2, which is the characteristic function of a subset $Y_{\alpha+m}$ of $\omega_1$ with the following property:

$(**)_\alpha$: For every $\beta < \omega_1$ and any suitable $M$ such that $\omega_1^M = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $M$, we have $M \models \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \land \phi(\omega_1, \omega_2, X_\alpha \cap \beta)$.

**Lemma 1.** The poset $\mathbb{P}_0 := \mathbb{P}^0 \ast \mathbb{P}^1 \ast \mathbb{P}^2$ is $\omega$-distributive.

**Proof.** Given a condition $p_0 \in \mathbb{P}_0$ and a collection $\{O_n\}_{n \in \omega}$ of open dense subsets of $\mathbb{P}_0$, choose the least countable elementary submodel $\mathcal{N}$ of some large $L_\theta$ ($\theta$ regular) such that $\{p_0\} \cup \{\mathbb{P}_0\} \cup \{O_n\}_{n \in \omega} \subset \mathcal{N}$. Build a subfilter $g$ of $\mathbb{P}_0 \cap \mathcal{N}$, below $p_0$, which hits all dense subsets of $\mathbb{P}_0$ which belong to $\mathcal{N}$. Write $g$ as $g(0) \ast g(1) \ast g(2)$. Now $g(0) \ast g(1)$ has a greatest lower bound $p(0) \ast p(1)$ because the forcing $\mathbb{P}^0 \ast \mathbb{P}^1$ is $\omega$-closed. The condition $(p(0), p(1))$ is obviously $(\mathcal{N}, \mathbb{P}^0 \ast \mathbb{P}^1)$-generic.
On each component $\alpha + m \in \mathcal{N} \cap \omega_3$, where $\alpha \in \text{Lim}(\omega_3)$, $m \in \omega$, define $p(2)(\alpha + m) = \bigcup g(2)(\alpha + m)$. It suffices to verify that $p(2)(\alpha + m)$ is a condition in $\mathbb{P}_\alpha^{\omega_m} \ast \mathbb{P}_\alpha^{1 \omega_m} \ast \mathbb{P}_\alpha^{1 \omega_m}$, for this will give us a condition $p(2)$ so that $p(0) \ast p(1) = p(2)$ meets each of the $O_\alpha$’s.

As $(p(0)(\alpha), p(0)(\alpha + m), p(1)(\alpha), p(1)(\alpha + m))$ is a $(\mathcal{N}, \mathbb{P}_\alpha^{0 \omega_m} \ast \mathbb{P}_\alpha^{0 \omega_m} \ast \mathbb{P}_\alpha^{1 \omega_m} \ast \mathbb{P}_\alpha^{1 \omega_m})$-generic condition, if

$$G := G(0)(\alpha) \ast G(0)(\alpha + m) \ast G(1)(\alpha) \ast G(1)(\alpha + m)$$

is a $\mathbb{P}_\alpha^{0 \omega_m} \ast \mathbb{P}_\alpha^{0 \omega_m} \ast \mathbb{P}_\alpha^{1 \omega_m}$-generic filter over $L$ containing it, then the isomorphism $\pi$ of the transitive collapse $\dot{\mathcal{N}}$ of $\mathcal{N}$, onto $\mathcal{N}$ extends to an elementary embedding from

$$\dot{\mathcal{N}}_0 := \dot{\mathcal{N}}[\bar{g}(0)(\bar{a}) \ast \bar{g}(0)(\bar{a} + m) \ast \bar{g}(1)(\bar{a}) \ast \bar{g}(1)(\bar{a} + m)]$$

into $L_\theta[G]$. Here $\bar{g}(i) = \pi^{-1}(g(i))$, $i \in 2$, and $\bar{\xi} = \pi^{-1}(\xi)$ for all $\xi \in \mathcal{N} \cap \text{Ord}$. By the genericity of $G$ we know that, letting $X_\eta = \bigcup (G(1)(\alpha), X_{\alpha + m} = \bigcup (G(1)(\alpha + m), \text{properties } (**)_\alpha \text{ and } (**)_\alpha + m \text{ hold. By elementarity, } \dot{\mathcal{N}}_0 \text{ is a suitable model and } \dot{\mathcal{N}}_0 \models \phi(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m}), \text{ where } x_{\bar{a}} = \bigcup (g(1)(\alpha) = \bigcup (\bar{g}(1)(\bar{a}) \land x_{\bar{a} + m} = \bigcup (g(1)(\alpha + m) = \bigcup (\bar{g}(1)(\bar{a} + m)). \text{ By the construction of } \mathbb{P}_0, \mathcal{N}_0 = \mathcal{N}[x_{\bar{a}}, x_{\bar{a} + m}] \text{ and hence } \mathcal{N}_0 = \mathcal{N}[x_{\bar{a}}, x_{\bar{a} + m}] \models \phi(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m}).$

Let $\xi$ be such that $\dot{\mathcal{N}} = L_\xi$ and let $M$ be any suitable model containing $p(2)(\alpha)$, $p(2)(\alpha + m)$, and such that $\omega_1^M = \omega_1 \cap \mathcal{N}$. We have to show that $\mathcal{M} = p(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m})$. Set $\eta = \mathcal{M} \cap \text{Ord}$ and consider the chain $M_0 \subseteq M_1 \subseteq \mathcal{M}$ of suitable models, where $M_2 = L_{\eta}[x_{\bar{a}}, x_{\bar{a} + m}]$ and $M_1 = L_{\eta}[p(2)(\alpha), p(2)(\alpha + m)]$. Three cases are possible.

**Case a.** $\eta > \xi$. Since $\mathcal{N}$ was chosen to be the least countable elementary submodel of $L_\theta$ containing the initial condition, the poset and the sequence of dense sets, it follows that $\xi$ (and therefore also $\delta$) is collapsed to $\omega$ in $L_\xi + 2$, and hence this case cannot happen.

**Case b.** $\eta = \xi$. In this case $M_2 \models \phi(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m})$. Indeed, $M_2 = L_{\eta}[x_{\bar{a}}, x_{\bar{a} + m}]$. Since $\phi$ is a $\Sigma_1$-formula, $\omega_1^{M_2} = \omega_1^M$, and $\omega_2^{M_2} = \omega_2^M$, we have $\mathcal{M} = p(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m}).$

**Case c.** $\eta < \xi$. In this case $M_2$ is an element of $\mathcal{N}[x_{\bar{a}}, x_{\bar{a} + m}]$. Since $L_\theta[G]$ satisfies $(**)_{\alpha}$ and $(**)_{\alpha + m}$, by elementarity so does the model $\mathcal{N}[x_{\bar{a}}, x_{\bar{a} + m}]$ with $x_{\bar{a}}$ replaced by $x_{\bar{a}}$ and $x_{\bar{a} + m}$ replaced by $x_{\bar{a} + m}$. In particular, $M_2 \models \phi(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m})$. Since $\phi$ is a $\Sigma_1$-formula, $\omega_1^{M_2} = \omega_1^M$, and $\omega_2^{M_2} = \omega_2^M$, we have $\mathcal{M} = \phi(\omega_1, \omega_2, x_{\bar{a}}) \land \phi(\omega_1, \omega_2, x_{\bar{a} + m})$, which finishes our proof.

Set $\mathbb{P}_0 = \mathbb{P}_0 \ast \mathbb{P}_1 \ast \mathbb{P}_2$. Let us fix $\xi \in \omega_3$ and denote by $\mathbb{P}_0^{\alpha \xi}, \mathbb{P}_1^{\alpha \xi}, \mathbb{P}_2^{\alpha \xi}$ the following posets in $L, L^{\mathbb{P}_0^{\alpha \xi}}, \text{ and } L^{\mathbb{P}_1^{\alpha \xi}}, \mathbb{P}_2^{\alpha \xi}$, respectively:

$$\prod_{\sigma \in \omega_1 \setminus \{\xi\}} \mathbb{P}_0^{\alpha} \text{ with supports of size } \omega_1;$$
$$\prod_{\sigma \in \omega_1 \setminus \{\xi\}} \mathbb{P}_0^{\alpha} \text{ with countable supports; and}$$
\[ \prod_{\alpha \in \omega_1 \setminus \{\xi\}} \mathbb{P}_\alpha^2 \] with countable supports.

Observe that \( \mathbb{P}_0^\xi := \mathbb{P}_0^0 \cdot \mathbb{P}_0^1 \cdot \mathbb{P}_0^2 \cdot \mathbb{P}_0^\xi \) for posets \( \mathbb{P} \subseteq \mathbb{Q} \). The notation \( \mathbb{P} \prec \mathbb{Q} \) means that the identity embedding from \( \mathbb{P} \) to \( \mathbb{Q} \) is complete.\(^2\) Let \( \mathbb{R} \) be the quotient poset \( \mathbb{P}_0^\mathbb{P}_0 \). Thus \( \mathbb{P}_0^\mathbb{P}_0 = \mathbb{P}_0 \).

**Step 3.** We begin with fixing some terminology. For \( \alpha : 1 < \alpha < \omega_3 \) we will say that there is a stationary kill of \( S_\alpha \), if there is a closed unbounded set \( C \) disjoint from \( S_\alpha \).

We will say that the stationary kill of \( S_\alpha \) is coded by a real, if there is a closed unbounded set disjoint from \( S_\alpha \) which is constructible from this real.

Fix a nicely definable sequence \( \tilde{B} = \langle B_{\xi,m} : \xi < \omega_1, m \in \omega \rangle \) of almost disjoint subsets of \( \omega \). We will define a finite support iteration \( \langle \mathbb{P}_\alpha, \mathbb{Q}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle \) such that \( \mathbb{P}_0 \) is as above, \( \mathbb{Q}_\alpha \) is a \( \mathbb{P}_\alpha \)-name for a \( \sigma \)-centered poset, in \( L^{\mathbb{P}_\alpha} \) there is a \( \Delta^1_2 \)-definable wellorder of the reals and \( c = b = \mathbb{N}_3 \). Every \( \mathbb{Q}_\alpha \) is going to add a generic real whose \( \mathbb{P}_\alpha \)-name will be denoted by \( \dot{u}_\alpha \) and we shall prove that \( L[\mathcal{G}_\alpha] \cap \omega^\omega = L[\langle \dot{u}_\alpha^G : \xi < \alpha \rangle] \cap \omega^\omega \) for every \( \mathbb{P}_\alpha \)-generic filter \( \mathcal{G}_\alpha \) (see Lemma 2). This gives us a canonical wellorder of the reals in \( L[\mathcal{G}_\alpha] \), which depends only on the sequence \( \langle \dot{u}_\alpha^G : \xi < \alpha \rangle \), whose \( \mathbb{P}_\alpha \)-name will be denoted by \( \dot{\gamma}_\alpha \). We can additionally arrange that for \( \alpha < \beta \) we have that \( 1_{\mathbb{P}_\beta} \) forces \( \dot{\gamma}_\alpha \) to be an initial segment of \( \dot{\gamma}_\beta \). Then if \( G \) is a \( \mathbb{P}_{\omega_3} \)-generic filter over \( L, \langle \dot{\gamma}_\alpha : \alpha < \omega_3 \rangle \) will be the desired wellorder of the reals. Furthermore this wellorder will not depend on the generic set \( G \) (see Lemmas 4 and 5).

We proceed with the recursive construction of \( \mathbb{P}_{\omega_1} \). Along this construction we shall also define a sequence \( \langle A_\alpha : \alpha \in \text{Lim}(\omega_3) \rangle \), where \( A_\alpha \) is a \( \mathbb{P}_\alpha \)-name for a subset of \( [\alpha, \alpha + \omega) \). For every \( \omega_2 \leq \nu < \omega_3 \) fix a bijection \( \iota_\nu : \{0,1,2\} \times \nu \rightarrow \text{Lim}(\omega_3) \). If \( \mathcal{G}_\alpha \) is \( \mathbb{P}_\alpha \)-generic over \( L \), \( \dot{\gamma}_\alpha = \dot{\gamma}_{\alpha}^G \) and \( x, y \) are reals in \( L[\mathcal{G}_\alpha] \) such that \( x <_\alpha y \), let \( x \cdot y = (2n : n \in x) \cup (2n + 1 : n \in y) \) and \( \Delta(x \cdot y) = (2n + 2 : n \in x \cdot y) \cup (2n + 1 : n \notin x \cdot y) \).

Suppose \( \mathbb{P}_\alpha \) has been defined and fix a \( \mathbb{P}_\alpha \)-generic filter \( \mathcal{G}_\alpha \).

**Case 1.** Suppose \( \alpha \) is a limit ordinal and write it in the form \( \omega_2 \cdot \alpha' + \xi \), where \( \xi < \omega_2 \). If \( \alpha' > 0 \), let \( i = \iota_{\omega_2 \cdot \xi} \cdot (\omega_2, \kappa) \) and \( \dot{\xi}_0, \dot{\xi}_1 = \iota^{-1}(\xi) \). Let \( A_\alpha := A_\alpha^G \) be the set \( \alpha + (\omega \setminus \Delta(x_{\xi_0} \cdot x_{\xi_1})) \), where \( x_{\xi} \) is the \( \xi \)-th real in \( L[\mathcal{G}_{\omega_2 \cdot \alpha'}] \cap [\omega]^{\omega_2} \) according to the wellorder \( \dot{\gamma}_{\omega_2 \cdot \alpha'}^G \) (here \( G_{\omega_2 \cdot \alpha'} = G_\alpha \cap \mathbb{P}_{\omega_2 \cdot \alpha'} \)). Let also

\[ Q_\alpha = \{ \langle s_0, s_1 \rangle : s_0 \in [\omega]^{<\omega}, s_1 \in \bigcup_{m \in \Delta(x_{\xi_0} \cdot x_{\xi_1})} Y_{\alpha+m} \times \{ m \} \}_{<\omega} \],

where \( \langle t_0, t_1 \rangle = \langle s_0, s_1 \rangle \) if and only if \( s_1 \subseteq t_1, s_0 \) is an initial segment of \( t_0 \) and \( (t_0 \setminus s_0) \cap B_{\xi,m} = \emptyset \) for all \( \langle \xi, m \rangle \in s_1 \).

\(^2\) It might seem unclear why we denote \( \mathbb{P}_0^0 \cdot \mathbb{P}_0^1 \cdot \mathbb{P}_0^2 \cdot \mathbb{P}_0^\xi \) by \( \mathbb{P}_0^\mathbb{P}_0 \) and not simply by \( \mathbb{P}_0 \). It is to reserve the notation \( \mathbb{P}_0^\mathbb{P}_0 \) for a certain restriction of \( \mathbb{P}_0^0 \cdot \mathbb{P}_0^1 \cdot \mathbb{P}_0^2 \cdot \mathbb{P}_0^\xi \) appearing naturally in the proof of Lemma 3.
Case 2. If $\alpha$ is not of the form above, i.e. $\alpha$ is a successor or $\alpha < \omega_2$, then $A_\alpha$ is a name for the empty set and $Q_\alpha$ is a name for the following poset adding a dominating real:

$$Q_\alpha = \{(s_0, s_1) : s_0 \in \omega^{<\omega}, s_1 \in [0.t.(\zeta^G_\alpha)]^{<\omega}\},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ if and only if $s_0$ is an initial segment of $t_0$, $s_1 \subset t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \text{dom}(t_0) \setminus \text{dom}(s_0)$ and $\xi \in s_1$, where $x_\xi$ is the $\xi$-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $<^G_\alpha$.

In both cases $Q_\alpha$ adds the generic real\(^3\) $u_\alpha = \bigcup\{s_0 : \exists s_1 \langle s_0, s_1 \rangle \in g_\alpha\}$, where $g_\alpha$ is $Q_\alpha$-generic over $V[G_\alpha]$ and $L[G_\alpha][u_\alpha] = L[G_\alpha][g_\alpha]$.

With this the definitions of $\mathbb{P} = \mathbb{P}_\alpha$ and $\langle A_\alpha : \alpha \in \text{Lim}(\omega_3) \rangle$ are complete.

Remark 1. Note that if the first case in the definition of $Q_\alpha$ above takes place, then in $L^{\mathbb{P}_\alpha}$ the poset $Q_\alpha$ produces a real $r_\alpha$, which for certain reals $x, y$ codes $\gamma_{\alpha + m}$ for all $m \in \Delta(x \ast y)$.

Let $H$ be a poset. An $H$-name $\dot{f}$ is called a nice name for a real if $\dot{f} = \bigcup_{i \in \omega} (\{i, \dot{f}_i\}, p)$ : $p \in A(\dot{f})$ where for all $i \in \omega$, $A(\dot{f})$ is a maximal antichain in $H$, $\dot{f}_i \in \omega$ and for all $p \in A(\dot{f})$, $p \vdash \dot{f}(i) = \dot{f}_i$. From now on we will assume that all names for reals are nice.

Using the fact that for every $p \in \mathbb{P}$ and $\alpha > 0$ the coordinate $p(\alpha)$ is a $\mathbb{P}_\alpha$-name for a finite set of ordinals, one can show that the set $D$ of conditions $p$ fulfilling the following properties is dense in $\mathbb{P}$:

- For every $\alpha > 0$ in the support of $p$, $p(\alpha) = \langle s_0, s_1 \rangle$ for some $s_1 \in [\text{Ord}]^{<\omega}$ and $s_0 \in [\omega]^{<\omega}$ or $s_0 \in \omega^{<\omega}$ depending on $Q_\alpha$.

Lemma 2. Let $\gamma \leq \omega_3$ and let $G_\gamma$ be a $\mathbb{P}_\gamma$-generic filter over $L$. Then $L[G_\gamma] \cap \omega^\omega = L[\{\dot{u}_\xi^{G_\gamma} : \xi < \gamma\}] \cap \omega^\omega$.

Proof. Let $\dot{f} = \bigcup_{i \in \omega} (\{i, \dot{f}_i\}, p) : p \in A(\dot{f})$ be a nice $\mathbb{P}_\gamma$-name for a real such that $\bigcup_{i \in \omega} A(\dot{f}) \subset D$, $\dot{f} = \dot{f}_\gamma^G$, and let $p_1$ be the unique element of $A(\dot{f}) \cap G_\gamma$. Let $u_\xi = u_\xi^{G_\gamma}$ for all $\xi < \gamma$. Since $\mathbb{P}_0$ is countably distributive, there exists $q \in \mathbb{P}_0 \cap G_\gamma$ such that $q \leq p_1(0)$ for all $i \in \omega$.

Observe that $\langle i, j \rangle \in f$ if and only if there exists $p \in A(\dot{f})$ such that $p(0) \geq q$ and for every $\alpha$ in the support of $p$ the following holds:

If $p \upharpoonright \alpha$ forces $Q_\alpha$ to be an almost disjoint coding, i.e. $\alpha = \omega_2 \cdot \alpha' + i(\beta_0, \beta_1)$ for some $\alpha' > 0$ and $\beta_0 < \beta_1 < o.t.(\zeta^G_\alpha)$ and $Q_\alpha$ produces a real coding a stationary kill of $S_{\alpha + m}$ for all $m \in \Delta(x_{\beta_0} \ast x_{\beta_1})$, then $p(\alpha)_0$ is an initial segment of $u_\alpha$ and $u_\alpha \setminus p(\alpha)_0$ is disjoint from $B_{\zeta, m}$ for all $\langle \zeta, m \rangle \in p(\alpha)_1$; and

\(^3u_\alpha \in [\omega]^{<\omega}$ in the first case and $u_\alpha \in \omega^{<\omega}$ in the second case.
If $p \upharpoonright \alpha$ forces $\dot{Q}_\alpha$ to be a poset adding a dominating function, i.e. $Q_\alpha$ produces a real $u_\alpha$ dominating all reals in $L(\langle u_\xi : \xi < \alpha \rangle)$, then $p(\alpha)_0$ is an initial segment of $u_\alpha$ and $u_\alpha(n) > x_\alpha(n)$ for all $\xi < p(\alpha)$ and $n \geq \text{dom}(p(\alpha)_0)$, where $x_\xi$ is the $\xi$-th real in $L(\langle u_\xi : \xi < \alpha \rangle)$ according to the wellorder $\dot{\langle}^G_\alpha$.

Since $\dot{\langle}^G_\alpha$ depends only on the sequence $\langle u_\xi : \xi < \beta \rangle$ for all $\beta < \gamma$, the definition of $f$ above implies that $f \in L(\langle u_\xi : \xi < \gamma \rangle)$, which finishes our proof. \hfill $\square$

**Lemma 3.** Let $G$ be a $\mathbb{P}$-generic filter over $L$. Then for $\xi \in \bigcup_{\alpha \in \text{Lim}(\omega_1)} A^G_\alpha$ there is no real coding a stationary kill of $S_{\xi}$.

**Proof.** Let $p \in G$ be a condition forcing

$$\xi \in \bigcup_{\alpha \in \text{Lim}(\omega_1)} A^G_\alpha.$$  

Suppose that $\xi = \beta + 2n - 1$ for some limit $\beta$ and $n \in \omega$. Without loss of generality, $p \in \mathbb{P}_\beta \cap \mathcal{D}$.

We define a finite support iteration of a countably distributive poset followed by c.c.c. posets $\langle \mathbb{P}_\alpha, \dot{Q}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$, where $\mathbb{P}_0 = \mathbb{P}_0 \upharpoonright p(0)$ and in $L^{\mathbb{P}_0}$ we have $\dot{Q}_\alpha = Q_\alpha \upharpoonright p(\alpha)$. Such an iteration is just another way of thinking of the poset $\mathbb{P} \upharpoonright p$ which will appear useful for further considerations.

Let $p^\mathbb{P}_0, p^\dot{R}_0$ be such that $p^\mathbb{P}_0 \in \dot{\mathbb{P}}^\mathbb{P}_0$, $p^\dot{R}_0 \in \dot{\mathbb{R}}$ and $\langle p^\mathbb{P}_0, p^\dot{R}_0 \rangle = p(0)$, where $\dot{\mathbb{R}}$ is the quotient poset $\mathbb{P}_0/\mathbb{P}_0^{\mathbb{R}}$. Denote by $p^{\mathbb{P}_0^\mathbb{P}}$ the restriction $\mathbb{P}_0^{\mathbb{P}} \upharpoonright p^\mathbb{P}_0$ and let $\mathbb{R}$ be the $\mathbb{P}_0^\mathbb{P}$-name for $\dot{\mathbb{R}} \upharpoonright p^\mathbb{P}_0$. Note that $\mathbb{P}_0^\mathbb{P} \ast \mathbb{P}_0 = \mathbb{P}_0^\mathbb{P}$.

Now we define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{Q}_\gamma^P : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$, where $\mathbb{P}_0^\mathbb{P}$ is as above and $\dot{Q}_\gamma^P$ is a name for a $\sigma$-centered poset. Also we define a sequence $\langle A^\mathbb{P}_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$, where $A^\mathbb{P}_\alpha$ is a $\mathbb{P}_0^\mathbb{P}$-name for a subset of $[\alpha, \alpha + \omega)$. The intention is to show that in $\mathbb{P} = \mathbb{P}_\omega_3$ the components $\mathbb{P}_0^\mathbb{P}, \mathbb{P}_1^\mathbb{P}, \mathbb{P}_2^\mathbb{P}$ of $\mathbb{P}_0^\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2$, respectively, can be left out in a certain sense. Thus the iteration $\langle \mathbb{P}_\alpha^\mathbb{P}, \dot{Q}_\gamma^\mathbb{P} : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$ will be introduced along the lines of the definition of $\langle \mathbb{P}_\alpha, \dot{Q}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$. In particular, every $\dot{Q}_\alpha^\mathbb{P}$ will add a generic real with $\mathbb{P}_\alpha^{\mathbb{P}} \ast \dot{Q}_\alpha^\mathbb{P}$-name $\dot{u}_\alpha^\mathbb{P}$. Given a $\mathbb{P}_\alpha^{\mathbb{P}}$-generic filter $G = G^\mathbb{P}_\alpha$, this gives us a canonical wellorder of the reals in $L(\dot{u}_\alpha^{\mathbb{P}} : \xi < \alpha)$ which depends only on the sequence $\langle u_\xi^{\mathbb{P}} : \xi < \alpha \rangle$, whose $\mathbb{P}_\alpha^{\mathbb{P}}$-name will be denoted by $\dot{z}_\alpha^\mathbb{P}$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta}$ forces $\dot{z}_\alpha^\mathbb{P}$ to be an initial segment of $\dot{z}_\beta^\mathbb{P}$. Along the recursive construction for every $\gamma < \omega_3$ we will establish the following properties:

1. $\dot{z}_\gamma^\mathbb{P} <_{\mathbb{P}^\gamma} \dot{z}_\gamma^\mathbb{P};$

\footnote{In fact, one can prove that $\models_{\mathbb{P}_0^\mathbb{P}} \dot{\mathbb{R}} = \mathbb{P}_0^\mathbb{P} \ast \mathbb{P}_1^\mathbb{P} \ast \mathbb{P}_2^\mathbb{P}$, but this does not simplify the proof.}
2. $u_γ^{\pm x \gamma} = u_γ^H$, $\zeta_γ^{\pm x \gamma} = \zeta_γ^H$ and $A_γ^H = A_γ^{\pm x \gamma}$ for limit $γ$, where $H_γ^{\pm x} \subseteq P_γ^{\pm x}$ is the preimage of the $\bar{P}_γ$-generic filter $H_γ$ under the complete embedding from (1);

3. Let $P_γ^{\pm x} \cap \bar{P}_γ^{\pm x}$ be the quotient posets $P_γ^{\pm x} / P_0^{\pm x}$ and $\bar{P}_γ / \bar{P}_0$ respectively. Then $\bar{P}_0 \cap P_γ^{\pm x} = \bar{P}_0$; and

4. $L[H_γ] \cap [\text{Ord}]^ω = L[H_γ^{\pm x}] \cap [\text{Ord}]^ω$ where $H_γ$, $H_γ^{\pm x}$ are as in (2).

For $γ = 0$ the properties above follow from the corresponding definitions. Suppose that (1)-(4) are established for all $η < γ$.

Case 1. If $γ$ is a limit, there is nothing to prove except for (4) (To see that $P_γ^{\pm x}$ is completely embedded in $\bar{P}_γ$ refer to the inductive hypothesis and [2, Lemma 10]). Let $H_0^{\pm x} = H_0^{\pm x} \cap P_0^{\pm x}$, $H_0 = H_γ \cap P_0$ and let $K$ be an $\mathbb{R}$-generic filter over $L[H_0^{\pm x}]$ such that $L[H_0] = L[H_0^{\pm x}] [K]$. Let $E$ be the poset $(P_0^{\pm x})^{\bar{P}_0} = P_0^{\bar{H}_0} \in L[H_0^{\pm x}]$ (the latter equality follows from (3)). Then $H_{(1,γ)}(= H_γ / H_0)$ is $E$-generic over $L[H_0^{\pm x}][K]$. Therefore $L[H_0^{\pm x}][K][H_{(1,γ)}] = L[H_0^{\pm x}][H_{(1,γ)}][K]$. The following standard fact may be compared to [9, Lemma 15.19].

Claim. Suppose that $P$, $Q$ are in $V$, $P$ is $ω$-distributive and $Q$ is c.c.c. in $V^P$. Then $P$ is $ω$-distributive in $V^Q$. In particular, if $P$ is $ω$-distributive and $Q$ is a finite support iteration of $σ$-centered posets, then $P$ is $ω$-distributive in $V^Q$.

Proof. Let $G \times H$ be $P \times Q$-generic. Let $f : ω → \text{Ord}$ be in $V[H][G] = V[G][H]$ and $σ$ be a $Q$-name for $f$ in $V[G]$. Without loss of generality, $σ$ is a nice name which can be written as $\bigcup_{i ∈ \text{Ord}} \{ (i, j^i_p, p) : p ∈ \mathcal{A}_i \}$, where $j^i_p$ is an ordinal and $\mathcal{A}_i \in G$ is a maximal antichain in $Q$. As $Q$ is c.c.c. in $V[G]$, each $\mathcal{A}_i$ is countable in $V[G]$, and hence $σ$ is countable in $V[G]$. Therefore $σ ∈ V$ by the countable distributivity of $P$. It follows that $f$ belongs to $V[H]$. □

By the above Claim, $\mathbb{R}$ is countably distributive in $L[H_0^{\pm x}][H_{(1,γ)}] = L[H_0^{\pm x}]$ and hence $L[H_0] \cap [\text{Ord}]^ω = L[H_0^{\pm x}] \cap [\text{Ord}]^ω$.

Case 2. $γ = η + 1$.

Let $H_0^{\pm x}_η$ be a $P_0^{\pm x}$-generic filter over $L$ and let $K$ be an $\mathbb{R}$-generic filter over $L[H_0^{\pm x}_η]$, where $H_0^{\pm x} = H_0^{\pm x} \cap P_0^{\pm x}$. In $L[H_0^{\pm x}]$, the quotient poset $P_{(1,0)} = P_0 / P_0$ is a finite support iteration of $σ$-centered posets. Since $P_0^{\pm x}$ has c.c.c. in $L[H_0^{\pm x}][K]$ and $R$ is $ω$-distributive, $H_0^{\pm x} \cap (1,0)$-generic over $L[H_0^{\pm x}][K]$. By (3), the equality $P_{(1,0)} = P_{(1,0)}$ holds in $L[H_0^{\pm x}][K]$. Therefore $H_0 := H_0^{\pm x} \ast K \ast H_0^{\pm x}$ is $P_0^{\pm x}$-generic over $L$.

Since $p ∈ D$, one of the following alternatives holds.
Case a). \(\bar{\mathcal{Q}}_\eta\) is a name for an almost disjoint coding below the condition \(p(\eta) = \langle \zeta_0^\eta, \zeta_1^\eta \rangle\).
Set \(\bar{\mathcal{Q}}_\eta = \bar{\mathcal{Q}}_\eta^H, u_\delta = u_\delta^H, A_\delta = A_\delta^H\), and \(\delta = \zeta^\eta_\delta\) for all \(\delta \leq \eta\).

It follows that:
- \(\eta\) is a limit ordinal that can be written in the form \(\eta = \omega_2 \cdot \nu + \zeta\), where \(\zeta = i(\zeta_0, \zeta_1)\) for some \(\zeta_0, \zeta_1 < 0.t.(\zeta^\eta_{\omega_2}, \nu)\);
- \(A_\eta = \eta + (\omega \setminus \Delta(x_{\zeta_0} \ast x_{\zeta_1}))\), where \(x_\epsilon\) is the \(\epsilon\)-th real in \(L[\langle u_\delta : \delta < \omega_2 \cdot \nu \rangle \cap \omega\omega\) according to the natural wellorder \(\langle \zeta^\eta_{\omega_2}, \nu \rangle\) of this set;
- \(\bar{\mathcal{Q}}_\eta = \{ (s_0, s_1) : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m \in \Delta(x_{\zeta_0} \ast x_{\zeta_1})} Y_{\eta+m} \times \{ m \}]^{<\omega}, s_0 \text{ end-extends } s_0^\eta, s_1 \supseteq s_1^\eta \text{ and } s_0 \setminus s_0^\eta \cap B_{\epsilon,m} = \emptyset \text{ for all } (\epsilon, m) \in \bigcup_{\eta} \text{ ordered as before.}\)

Our choice of \(p\) and the fact that the upwards closure of \(H_\eta\) in \(\mathbb{P}_\eta\) is a \(\mathbb{P}_\eta\)-generic filter containing \(p\) imply that \(Y_\xi\) is not among the \(Y_{\eta+m}\)'s involved into the definition of \(\bar{\mathcal{Q}}_\eta\). Thus \(\bar{\mathcal{Q}}_\eta \in H[\eta]^\zeta\). Moreover, \(\bar{\mathcal{Q}}_\eta\) is fully determined by the relevant \(Y_{\eta+m}\)'s, and the sequence \(\langle u_\delta : \delta < \eta \rangle\) which belongs to \(L[\eta]_{\omega_1}\) and does not depend on \(K\) by (2).

Therefore \(\bar{\mathcal{Q}}_\eta\) does not depend on \(K\) and hence we may set \(\bar{\mathcal{Q}}_{\eta}^{\sharp} = : \bar{\mathcal{Q}}_\eta, A_{\eta}^{\sharp} = : A_\eta\). Let \(\bar{\mathcal{Q}}_{\eta}^{\sharp}, A_{\eta}^{\sharp}\) be \(\mathbb{P}_\eta^{\sharp}\)-names for \(\bar{\mathcal{Q}}_\eta^{\sharp}\) and \(A_\eta^{\sharp}\) respectively. By the definition, (3) and the third part of (2) hold true.

The equality \(L[H_\eta] \cap \text{Ord}^\omega = L[H_\eta^{\sharp}] \cap \text{Ord}^\omega\) and the \(\sigma\)-centeredness of \(\bar{\mathcal{Q}}_\eta\) imply that any \(\mathbb{Q}_\eta^{\sharp}\)-generic over \(L[H_\eta^{\sharp}]\) is \(\mathbb{Q}_\eta^{\sharp}\)-generic over \(L[H_\eta]\) and vice versa. Therefore \(\mathbb{P}_{\eta+1}^{\sharp} \subset \mathbb{P}_{\eta+1}\) (note that \(H_\eta\) may be thought of as being an arbitrary \(\mathbb{P}_\eta^{\sharp}\)-generic filter over \(L\)). This establishes (1).

Let \(h_\eta\) be a \(\mathbb{Q}_\eta^{\sharp}\)-generic over \(L[H_\eta^{\sharp}]\) (or, equivalently, \(\mathcal{Q}_\eta^{\sharp}\)-generic filter over \(L[H_\eta]\)). Since a (nice) \(\mathbb{Q}_\eta^{\sharp}\)-name for a countable set of ordinals in \(L[H_\eta]\) can be naturally identified with a countable set of ordinals, every \(\mathbb{Q}_\eta^{\sharp}\)-name \(\sigma \in L[H_\eta]\) for a countable set of ordinals is in fact in \(L[H_\eta^{\sharp}]\). Therefore \(L[H_{\eta+1}] \cap \text{Ord}^\omega = L[H_{\eta+1}^{\sharp}] \cap \text{Ord}^\omega\), where \(H_{\eta+1} = H_\eta \ast h_\eta\). This proves (4).

Let us denote by \(u_{\eta, \delta}^{\sharp} \in [\omega]^{<\omega} \cap L[H_{\eta+1}^{\sharp}]\) the union of the first coordinates of elements of \(h_\eta\). By the maximality principle, this gives us a \(\mathbb{P}_{\eta+1}^{\sharp}\)-name \(u_\eta^{\sharp}\). By the definitions of \(\hat{u}_\delta\) and \(u_\delta\), \(u_\delta^H \ast h_\eta = u_\delta^H \ast H_{\eta}^{\sharp} \ast h_\eta\), which proves the first part of (2). By (4) and Lemma 2,

\[
L[H_\eta^{\sharp} \ast h_\eta] \cap [\omega]^{<\omega} = (L[H_\eta^{\sharp} \ast h_\eta] \cap [\text{Ord}^{\omega}]^{<\omega}) \cap [\omega]^{<\omega} = (L[H_\eta \ast h_\eta] \cap [\text{Ord}^{<\omega}]^{<\omega}) \cap [\omega]^{<\omega} = L[H_\eta \ast h_\eta] \cap [\omega]^{<\omega} = L[H_\eta \ast h_\eta] \cap [\omega]^{<\omega} = L[H_{\eta+1} \cap [\omega]^{<\omega} = L[H_{\eta+1} \ast h_\eta] \cap [\omega]^{<\omega},
\]

which implies the second equality in (2) and thus concludes Case a).

Case b). \(\bar{\mathcal{Q}}_\eta\) is a name for a poset adjoining a dominating function restricted to the condition \(p(\eta) = \langle s_0^\eta, s_1^\eta \rangle\). This case is analogous to, but easier than the Case a) (here we
do not have to worry about $Y_{ξ}$ and we leave it to the reader.

This finishes our construction of $⟨p^ξ _{α}, q^ξ _γ : α ≤ ω_3, γ < ω_1⟩$. Observe that conditions (1)-(4) hold for $γ = ω_3$. In particular, $L[G] ∩ ω_ω = L[G^ζ ] ∩ ω_ω$, where $G^ζ ∈ P^ζ$ is the preimage of the $P^{α_1}$-generic filter $G$ under the complete embedding from (1). So it is sufficient to show that in $L[G^ζ ]$ there is no real coding a closed unbounded sub-
set disjoint from $S_{ζ}$. Since $P^{ζ, ω_1}$ is a $P^{ζ}_0$-name for a c.c.c poset and $P^{ζ}_0, P^{ζ}_1$ are $P^{ζ}_0, P^{ζ}_1, P^{ζ}_2$-names for $ω_2$-c.c. posets, respectively, every closed unbounded sub-
set of $ω_2$ in $L[G^ζ ]$ contains a closed unbounded subset of $ω_2$ in $L[G^{0, ζ}_0]$, see [9, Lemma 22.25]. (Here $G^{0, ζ} = G^ζ ∩ P^{0, ζ}$ is the $P^{0, ζ}$-generic filter over $L$ induced by $G^ζ$). Thus it suffices to verify that $S_{ζ}$ is stationary in $L^{P^{0, ζ}}$. We shall use here an idea from [6].

Fix $p ∈ P^{0, ζ}$ and let $C$ be a name for a club in $ω_2$. We would like to find $q ∈ P^{0, ζ}$ such that $q ≤ p$ and $q ≠_p C ∩ S_{ζ} ≠ ∅$. Let $⟨M_i : i < ω_2⟩$ be a continuous chain of elementary submodels of some large $L_θ$ such that $M_0$ contains $p, α, C, ω_1 + 1, M_0$, $γ_1 := M_i ∩ ω_2 ∈ ω_2, cof(γ_1) = ω_1$, and $M_i^{<ω_1} ⊆ M_i$ for all $i ∈ ω_2$. Set $S^0_{ζ} = \{ i ∈ S_{ζ} : γ_1 = i \}$ and note that $S^0_{ζ}$ is stationary.

**Claim.** There exists $i ∈ S^0_{ζ}$ such that $i ∉ S_α$ for all $α ∈ M_i \{ ξ \}$.

**Proof.** Note that $α ∈ M_i$ is equivalent to $α < γ_1$, and hence to $α < i$ since $i ∈ S^0_{ζ}$. Suppose that for every $i ∈ S^0_{ζ}$ there exists $f(i) < i$ such that $i ∈ S_{f(i)}$ and $f(i) ≠ ξ$. By Fodor’s Lemma there exists $j ∈ ω_2$ and a stationary $T ⊆ S^0_{ζ}$ such that $f(i) ≡ j$ for all $i ∈ T$. It follows that $T ⊆ S_j$, and hence $T ⊆ S_j ∩ S_{ζ}$, a contradiction.

Choose $i$ as in the Claim above. We shall build an $ω_1$-sequence $p = p_0 ≥ p_1 ≥ ⋯$ with a lower bound forcing $ι ∈ C$. Let $⟨ι_α : α < ω_1⟩$ be an increasing continuous sequence of ordinals such that $sup_{α ∈ ω_1} i_α = i$. Given $p_α$, let $p_{α+1} ≤ p_α$ be such a condition in $P^{0, ζ} ∩ M_1$ such that $p_{α+1}$ forces some ordinal $j_{α+1} ∈ \{ i_α + 1, i \}$ to belong to $C$. For limit $α$ and $ξ ∈ i \{ ξ \}$ set

$$p_α(ξ) = \bigcup_{β < α} p_β(ξ) ∪ \{ sup_{β < α} p_β(ξ), i_α ∪ i \}.$$  

Since $S_{ζ}$'s consist of ordinals of cofinality $ω_1$ and $M_i$ is closed under countable se-
quences of its elements, $p_α ∈ P^{0, ζ} ∩ M_1$. This finishes our construction of the sequences $⟨p_α : α < ω_1⟩ ∈ M_i^{ω_1}$ and $⟨j_α : α < ω_1⟩ ∈ 2^{ω_1}$. Set $q(ξ) = ∪_{α ∈ ω_1} p_α(ξ) ∪ \{ i \}$ for all $ξ ∈ i \{ ξ \}$, and $q(ξ) ∩ S_{ζ} = ∅$ for all $ξ ∈ i \{ ξ \}$. From the above it follows that $q ∈ P^{0, ζ}$ and $q ⊨ P^{0, ζ}, i ∈ C$, which finishes our proof.

**Corollary 1.** Let $G$ be a $P$-generic filter over $L$ and let $x, y$ be reals in $L[G]$. Then $x ∩ G^ζ \ y$ if and only if there is $α < ω_3$ such that for all $m$, the stationary kill of $S_{α+m}$ is coded by a real iff $m ∈ Δ(x ∩ y)$.
Proof. Suppose that \( x \prec^G y \). Let \( \alpha' > 0 \) be minimal such that \( x, y \in L[G_{\omega_2, \alpha}] \) and let \( i = i_{\alpha, \alpha'} \). Find \( \xi \in \text{Lim}(\omega_2) \) such that \( i(\xi) = (\xi_x, \xi_y) \) where \( x \) and \( y \) are the \( \xi_x \)-th and \( \xi_y \)-th real respectively in \( L[G_{\omega_2, \alpha}] \) according to the wellorder \( \prec^G_{\omega_2, \alpha'} \). (By Lemma 2 such a \( \xi \) exists). Let \( \alpha = \omega_2 \cdot \alpha' + \xi \). Then \( Q_{\alpha} \) adds a real coding a stationary kill for \( S_{\alpha + m} \) for all \( m \in \Delta(x \ast y) \). On the other hand if \( m \notin \Delta(x \ast y) \), then \( \alpha + m \notin A_{3}^{\omega_3} = \alpha + (\omega_3 \Delta(x \ast y)) \) and so by Lemma 3, there is no real in \( L[G] \) coding the stationary kill of \( S_{\alpha + m} \).

Now suppose that there exists \( \alpha \) such that the stationary kill of \( S_{\alpha + m} \) is coded by a real iff \( m \in \Delta(x \ast y) \). Since the stationary kill of some \( \alpha + m \)'s is coded by a real in \( L[G] \), Lemma 3 implies that \( \bar{Q}_{\alpha} \) introduced a real coding stationary kill for all \( m \in \Delta(a \ast b) \) for some reals \( a \prec^G_{\omega_3} b \), while there are no reals coding a stationary kill of \( S_{\alpha + m} \) for \( m \notin \Delta(a \ast b) \). Therefore \( \Delta(a \ast b) = \Delta(x \ast y) \) and hence \( a = x \) and \( b = y \), and consequently \( x \prec^G y \).

Lemma 4. Let \( G \) be \( \mathbb{P} \)-generic over \( L \) and let \( x, y \) be reals in \( L[G] \). If \( x \prec^G y \), then there is a real \( r \) such that for every countable suitable model \( M \) such that \( r \in M \), there is \( \bar{a} < \omega_3^M \) such that for all \( m \in \Delta(x \ast y) \),

\[
(L[r])^M = S_{\bar{a} + m} \text{ is not stationary.}
\]

Proof. By Corollary 1, there exists \( \alpha < \omega_3 \) such that \( \bar{Q}_{\alpha} \) adds a real \( r \) coding a stationary kill of \( S_{\alpha + m} \) for all \( m \in \Delta(x \ast y) \). Let \( M \) be a countable suitable model containing \( r \). It follows that \( Y_{\alpha + m} \cap \omega_1^M \in M \) and hence \( X_{\alpha} \cap \omega_1^M, X_{\alpha + m} \cap \omega_1^M \) also belong to \( M \). Observe that these sets are actually in \( N := (L[r])^M \). Note also that \( N \) is a countable suitable model and consequently by the definition of \( L(X_{\alpha + m}, X_{\alpha}) \) we have that for every \( m \in \Delta(x \ast y) \),

\[
\text{"Using the sequence } \vec{A}, X_{\alpha + m} \cap \omega_1 \text{ (resp. } X_{\alpha} \cap \omega_1 \text{) almost disjointly codes a subset } \vec{Z}_m \text{ (resp. } \vec{Z}_0) \text{ of } \omega_2, \text{ whose even part } \text{Even}(\vec{Z}_m) \text{ (resp. } \text{Even}(\vec{Z}_0) \text{)} \text{ codes a tuple } (\bar{C}, \bar{W}_m, \bar{W}_0) \text{ (resp. } (\bar{C}, \bar{W}_0, \bar{W}_0) \text{), where } \bar{W}_m \text{ and } \bar{W}_0 \text{ are the } \bar{L}-\text{least codes of ordinals } \bar{\alpha}_m, \bar{\alpha}_0 < \omega_3 \text{ (resp. } \bar{\alpha}_0 = \bar{\alpha}_0 \text{ is the largest limit ordinal not exceeding } \bar{\alpha}_m \text{ and } \bar{C} \text{ is a club in } \omega_2 \text{ disjoint from } S_{\bar{a}_m}.\text{\"}  

\[
\text{Note that in particular for every } m \neq m' \in \Delta(x \ast y), \bar{a}_m = \bar{a}_{m'}. \quad \Box
\]

Lemma 5. Let \( G \) be \( \mathbb{P} \)-generic over \( L \) and let \( x, y \) be reals in \( L[G] \). If there is a real \( r \) such that for every countable suitable model \( M \) containing \( r \) as an element, there is \( \bar{a} < \omega_3^M \) such that for every \( m \in \Delta(x \ast y) \),

\[
(L[r])^M = S_{\bar{a} + m} \text{ is not stationary,}
\]

\[\text{In the above, } \vec{A}, S_{\alpha_0}, S_{\bar{a}_0}, \omega_1, \omega_2, \omega_3 \text{ refer of course to their interpretations in the model } N.\]
then \( x <^G y \).

**Proof.** Suppose that there is such a real \( r \). By the L"owenheim-Skolem theorem, it has the property described in the formulation with respect to all suitable models \( M \), in particular for \( \mathbb{R}_\Theta \), where \( \Theta \) is sufficiently large (here \( \mathbb{R}_\Theta \) denotes the set of all sets hereditarily of cardinality \( < \Theta \)). That is there is \( \alpha < \omega_3 \) such that for every \( m \in \Delta(x \ast y) \)

\[
L_\Theta[r] \models S_{\alpha + m} \text{ is not stationary.}
\]

Thus in particular the stationary kill of at least some \( S_{\alpha + m} \) was coded by a real. Lemma 3 implies that \( \dot{Q}^G_\alpha \) introduced a real \( u_\alpha \) (perhaps different from \( r \)) coding stationary kill for all \( m \in \Delta(a \ast b) \) for some reals \( a \dot{<}^G b \), while there are no reals coding a stationary kill of \( S_{\alpha + m} \) for \( m \notin \Delta(a \ast b) \). Therefore \( \Delta(a \ast b) \supset \Delta(x \ast y) \), which yields \( \Delta(a \ast b) = \Delta(x \ast y) \). From the above, it follows that \( a = x, b = y \) and hence \( x \dot{<}^G y \), which finishes our proof. \( \square \)

Combining Lemmata 4,5 and the fact that we have added dominating reals cofinally often, we get the following result.

**Theorem 1.** It is consistent with \( \kappa = \beta = \aleph_3 \), that there is a projective (indeed \( \Delta^1_3 \)-definable) wellorder of the reals.

3. **Projective mad families**

The main result of this section and of the whole paper is the following theorem which answers [7, Question 19] in the positive.

**Theorem 2.** It is consistent with \( \kappa = \beta = \aleph_3 \), that there is a \( \Delta^1_3 \)-definable wellorder of the reals and a \( \Pi^1_2 \)-definable \( \omega \)-mad subfamily of \( [\omega]^\omega \) (resp. \( \omega'' \)).

The proof is completely analogous to that of Theorem 2. Moreover, we believe that adding the argument responsible for \( \omega \)-mad families would just make the proof in the previous section messier without introducing any new ideas besides those used in the proof of Theorem 1 and in [7]. Therefore the proof of Theorem 2 is just sketched here. More precisely, we shall define the corresponding poset \( P_{\omega^\omega} \) and leave it to the reader to verify that the proof of Theorem 1 can be carried over.

Let \( \dot{B} = \langle B_{\xi,m} : \xi < \omega_1, m \in \omega \rangle \) be as in the proof of Theorem 1. We will define a finite support iteration \( \langle P_\alpha, \dot{Q}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle \), where \( \dot{Q}_\alpha \) is a \( P_\alpha \)-name for a \( \sigma \)-centered poset and in \( L[P_{\omega^\omega}] \) there is a \( \Delta^1_3 \)-definable wellorder of the reals, a \( \Pi^1_3 \)-definable \( \omega \)-mad subfamily of \( [\omega]^\omega \) (the case of subfamilies of \( \omega'' \) is completely analogous, see [7]), and \( \kappa = \beta = \aleph_3 \).
\( \mathbb{P}_0 \) is a three step iteration \( \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2 \), where \( \mathbb{P}^0 \) and \( \mathbb{P}^1 \) are exactly the same as in the proof of Theorem 1. The poset \( \mathbb{P}^2 \) uses the following modification of Definition 1, where \( \phi \) is as in \((**)_\eta\) from the previous section.

**Definition 2.** Let \( X, X' \subset \omega_1 \) be such that \( \phi(\omega_1, \omega_2, X) \) and \( \phi(\omega_1, \omega_2, X') \) hold in any suitable model \( \mathcal{M} \) with \( \omega_1^\mathcal{M} = \omega_1^\mathcal{L} \) containing \( X \) and \( X' \), respectively. Let also \( \eta \) be a countable limit ordinal. We denote by \( \mathcal{L}_\eta(X, X') \) the poset of all functions \( r : |r| \to 2 \), where the domain \(|r|\) of \( r \) is a countable limit ordinal such that:

1. \(|r| \geq \eta\)
2. if \( \gamma < \eta \) then \( r(\gamma) = 0 \)
3. if \( \gamma < |r| \) then \( \gamma \in X \) iff \( r(\eta + 3\gamma) = 1 \)
4. if \( \gamma < |r| \) then \( \gamma \in X' \) iff \( r(\eta + 3\gamma + 1) = 1 \)
5. if \( \gamma \leq |r| \), \( \mathcal{M} \) is a countable suitable model containing \( r \upharpoonright \gamma \) as an element and \( \gamma = \omega_1^\mathcal{M} \), then \( \mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma) \land \phi(\omega_1, \omega_2, X' \cap \gamma) \) holds in \( \mathcal{M} \).

The extension relation is end-extension.

For \( \alpha \in \text{Lim}(\omega_3) \setminus \omega_2 \) and \( m \in \omega \) set \( \mathbb{P}^2_{\alpha+m} = \prod_{\eta \in \text{Lim}(\omega_1)} \mathcal{L}_\eta(X_{\alpha+m}, X_\alpha) \). If \( \alpha \in \text{Lim}(\omega_2) \) and \( m \in \omega \), let \( \mathbb{P}^2_{\alpha+m} \) be the trivial poset. Then let

\[
\mathbb{P}^2 = \prod_{\alpha \in \text{Lim}(\omega_1)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}
\]

with countable supports. By the \( \Delta \)-system Lemma in \( L^{\mathbb{P}^0 * \mathbb{P}^1} \) the poset \( \mathbb{P}^2 \) has the \( \omega_2 \)-c.c. Analogously to Lemma 1 we conclude that \( \mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2 \) is \( \omega \)-distributive.

If \( \alpha \) is limit and \( m \in \omega \), we shall refer to the localizing set for \( X_{\alpha+m} \) produced by \( \mathcal{L}_\eta(X_{\alpha+m}, X_\alpha) \) as \( Y_{\alpha+m, \eta} \). That is \( Y_{\alpha+m, \eta} \subseteq \omega_1 \setminus \eta \) and \( Y_{\alpha+m, \eta} \) codes both \( X_{\alpha+m} \) and \( X_\alpha \).

Every \( \mathbb{Q}_\alpha \) is going to add a generic real whose \( \mathbb{P}_\alpha \)-name will be denoted by \( \dot{u}_\alpha \) and similarly to the proof of Lemma 2 one can prove that \( L[G_\alpha] \cap \omega^\omega = L[\langle \dot{u}_\xi^G : \xi < \alpha \rangle] \cap \omega^\omega \) for every \( \mathbb{P}_\alpha \)-generic filter \( G_\alpha \). This gives us a canonical wellorder of the reals in \( L[G_\alpha] \), which depends only on the sequence \( \langle \dot{u}_\xi^G : \xi < \alpha \rangle \), whose \( \mathbb{P}_\alpha \)-name will be denoted by \( \dot{\xi}_\alpha \). We can additionally arrange that for \( \alpha < \beta \) we have that \( 1_{\mathbb{P}_\beta} \) forces \( \dot{\xi}_\alpha \) to be an initial segment of \( \dot{\xi}_\beta \). Then if \( G \) is a \( \mathbb{P}_{\omega_1} \)-generic filter over \( L \), \( \dot{G} = \bigcup \{ \dot{G}_\alpha : \alpha < \omega_1 \} \) will be the desired wellorder of the reals.

We proceed with the recursive construction of \( \mathbb{P}_{\omega_1} \). Along this construction we shall also define a sequence \( \langle \dot{A}_\alpha : \alpha \in \text{Lim}(\omega_3) \rangle \), where \( \dot{A}_\alpha \) is a \( \mathbb{P}_\alpha \)-name for a subset of \( [\alpha, \alpha + \omega) \). Let \( i : \omega \times \omega \to \omega \) and

\[
j_\nu : \nu \cup \{ (\xi, \xi) : \xi < \xi < \nu \} \to \text{Lim}(\omega_2)
\]
be some bijections, where $\nu \in [\omega_2, \omega_3]$. Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter $G_\alpha$.

Case 1. $\alpha$ is a limit ordinal that can be written in the form $\omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$, $\xi < \omega_2$, and the preimage $j^{-1}(\xi)$ is a tuple $(\xi_0, \xi_1)$ for some $\xi_0 \lesssim_{\omega_2 \cdot \alpha'} \xi_1$, where $j = j_{\alpha, t}(\xi_0)$. In this case the definition of $\bar{Q}_\alpha$ is the same as in the proof of Theorem 1.

Case 2. $\alpha$ is a limit ordinal that can be written in the form $\omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$ and the preimage $j^{-1}(\xi)$ is an ordinal $\zeta \in o.t.(\bar{G}_\alpha^{\omega_2 \cdot \alpha'})$, where $j = j_{\alpha, t}(\zeta)$. In this case we use a simplified version of the poset from [7, Theorem 1]. More precisely, ordinals fulfilling the condition above will be used for the construction of a $\Pi^1_2$ definable $\omega$-mad family $\mathcal{A}$.

For a subset $s$ of $\omega$ and $l \in |s| (= \text{card}(s) \leq \omega)$ we denote by $s(l)$ the $l$-th element of $s$. In what follows we shall denote by $E(s)$ and $O(s)$ the sets $\{s(2i) : 2i \in |s|\}$ and $\{s(2i + 1) : 2i + 1 \in |s|\}$, respectively. Let $\mathcal{A}_\alpha$ be the approximation to $\mathcal{A}$ constructed thus far. Suppose also that

$$\forall \mathcal{D} \in [\mathcal{A}_\alpha]^{<\omega} \forall B \in \bar{B} ([E(B) \cup \mathcal{D}]) = |O(B) \cup \mathcal{D}| = \omega).$$

Observe that equation (*) yields $|E(B) \cup \mathcal{D}| = |O(B) \cup \mathcal{D}| = \omega$ for every $\mathcal{D} \in [\bar{B} \cup \mathcal{A}_\alpha]^{<\omega}$ and $B \in \bar{B} \setminus \mathcal{D}$. Let $\chi$ be the $\zeta$-th real in $L[G_\alpha^{\omega_2 \cdot \alpha'}] \cap [\omega]^{<\omega}$ according to the wellorder $\preceq_{\omega_2 \cdot \alpha'}$. Set $C_n = \{\chi(n, m) : m \in \omega\} \in [\omega]^{<\omega}$ and $C = \{C_n : n \in \omega\}$. Unless the following holds, $\bar{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset: none of the $C_n$’s is covered by a finite subfamily of $\mathcal{A}_\alpha$. In the latter case $\bar{Q}_\alpha := \bar{Q}_G^{G_\alpha}$ is defined as follows.

Let us fix a limit ordinal $\eta_\alpha \in \omega_1$ such that there are no finite subsets $I, E$ of $(\omega_1 \setminus \eta_\alpha) \times \omega$, $\mathcal{A}_\alpha$, respectively and $n \in \omega$, such that $C_n \subset \bigcup_{(q, m) \in I} B_{q,m} \cup \bigcup_{E}$. (The almost disjointness of the $B_{q,m}$’s imply that if $C_n \subset \bigcup B' \cup \bigcup \mathcal{A}'$ for some $B' \in [\bar{B}]^{<\omega}$ and $\mathcal{A}' \in [\mathcal{A}_\alpha]^{<\omega}$, then $C_n \setminus \bigcup \mathcal{A}'$ has finite intersection with all elements of $\bar{B} \setminus B'$. This easily yields the existence of such an $\eta_\alpha$.) Let $I_\alpha$ be an infinite subset of $\omega$ coding a surjection from $\omega$ onto $\eta_\alpha$. For a subset $s$ of $\omega$ we denote by $\Delta_s$ the set $\{2k + 1 : k \in \text{sup } s \setminus s\} \cup \{2k + 2 : k \in s\}$.

In $V[G_\alpha]$, $\bar{Q}_\alpha$ consists of pairs $(s, s')$ such that $s \in [\omega]^{<\omega}$, $s' \in [[B_{q,m} : m \in \Delta(s), \beta \in Y_{a+m, \eta_\alpha}] \cup \mathcal{A}_\alpha]^{<\omega}$, and for every $2n \in |s \cap B_{0,0}|$, $n \in I_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap B_{0,0})(2n) = B_{0,0}(2m)$. For conditions $p = (s, s')$ and $q = (t, t')$ in $\bar{Q}_\alpha$, we let $q \leq p$ if and only if $t$ is an end-extension of $s$ and $t \setminus s$ has empty intersection with all elements of $s'$.

Let $h_\alpha$ be a $\bar{Q}_\alpha$-generic filter over $L[G_\alpha]$. Set $u_\alpha = \bigcup_{(s, s') \in h_\alpha} s$, $A_\alpha = \alpha + (\omega \setminus \Delta(u_\alpha))$, and $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{u_\alpha\}$. As a consequence of the definition of $\bar{Q}_\alpha$ and the genericity of
we get\(^6\)

1. \( u_\alpha \in [\omega]^\omega \), \( u_\alpha \) is almost disjoint from all elements of \( \mathcal{A}_\alpha \), and has infinite intersection with \( C_\alpha \) for all \( n \in \omega \);

2. If \( m \in \Delta(u_\alpha) \), then \( |u_\alpha \cap B_{\beta,m}| < \omega \) if and only if \( \beta \in Y_{\alpha+m,\eta_\alpha} \);

3. For every \( n \in \omega \), \( n \in I_\alpha \) if and only if there exists \( m \in \omega \) such that \( (u_\alpha \cap B_{0,0})(2n) = B_{0,0}(2m) \); and

4. Equation \((*)\) holds for \( \alpha + 1 \), i.e. for every \( B \in \bar{B} \) and a finite subfamily \( \mathcal{A}' \) of \( \mathcal{A}_{\alpha+1} \), \( \mathcal{A}' \) covers neither a cofinite part of \( E(B) \) nor of \( O(B) \).

By (2) \( u_\alpha \) codes \( Y_{\alpha+m,\eta_\alpha} \) for all \( m \in \Delta(u_\alpha) \).

**Case 3.** If \( \alpha \) is not of the form above, i.e. \( \alpha \) is a successor or \( \alpha < \omega_2 \), then \( \dot{A}_\alpha \) is a name for the empty set and \( \dot{Q}_\alpha \) is a name for the poset adding a dominating real defined in Case 2 of the proof of Theorem 1.

With this the definitions of \( P = P_{\omega_1} \) and \( \langle \dot{A}_\alpha : \alpha \in \text{Lim}(\omega_3) \rangle \) are complete. Let \( G \) be a \( P \)-generic over \( L \).

Just as in the proof of Theorem 1 one can verify that Lemmata 2 and 3 hold true. These were of crucial importance for the proof of Corollary 1, which in turn was used in the proofs of Lemmata 4 and 5. Again, a direct verification shows that all of these statements still hold and hence \( G \) is a \( \Delta^1_3 \)-wellorder of the reals in \( L[G] \).

Lemmas 2 implies that the family \( \mathcal{A} \) we construct in the instances of **Case 2** is an \( \omega \)-mad subfamily of \( [\omega]^\omega \). Condition (3) above yields \( \eta_\alpha < \omega_1^M \) for all countable suitable models \( M \) containing \( \dot{\alpha}_{\alpha} \) provided that at stage \( \alpha \), **Case 2** took place (i.e., there is a condition in \( G \) which forces this). Combining this with the ideas of the proofs of Lemmata 4 and 5 we get that \( a \in \mathcal{A} \iff \) for every countable suitable model \( M \) containing \( a \) as an element there exists \( \bar{\alpha} < \omega_3^M \) such that \( S_{\bar{\alpha}+k}^M \) is nonstationary in \( (L[a])^M \) for all \( k \in \Delta(a) \). This provides a \( \Pi^1_2 \) definition of \( \mathcal{A} \), which finishes our proof of Theorem 2.

4. Questions

The consistency of the existence of a \( \Delta^1_3 \)-definable wellorder of the reals in the presence of \( c \geq \aleph_3 \) and MA, is still open. A second question naturally emerging from the developed techniques is the existence of a model in which a desired inequality between the cardinal characteristics of the real line holds, there is a \( \Delta^1_3 \)-definable wellorder of the reals...
reals and $\mathfrak{c} \geq \aleph_3$. Note that the bookkeeping argument which we have used in Theorems 1 and 2 allows only for handling of countable objects, which presents an additional difficulty in obtaining such models.

References