ON THE FUNDAMENTAL GROUP OF THE SIERPIŃSKI-GASKET

S. AKIYAMA, G. DORFER, J. M. THUSWALDNER, AND R. WINKLER

Abstract. We are giving a description of the fundamental group of the Sierpiński gasket.

1. Introduction

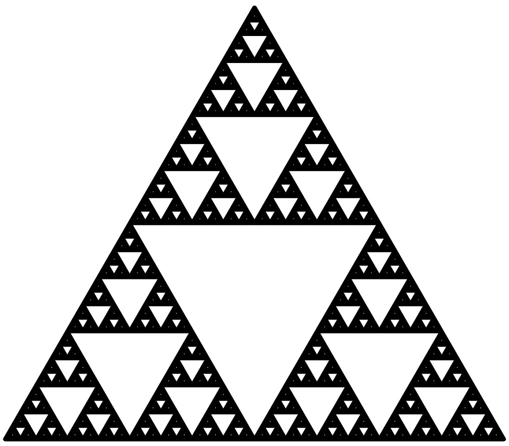

Figure 1. The Sierpiński-gasket

In an attempt to describe the fundamental group of the Sierpiński-gasket $\triangle$ it is an evident idea to consider for a loop f in $\triangle$ the sequence of homotopy classes $[f]_n$ of f in the approximating spaces $\triangle_n$ that arise when the usual construction process of recursively removing the open middle triangle is stopped at level n. By a result of Eda and Kawamura [4] the sequence $([f]_n)_{n\geq 0}$ characterizes f exactly up to homotopy. The natural ambient space for the sequences $([f]_n)_{n\geq 0}$ is the inverse limit $\varprojlim G_n$ of the fundamental groups G_n of $\triangle_n$. With an easy example (see Example 2.13) it becomes clear that $\varprojlim G_n$ contains elements which do not represent homotopy classes for loops in $\triangle$. So the objective

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appears to describe the subgroup of $\varprojlim G_n$ that corresponds to the fundamental group of $\triangle$.

Our approach to this task pursues the following strategy: Instead of investigating the problem directly in $\varprojlim G_n$ we consider an intermediate semigroup structure $\varprojlim S_n$ in which the set $S(\triangle)$ of all loops in $\triangle$ is described up to re-parametrization (see Figure 2).

$$\begin{array}{ccc} S(\triangle) & \stackrel{\sigma}{\to} & \varprojlim S_n \\ \downarrow [\cdot] & & \text{Red} \downarrow \\ \pi(\triangle) & \stackrel{\varphi}{\hookrightarrow} & \varprojlim G_n \end{array}$$

FIGURE 2.

To this end at every approximation level n we represent a loop f by a (finite) word $\sigma_n(f)$ consisting of the sequence of transition points (later called dyadic points) between the subtriangles of $\triangle_n$ that the loop passes. An appropriate reduction process on $\sigma_n(f)$ leads then to a canonical representative of the homotopy class $[f]_n$ which as a byproduct gives rise to an adequate representation of the elements in $\varprojlim G_n$.

We finally succeed in characterizing the elements of the fundamental group of $\triangle$ by a, after all, surprisingly simple Mittag-Leffler like stabilizing condition in the inverse semigroup limit $\varprojlim S_n$. The crucial step towards this result is the fact that though σ is not surjective the reduction map $\varprojlim S_n \to \varprojlim G_n$ does not distinguish the elements in the range of σ compared to entire $\varprojlim S_n$.

Moreover, we employ considerable effort to completely describe the kernel and the range of σ to enlighten the relevance of $\varprojlim S_n$ independently of its expedience with respect to the description of the fundamental group of $\triangle$: The elements in the range of σ are characterized by a completeness condition and they precisely describe the set of all loops in $\triangle$ up to re-parametrization.

The paper is organized as follows: In Section 2.1 we introduce a digital representation for the points of the Sierpiński-gasket $\triangle$ by retracing the usual construction process of recursively removing the open middle triangle. Thereby we obtain two sequences of approximating spaces to $\triangle$ and the points in $\triangle$ naturally split into the two classes of dyadic and generic points. In Section 2.2 it is explicated how a loop in $\triangle$ can be represented by a finite word over the alphabet of dyadic points of order $\leq n$ at every approximation level n. In Section 2.3 we introduce the inverse limit of semigroups $\varprojlim S_n$ and show that the semigroup $S(\triangle)$ of all loops in $\triangle$ can be mapped by a homomorphism into $\varprojlim S_n$ by means of the sequence of representations of a loop attained in Section 2.2.

In Section 2.4 we introduce the set of reduced words G_n which turns out to be isomorphic to the fundamental group of $\triangle_n$. The G_n, $n \geq 0$, form an inverse limit of groups $\varprojlim G_n$ and an appropriate reduction map on elements of $\varprojlim S_n$ is defined such that the diagram in Figure 2 commutes. Employing a result of Eda and Kawamura [4] we see that φ is injective and thus the fundamental group of $\triangle$ is a subgroup of $\varprojlim G_n$. Example 2.13 is presented demonstrating that φ is not surjective which provided the initial motivation for considering $\varprojlim S_n$.

In Section 3.1 we develop the machinery to study the range and the kernel of σ which is accomplished in Propositions 3.3–3.5 in full detail. In Section 3.2 we finally present the characterization of the elements in $\varprojlim G_n$ representing a homotopy class in $\pi(\triangle)$.

2. Preliminaries

2.1. **Digital representations of the Sierpiński-gasket $\triangle$.** For our purposes we need a digital representation of the points of the Sierpiński-gasket $\triangle$. To this end we follow the construction process of $\triangle$ that recursively removes the open middle triangle at each stage. We start with a triangle (including its inside) $\triangle_0$ in the plane. Just to have a concrete metric at hand we assume that $\triangle_0$ is equilateral with side length 1. The vertices of $\triangle_0$ are denoted by 0, 1 and 2. By joining the midpoints of the sides $\triangle_0$ is subdivided in four smaller triangles $\langle 0\rangle$, $\langle 1\rangle$, $\langle 2\rangle$ and the middle triangle, where $\langle i\rangle$ is the subtriangle that contains the vertex i. Removing the interior of the middle triangle from $\triangle_0$ we obtain the first approximation $\triangle_1$, i.e.

$$\triangle_1 = \langle 0\rangle \cup \langle 1\rangle \cup \langle 2\rangle.$$

With the remaining triangles $\langle i\rangle$, $i = 0, 1, 2$, we proceed the same way: $\langle i\rangle$ is divided into the four subtriangles $\langle i0\rangle$, $\langle i1\rangle$, $\langle i2\rangle$, and the middle triangle the interior of which is cut out in the next step. Thus we get the second approximation

$$\triangle_2 = \bigcup_{i,j\in\{0,1,2\}} \langle ij\rangle,$$

and so on and so forth. We obtain a decreasing sequence $\triangle_0 \supset \triangle_1 \supset \triangle_2 \ldots$ of compact spaces and hence the intersection $\triangle = \bigcap_{n\in\mathbb{N}} \triangle_n$, the Sierpiński gasket, is a compact space as well. $\triangle$ consists of two types of points:

Dyadic points: these are points P which lie in two different subtriangles at some stage (and consequently in all the following stages) in the construction process described before. The smallest level at which

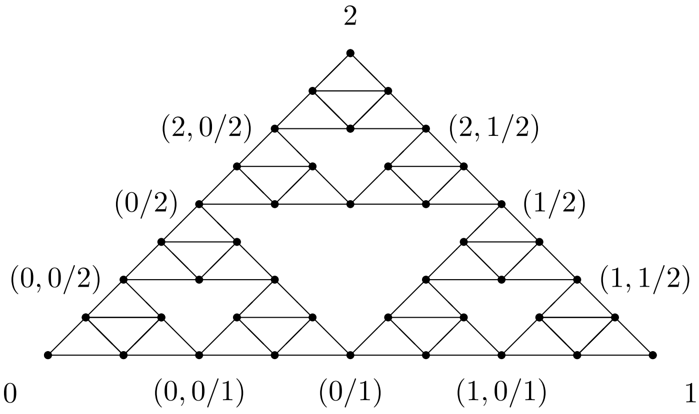

FIGURE 3

P appears as a vertex of two different subtriangles is called the order of P. For instance $P = \langle 01 \rangle \cap \langle 02 \rangle = \langle 012 \rangle \cap \langle 021 \rangle = \dots$ is of order 2. We represent P as $(0, 1/2)$ or $(0, 2/1)$. In general a dyadic point of order n has a finite representation of the form

$$P = (a_1, a_2, \dots, a_{n-1}, a/b) = (a_1, a_2, \dots, a_{n-1}, b/a)$$

with $a_i, a, b \in \{0, 1, 2\}$ and $a \neq b$, and this means $P = \langle a_1 a_2 \dots a_{n-1} a \rangle \cap \langle a_1 a_2 \dots a_{n-1} b \rangle$. We consider the vertices $0, 1, 2$ of $\triangle_0$ as dyadic points of order 0. Let in the following D_n denote the set of all dyadic points of order $\leq n$. In D_n there is a natural relation $\sim_n$ describing the neighborhood of dyadic points at level n: for $P, Q \in D_n$ we have $P \sim_n Q$ if and only if $P \neq Q$ and there is a subtriangle $\langle a_1 \dots a_n \rangle$ of $\triangle_n$ to which P and Q belong. At every stage n a dyadic point $P \neq 0, 1, 2$ has exactly four neighbours, and the points $0, 1$ and 2 have exactly two neighbors each.

Generic points: these are points P of $\triangle$ such that at every stage n there is a unique subtriangle of $\triangle_n$ to which P belongs. If $P \in \langle a_1 a_2 \dots a_n \rangle$, $n \in \mathbb{N}$, then P has the infinite representation $P = (a_1, a_2, \dots)$ with $a_i \in \{0, 1, 2\}$, where the sequence $(a_n)_{n \in \mathbb{N}}$ is not ultimately constant.

Formally $\triangle$ can be obtained as the quotient space of the compact space X of one-sided infinite sequences over the three letter alphabet $\{0, 1, 2\}$, i.e. $X = \{0, 1, 2\}^{\mathbb{N}}$ with the discrete topology on the factors, where a pair of points $P = (a_n)_{n \in \mathbb{N}}$ and $Q = (b_n)_{n \in \mathbb{N}}$ is identified if and only if there is an n_0 such that $a_n = b_n$ for $n < n_0$ and $a_n = b_{n_0} \neq a_{n_0} = b_n$ for $n > n_0$. In the approach described before this means that $P = Q = (a_1, a_2, \dots, a_{n_0 - 1}, a_{n_0} / b_{n_0})$ is a dyadic point of order n_0.

The $\triangle_n$, $n \geq 0$, provide an encasing approximation to the Sierpiński-gasket. In the following we will also consider an approximation from inside. Let $\triangle^n$ denote the boundary of $\triangle_n$ considered as a subspace of the plane. Then $\triangle = \overline{\bigcup_{n\in\mathbb{N}} \triangle^n}$ where the bar means the closure operator in the plane: $\bigcup_{n\in\mathbb{N}} \triangle^n$ contains exactly those points $P = (a_n)$ such that eventually the digits a_n are out of a two-element subset of $\{0, 1, 2\}$, in particular this set contains all dyadic points. On the other hand every generic point of $\triangle$ is the limit of a sequence of dyadic points.

Concerning homotopy the spaces $\triangle_n$ and $\triangle^{n-1}$, $n \geq 1$, provide the same level of approximation to the Sierpiński gasket $\triangle$. There exists a deformation p_n that retracts $\triangle_n$ to $\triangle^{n-1}$: For every subtriangle $T = \langle a_1a_2 \dots a_{n-1}\rangle$ of $\triangle_{n-1}$ the map p_n projects the points of $\triangle_n \cap T$ from the center of T to the boundary of T. Hence the fundamental groups $\pi(\triangle_n)$ and $\pi(\triangle^{n-1})$ are isomorphic (cf. [6, Theorem 1.22 and Theorem 3.10]).

2.2. **Representation of loops in $\triangle$.** To describe the fundamental group $\pi(\triangle)$ we have to consider continuous loops $f : [0, 1] \to \triangle$. Since $\triangle$ is path connected we will always assume $f(0) = f(1) = 0$. Our next aim is to represent f by a finite word over the alphabet D_n for every n.

Let us fix n. The pre-images $\{f^{-1}(P)|P \in D_n\}$ form a finite family of disjoint compact subsets of the interval $[0, 1]$. Therefore this family is separated, i.e. there is $m \in \mathbb{N}$ such that for all $i = 1, 2, \dots, m-1$ the set $f^{-1}(P) \cap [\frac{i-1}{m}, \frac{i}{m}]$ is non-empty for at most one P. We list these points P as i increases and in the arising sequence we cancel out consecutive repetitions. Thus we obtain a finite word $P_1P_2 \dots P_k =: \sigma_n(f)$ over D_n which is independent of the chosen m and represents f at approximation level n. Obviously $\sigma_n(f)$ has the following properties:

$$P_1 = P_k = 0, \tag{2.1}$$

$$P_i \sim_n P_{i+1} \text{ for all } i = 1, \dots k-1. \tag{2.2}$$

In the following we will also consider the loop that emerges from $\sigma_n(f)$ by connecting the listed points straight-lined in the order they appear and call it the piecewise linear loop corresponding to $\sigma_n(f)$. In order to disburden the notation we will not distinguish between the string $\sigma_n(f)$ and the associated loop as long as no confusion can arise.

Proposition 2.1. *In $\triangle_n$ the loop f and the piecewise linear loop $\sigma_n(f)$ are homotopic.*

Proof. Let $\sigma_n(f) = P_1 \dots P_k$. For every $i = 1, \dots, k$ there is a maximal interval $[s_i, t_i]$ such that $f(s_i) = f(t_i) = P_i$, $f([s_i, t_i]) \cap D_n = \{P_i\}$ and $0 = s_1 \leq t_1 < s_2 \leq t_2 < \dots < s_k \leq t_k = 1$. This means that $f([s_i, t_i])$ is contained in the interior – as a subset of $\triangle_n$ – of the union of the two subtriangles of $\triangle_n$ that intersect in P_i. Since this set is simply connected $f{\restriction}[s_i, t_i]$ is homotopic to the constant loop in P_i.

Moreover, the conditions on s_i and t_i imply that $f([t_i, s_{i+1}])$ is a subset of the subtriangle of $\triangle_n$ that contains P_i and P_{i+1} and hence $f \upharpoonright [t_i, s_{i+1}]$ is homotopic to the straight line between P_i and P_{i+1}.

Putting the pieces together we obtain the assertion. □

In order to describe the fundamental group of $\triangle$, Proposition 2.1 suggests to represent a loop f, as a first step, by the sequence $(\sigma_n(f))_{n\geq 0}$. In the next section we will elaborate an ambient space where the sequence $(\sigma_n(f))_{n\geq 0}$ has its appropriate position.

2.3. **The inverse system (S_n, γ_n) of semigroups.** The semigroups S_n, $n \geq 0$, are defined in the following way: The elements of S_n are finite words $\omega_n = P_1 \dots P_k$ over the alphabet D_n such that (2.1) and (2.2) are satisfied. These words are called admissible. (2.1) means that we consider only cyclic paths with base point 0, (2.2) reflects that with respect to homotopy constant parts of paths do not matter and that in a continuous path a dyadic point can only be followed by a neighboring dyadic point.

The semigroup operation $\cdot$ on S_n is defined by concatenation of words and cancelation of one of the adjacent letters 0 at the interface:

$$P_1 \dots P_k \cdot Q_1 \dots Q_l = P_1 \dots P_k Q_2 \dots Q_l.$$

The mapping $\gamma_n : S_n \to S_{n-1}$, $n \geq 1$, eliminates from an element of S_n all points of order n, and then cancels consecutive repetitions of points of order $< n$ arising in this process. Obviously the result is an admissible word in S_{n-1} and γ_n is a semigroup epimorphism. Thus we may consider the inverse semigroup-limit

$$\varprojlim S_n = \{(\omega_n)_{n\geq 0} \mid \gamma_k(\omega_k) = \omega_{k-1} \text{ for all } k \geq 1\}$$

corresponding to the sequence $(S_n, \gamma_n)_{n\geq 0}$.

Let $(S(\triangle), \cdot)$ denote the semigroup of continuous loops $f : [0, 1] \to \triangle$ where multiplication $\cdot$ is just concatenation of loops without taking care of homotopy. As a general principle we denote the semigroup operations in $S(\triangle)$, S_n and $\varprojlim S_n$ by $\cdot$ (or omit the operation symbol), whereas for the group operations, for instance in the fundamental group $\pi(\triangle)$, we use the notation $*$.

Next we will provide a combinatorial description of loops f at the semigroup level.

Proposition 2.2. *The map*

$$\sigma : \begin{cases} S(\triangle) & \to & \varprojlim S_n \\ f & \mapsto & (\sigma_n(f))_{n\geq 0} \end{cases}$$

is a semigroup homomorphism.

Proof. Firstly we show that σ is well defined: Let f be an element of $S(\triangle)$. Then the word $\sigma_n(f)$ contains the dyadic points of D_n which are passed by the loop f in the order they appear in f without consecutive repetitions. When we apply γ_n to $\sigma_n(f)$ obviously we end up with the same word in S_{n-1} we obtain when we list the dyadic points f passes at level $n-1$, i.e. $\gamma_n(\sigma_n(f)) = \sigma_{n-1}(f)$, and thus $\sigma(f) \in \varprojlim S_n$.

σ is a homomorphism since concatenation of loops in $S(\triangle)$ correlates exactly to the concatenation of words in the components S_n, $n \geq 0$. To put it more formally, for $f, g \in S(\triangle)$ we have:

$$\sigma(f \cdot g) = (\sigma_n(f \cdot g))_{n\geq 0} = (\sigma_n(f) \cdot \sigma_n(g))_{n\geq 0} = \\ (\sigma_n(f))_{n\geq 0} \cdot (\sigma_n(g))_{n\geq 0} = \sigma(f) \cdot \sigma(g).$$

□

2.4. **The inverse system (G_n, δ_n) of groups.** To describe the homotopy of loops in $\triangle$ we have to consider an appropriate reduction process for the semigroup words in $\varprojlim S_n$. In the following for $f : [0,1] \to \triangle$ let $[f]$ denote the homotopy class of f in $\triangle$, and let $[f]_n$ denote the homotopy class of f in $\triangle_n$, i.e. f is considered as a map with range $\triangle_n$.

In a first step we will describe the elements of the fundamental group of $\triangle_n$. Very briefly we recall here the standard approach to the fundamental group of a simplicial complex (cf. [6, chapter 7]): One considers edge paths in $\triangle_n$ which start and end in the same vertex, say in 0. In principle an edge path is the same as an admissible word over D_n, i.e. an element of S_n, except that also constant edges are allowed. Two edge paths are defined to be equivalent if one can be obtained from the other by a finite number of elementary moves. In our language an elementary move is a substitution on subwords consisting of consecutive letters of the form

$$PQP \longleftrightarrow P \quad \text{or} \quad PQR \longleftrightarrow PR \tag{2.3}$$

where P, Q, R are the distinct vertices of a simplex in the simplicial complex which in our case means that P, Q, R form a subtriangle of $\triangle_n$. As the arrows indicate these transformations may be performed in both directions. The equivalence classes of edge paths then constitute the elements of the fundamental group with concatenation as the group operation (cf. [6, Theorem 7.36]).

In our case we proceed slightly different: We call an element $\omega_n \in S_n$ reduced if ω_n cannot be shortened by an elementary move as described in (2.3). A reduced word in S_n can be identified with a sequence of subtriangles of $\triangle_n$ such that any three consecutive subtriangles are pairwise different. Let G_n denote the set of all reduced words of S_n and $\mathrm{Red}_n : S_n \to G_n$ the mapping that performs elementary moves until the word is reduced.

Proposition 2.3. Red_n *is well defined and for* $\omega_n \in S_n$ *the loop corresponding to* $\mathrm{Red}_n(\omega_n)$ *forms a canonical representative of the homotopy class of the loop corresponding to* ω_n *in* $\triangle_n$*.*

Proof. Obviously, by performing an elementary move on an element of S_n we stay in the same homotopy class for the corresponding loops. All we have to show is that two different reduced words correspond to non-homotopic loops. Here we use the fact that $\triangle_n$ and $\triangle^{n-1}$ have isomorphic homotopy groups ($\triangle^{n-1}$ is a deformation retract of $\triangle_n$). Since $\triangle^{n-1}$ is a connected 1-complex its homotopy group is a free group, freely generated by the edges not contained in a fixed spanning tree T (cf. [6, Corollary 7.35]). Starting with two different reduced words $\omega_n \neq \bar{\omega}_n$ in G_n by retracting to $\triangle^{n-1}$ we end up with two different words $\alpha_{n-1} \neq \bar{\alpha}_{n-1}$ over the alphabet D_{n-1} such that any three consecutive letters of these words are pairwise different elements of D_{n-1} (reduced word in $G_n \leftrightarrow$ sequence of subtriangles in $\triangle_n$; every subtriangle in $\triangle_n$ contains exactly one vertex in D_{n-1}, the sequence of these vertices is exactly what we obtain by the retraction). Suppose the two emerging loops corresponding to α_{n-1} and $\bar{\alpha}_{n-1}$ are homotopic in $\triangle^{n-1}$, then due to the fact that the homotopy group of $\triangle^{n-1}$ is a free group the two words must contain the same edges not contained in the tree T in the corresponding order. Moreover, there is a unique path in the tree connecting these edges. Since α_{n-1} and $\bar{\alpha}_{n-1}$ do not contain subwords of the form PQP, α_{n-1} and $\bar{\alpha}_{n-1}$ must be identical in the parts connecting the edges not in T, and hence they must coincide on the whole, which is a contradiction. □

Now it is obvious how to define the group operation for $\omega_n, \bar{\omega}_n \in G_n$:

$$\omega_n * \bar{\omega}_n = \mathrm{Red}_n(\omega_n \cdot \bar{\omega}_n),$$

where $\omega_n \cdot \bar{\omega}_n$ is the product in S_n. Together with the results in [6, chapter 7] we obtain:

Proposition 2.4. $(G_n, *)$ *is isomorphic to the fundamental group* $(\pi(\triangle_n), *)$ *with the isomorphism* $\varphi_n : [f]_n \mapsto \mathrm{Red}_n(\sigma_n(f))$ *where* $f \in S(\triangle_n)$ *is continuous.*

$\mathrm{Red}_n : S_n \to G_n$ *is a semigroup epimorphism associating to every admissible word in* S_n *its reduced form, i.e.* $(G_n, *)$ *is isomorphic to* $(S_n / \ker(\mathrm{Red}_n), \cdot)$*.*

Now we elaborate a connection between the groups G_n, $n \geq 1$.

Lemma 2.5. *The map*

$$\delta_n : \begin{cases} G_n & \to \quad G_{n-1} \\ \omega_n & \mapsto \quad \mathrm{Red}_{n-1}(\gamma_n(\omega_n)) \end{cases}$$

is a group epimorphism.

Proof. Let $\omega_n, \bar{\omega}_n \in G_n$. We have

$$\delta_n(\omega_n * \bar{\omega}_n) = \mathrm{Red}_{n-1}(\gamma_n(\mathrm{Red}_n(\omega_n \cdot \bar{\omega}_n))).$$

On the other hand we have

$$\delta_n(\omega_n) * \delta_n(\bar{\omega}_n) = \mathrm{Red}_{n-1}(\mathrm{Red}_{n-1}(\gamma_n(\omega_n)) \cdot \mathrm{Red}_{n-1}(\gamma_n(\bar{\omega}_n))) = \mathrm{Red}_{n-1}(\gamma_n(\omega_n) \cdot \gamma_n(\bar{\omega}_n)) = \mathrm{Red}_{n-1}(\gamma_n(\omega_n \cdot \bar{\omega}_n)).$$

Due to Proposition 2.3 it is sufficient to show that $\gamma_n(\mathrm{Red}_n(\omega_n \cdot \bar{\omega}_n))$ and $\gamma_n(\omega_n \cdot \bar{\omega}_n)$ are homotopic in $\triangle_{n-1}$. It is obvious by the construction of γ_n that for every $\alpha_n \in S_n$ we have $[\alpha_n]_{n-1} = [\gamma_n(\alpha_n)]_{n-1}$. Further we have $[\alpha_n]_n = [\mathrm{Red}_n(\alpha_n)]_n$ and hence also $[\alpha_n]_{n-1} = [\mathrm{Red}_n(\alpha_n)]_{n-1}$. Altogether we obtain

$$[\gamma_n(\omega_n\bar{\omega}_n)]_{n-1} = [\omega_n\bar{\omega}_n]_{n-1} = [\mathrm{Red}_n(\omega_n\bar{\omega}_n)]_{n-1} = [\gamma_n(\mathrm{Red}_n(\omega_n\bar{\omega}_n))]_{n-1}$$

and we are done.

δ_n is surjective: Suppose $\omega_{n-1} = P_1P_2 \dots P_k$ in G_{n-1} is given. Put $\omega_n = P_1Q_1P_2Q_2 \dots Q_{k-1}P_k$, where Q_i is the (unique) element of D_n with $P_i \sim_n Q_i \sim_n P_{i+1}$. One can check easily that ω_n is reduced and $\mathrm{Red}_n(\omega_n) = \omega_{n-1}$. □

As a consequence of the last lemma we can consider the inverse group-limit

$$\varprojlim G_n = \{(\omega_n)_{n\geq 0} \mid \delta_k(\omega_k) = \omega_{k-1} \text{ for all } k \geq 1\}.$$

Next we show that the reduction maps $\mathrm{Red}_n : S_n \to G_n$ can be lifted to a map on the inverse limits.

Lemma 2.6. *For every $n \geq 1$ the following diagram commutes:*

$$\begin{array}{ccc} S_n & \xrightarrow{\gamma_n} & S_{n-1} \\ \downarrow \mathrm{Red}_n & & \mathrm{Red}_{n-1} \downarrow \\ G_n & \xrightarrow{\delta_n} & G_{n-1} \end{array}$$

Proof. Let ω_n be in S_n. We have to show that $\delta_n(\mathrm{Red}_n(\omega_n)) = \mathrm{Red}_{n-1}(\gamma_n(\omega_n))$. Since $\delta_n(\mathrm{Red}_n(\omega_n)) = \mathrm{Red}_{n-1}(\gamma_n(\mathrm{Red}_n(\omega_n)))$ it suffices to prove that $\gamma_n(\omega_n)$ and $\gamma_n(\mathrm{Red}_n(\omega_n))$ are homotopic in $\triangle_{n-1}$. From here the proof is identical to the one of Lemma 2.5, so we omit it. □

Proposition 2.7. *The map*

$$\mathrm{Red} : \begin{cases} \varprojlim S_n & \to & \varprojlim G_n \\ (\omega_n)_{n\geq 0} & \mapsto & (\mathrm{Red}_n(\omega_n))_{n\geq 0} \end{cases}$$

is a well defined semigroup homomorphism.

Proof. If $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ then $\gamma_n(\omega_n) = \omega_{n-1}$ for every n. This yields

$$\delta_n(\mathrm{Red}_n(\omega_n)) = \mathrm{Red}_{n-1}(\gamma_n(\mathrm{Red}_n(\omega_n))) = \\ \mathrm{Red}_{n-1}(\gamma_n(\omega_n)) = \mathrm{Red}_{n-1}(\omega_{n-1})$$

where the penultimate identity was derived in the proof of Lemma 2.6. This shows that Red is well defined. The proof that Red is a homomorphism is straightforward. □

Now we figure out that the fundamental group $(\pi(\triangle)), *)$ can be embedded into the group-limit $(\varprojlim G_n, *)$.

Proposition 2.8. *The map*

$$\varphi : \begin{cases} \pi(\triangle) & \rightarrow & \varprojlim G_n \\ [f] & \mapsto & \mathrm{Red}(\sigma(f)) \end{cases}$$

is a well defined group homomorphism.

Proof. Let f, g be a continuous loops in $\triangle$. We recall (Proposition 2.4) that $\mathrm{Red}_n(\sigma_n(f))$ (considered as a piecewise linear path) is the canonical representative of $[f]_n$, the homotopy class of f in $\triangle_n$, $n \geq 0$. If $[f] = [g]$ then, of course, $[f]_n = [g]_n$ for all n. But this means $\mathrm{Red}_n(\sigma_n(f)) = \mathrm{Red}_n(\sigma_n(g))$ for all n and hence $\mathrm{Red}(\sigma(f)) = \mathrm{Red}(\sigma(g))$. This shows that φ is well defined.

With the results we already have it is straightforward to prove that φ is a homomorphism. □

In a next step we want to prove the injectivity of φ. To this matter we first construct the Čech homotopy group $\check{\pi}(\triangle)$ of $\triangle$ (see e.g. [5, p. 130][1] or [4, Appendix A] for a definition of $\check{\pi}$). Since it will turn out that $\check{\pi}(\triangle) = \varprojlim G_n$, the injectivity of φ follows from the fact that the fundamental group of a one-dimensional space can be embedded in its Čech homotopy group (cf. [4, Theorem 1.1]) and φ is the corresponding canonical embedding. Before we give the details we have to set up some notations.

Note that each of the equilateral triangles $T = \langle a_1 a_2 \dots a_n\rangle$ has side length $\frac{1}{2^n}$. For a small $\varepsilon > 0$ let U_T be the open equilateral triangle of side length $(1+\varepsilon)\frac{1}{2^n}$ having the same midpoint and parallel sides to T. For each $n \geq 0$ define the collections

$$\mathcal{U}_n := \{U_T \mid T = \langle a_1 \dots a_n\rangle \text{ with } a_1, \dots, a_n \in \{0,1,2\}\}.$$

Now choose ε in a way that $\mathcal{U}_n$ is a cover of order 2 for each n (i.e. for all pairwise distinct sets $U, U', U'' \in \mathcal{U}_n$ we have $U \cap U' \cap U'' = \emptyset$). This implies that

$$U_{\langle a_1 \dots a_n\rangle} \cap U_{\langle a'_1 \dots a'_n\rangle} \neq \emptyset \iff \langle a_1 \dots a_n\rangle \cap \langle a'_1 \dots a'_n\rangle \neq \emptyset. \tag{2.4}$$

[1]Note that the Čech homotopy group is called *shape group* in this text.

Employing compactness of $\triangle$ we know that the Čech homotopy group can be defined in terms of finite covers. $\mathcal{U}_n$ is a finite, open cover of $\triangle_n$ and thus also of $\triangle$. Moreover, it is easy to see that $(\mathcal{U}_n)_{n\geq 0}$ is a sequence of covers of $\triangle$ which is cofinal in the set of all finite open covers of $\triangle$. Thus, denoting the nerve of a cover $\mathcal{C}$ by $\mathcal{N}(\mathcal{C})$, we have

$$\check{\pi}(\triangle) \cong \varprojlim \pi(\mathcal{N}(\mathcal{U}_n)). \tag{2.5}$$

Now we are in a position to prove that the nerve $\mathcal{N}(\mathcal{U}_n)$ is homotopy equivalent to $\triangle_n$ for each $n \geq 1$. (In what follows, homotopy equivalence will be denoted by "$\simeq$".)

Lemma 2.9. *For each $n \geq 1$ we have*

$$\mathcal{N}(\mathcal{U}_n) \simeq \triangle^{n-1} \simeq \triangle_n.$$

Proof. We start with proving the first homotopy equivalence by induction on n. For $n = 1$ the sets $\mathcal{N}(\mathcal{U}_n)$ and $\triangle^{n-1}$ are both homeomorphic to a circle and thus homotopy equivalent.

Assume that the result is proved for a certain n. Now we are going to construct the nerve of $\mathcal{U}_{n+1}$. Consider the subdivision

$$\mathcal{U}_{n+1} = \mathcal{U}_{n+1}^{(0)} \cup \mathcal{U}_{n+1}^{(1)} \cup \mathcal{U}_{n+1}^{(2)}$$

with

$$\mathcal{U}_{n+1}^{(i)} := \{U_T \mid T = \langle i a_2 \dots a_{n+1} \rangle \text{ with } a_2, \dots, a_{n+1} \in \{0,1,2\}\}.$$

Then $\mathcal{N}(\mathcal{U}_{n+1}^{(i)})$ is homeomorphic to $\mathcal{N}(\mathcal{U}_n)$. Thus $\mathcal{N}(\mathcal{U}_{n+1})$ contains three copies of $\mathcal{N}(\mathcal{U}_n)$. Each of these copies is homotopy equivalent to $\triangle^{n-1}$ by induction (see Figure 4 where the situation is depicted for $n + 1 = 3$).

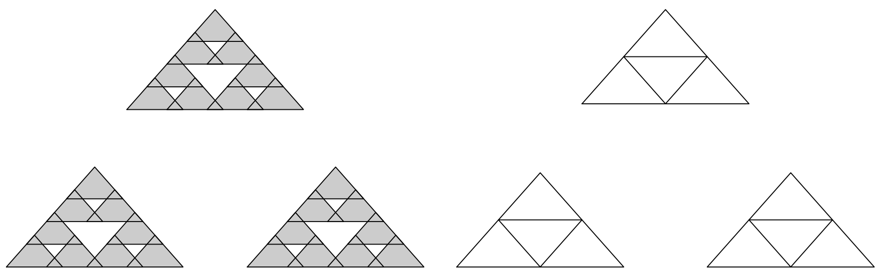

FIGURE 4. On the left hand side the covers $\mathcal{U}_3^{(i)}$ for $i \in \{0, 1, 2\}$ are depicted separately. Right you can see (deformation retracts of) the associated nerves $\mathcal{N}(\mathcal{U}_3^{(i)})$.

Now we have to determine the overlaps between the elements of the covers $\mathcal{U}_{n+1}^{(i)}$. Let U_T, U_T' be two elements of $\mathcal{U}_{n+1}$ having non-empty intersection.

If both of these sets are contained in $\mathcal{U}_{n+1}^{(i)}$ for the same $i \in \{0,1,2\}$ the 1-simplex caused by this intersection in $\mathcal{N}(\mathcal{U}_{n+1})$ is already contained in $\mathcal{N}(\mathcal{U}_{n+1}^{(i)})$.

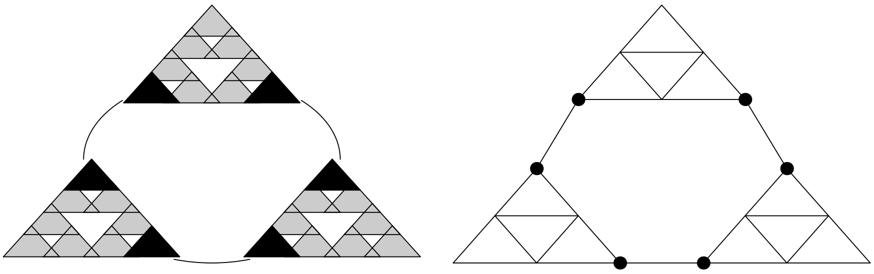

FIGURE 5. On the left hand side we illustrate the intersections between the covers $\mathcal{U}_3^{(i)}$ $(0 \le i \le 2)$. The pairs of black triangles connected by a line segment intersect in $\mathcal{U}_3$. This leads (see right) to additional 1-simplices between the nerves of $\mathcal{U}_3^{(i)}$.

Suppose on the other hand that $U_T, \in \mathcal{U}_{n+1}^{(i)}$ and $U'_T, \in \mathcal{U}_{n+1}^{(i')}$ for $i \neq i'$. From (2.4) we see that this implies that one of the following three constellations holds

(C.1) $T = \langle 01\dots1\rangle$ and $T' = \langle 10\dots0\rangle$ (or vice versa),
(C.2) $T = \langle 12\dots2\rangle$ and $T' = \langle 21\dots1\rangle$ (or vice versa),
(C.3) $T = \langle 02\dots2\rangle$ and $T' = \langle 20\dots0\rangle$ (or vice versa).

The constellation in (C.i) gives rise to a 1-simplex leading from $\mathcal{N}(\mathcal{U}_{n+1}^{(i-1)})$ to $\mathcal{N}(\mathcal{U}_{n+1}^{(i \bmod 3)})$ in $\mathcal{N}(\mathcal{U}_{n+1})$ (see Figure 5 for an illustration in the case $n+1=3$).

Summing up we have shown that $\mathcal{N}(\mathcal{U}_{n+1})$ is the union of three homotopic copies of $\mathcal{N}(\mathcal{U}_n) \simeq \triangle^{n-1}$ connected cyclically by 1-simplices. By shrinking these three 1-simplices to points it is easy to see that $\mathcal{N}(\mathcal{U}_{n+1})$ deformation retracts to $\triangle^{n-1}$ and the first assertion is proved. The second assertion was already verified in the last paragraph of Section 2.1. □

Lemma 2.10.

$$\check{\pi}(\triangle) \cong \varprojlim G_n.$$

Proof. This follows by combining Lemma 2.9 with (2.5) and Proposition 2.4. □

Proposition 2.11. *The group homomorphism φ defined in Proposition 2.8 is injective.*

Proof. Composing $\varphi : \pi(\triangle) \to \varprojlim G_n$ with the canonical isomorphism between $\varprojlim G_n$ and $\check{\pi}(\triangle)$ (see Lemma 2.10) we obtain the canonical homomorphism from $\pi(\triangle)$ to $\check{\pi}(\triangle)$. Since $\triangle$ is a one-dimensional continuum, [4, Corollary 1.2] implies that this canonical homomorphism is injective and so is φ.

we can confine ourselves to proving that $\pi(\triangle)$ is isomorphic to a subgroup of $\check{\pi}(\triangle)$. However, this is the content of [4, Corollary 1.2]. □

The next theorem gives an interim survey of what we have established up to this point.

Theorem 2.12. *The fundamental group* $(\pi(\triangle), *)$ *of the Sierpiński gasket is a subgroup of* $(\varprojlim G_n, *)$. *Moreover, the following diagram commutes:*

$$\begin{array}{ccc} S(\triangle) & \xrightarrow{\sigma} & \varprojlim S_n \\ \downarrow [\cdot] & & \text{Red} \downarrow \\ \pi(\triangle) & \overset{\varphi}{\hookrightarrow} & \varprojlim G_n \end{array}$$

However, the next example shows that φ is not surjective:

Example 2.13. Let C_0 be the (piecewise linear) loop that starting at 0 passes around the boundary of $\triangle_0$ in positive direction (i.e. passing from 0 to 1, then 2 and back to 0). By C_0^{-1} we mean the same cycle passed in the opposite direction. C_1 denotes the loop around the subtriangle $\langle 0\rangle$ in $\triangle_1$ (i.e. passing through 0, (0/1), (0/2) and 0), C_2 the loop around $\langle 01\rangle$ in $\triangle_2$, and so on. Now we consider the following sequence of words:

$$\begin{aligned} \omega_0 &= \omega_1 = 0 \\ \omega_2 &= \text{Red}_2(\sigma_2(C_0C_1C_0^{-1})) \\ \omega_3 &= \text{Red}_3(\sigma_3(C_0C_1C_0^{-1}C_2)) \\ \omega_4 &= \text{Red}_4(\sigma_4(C_0C_1C_0^{-1}C_2C_0C_3C_0^{-1})) \\ \omega_5 &= \text{Red}_4(\sigma_4(C_0C_1C_0^{-1}C_2C_0C_3C_0^{-1}C_4)) \\ &\dots \end{aligned}$$

It can be checked easily that $(\omega_n)_{n\geq 0}$ is a element of $\varprojlim G_n$. For instance, if we apply δ_4 to ω_4, the loop C_3 disappears since it is nullhomotopic in $\triangle_3$, and consequently also the C_0 and C_0^{-1} neighboring C_3 cancel out and we arrive at ω_3.

Obviously, there exists no f in $S(\triangle)$ such that $\varphi([f]) = (\omega_n)_{n\geq 0}$: if so then due to the construction of $\omega_n = [f]_n$ the loop f would have to go around the circle C_0 infinitely many times, which is not possible.

Maybe it is instructive to see here that $(\omega_n)_{n\geq 0}$ is even not in $\text{Red}(\varprojlim S_n)$. Suppose there is $(\alpha_n)_{n\geq 0}$ in $\varprojlim S_n$ with $\text{Red}((\alpha_n)_{n\geq 0}) = (\omega_n)_{n\geq 0}$. If we consider just the dyadic points of order 1 that appear in ω_{2n}, we see that the sequence (0/1) (1/2) (0/2) (1/2) (0/2) repeats n

times. This means that at least this sequence of $5n$ points of order 1 also appears in α_{2n} (maybe some more which cancel out by performing Red_{2n}). However, when projecting down from S_{2n} to S_1 in $\varprojlim S_n$ no cancelation in between this $5n$ points can occur. As a consequence α_1 would contain infinitely many points which is a contradiction.

We aim at describing the fundamental group of the Sierpiński gasket. Retrospectively, Theorem 2.12 provides the motivation for investigating the semigroup limit $\varprojlim S_n$: $\pi(\triangle) \cong \varphi(\pi(\triangle)) = \mathrm{Red}(\sigma(S(\triangle)))$. Therefore we have to study the range of σ in $\varprojlim S_n$ and the range of Red in $\varprojlim G_n$. This will be accomplished in the next section.

3. A description of the elements in $\varphi(\pi(\triangle))$

3.1. **The range and the kernel of $\boldsymbol{\sigma}$.** We associate to a fixed element $(\omega_n)_{n\geq 0} = (P_{n1}P_{n2}\dots P_{nk_n})_{n\geq 0}$ in $\varprojlim S_n$ a graph $G = (V, E)$ with vertices V and directed edges E. We think of the graph G as organized in rows: in the nth row, $n \geq 0$, we have for every letter appearing in the word ω_n a corresponding vertex, i.e. $V = \{(n, j) \mid n \geq 0, 1 \leq j \leq k_n\}$. Edges connect certain vertices from row n to vertices in row $n + 1$, namely, $((n, i), (n + 1, j)) \in E$ if and only if $P_{ni} = P_{n+1,j}$ and in the course of γ_{n+1} that maps ω_{n+1} to ω_n the point $P_{n+1,j}$ is projected to P_{ni}. Consequently any vertex (n, i) in row n has at least one successor up to a finite number of successors (not bounded from above for growing n) in row $n + 1$, and (n, i) has exactly one predecessor in row $n - 1$ if and only if the order of P_{ni} is $< n$.

Example 3.1. We consider the following element in $\varprojlim S_n$ one can think of as a "pseudo-path" that passes from 0 on the baseline of $\triangle^0$ arbitrarily near to 1 without touching 1 and then goes the same way back to 0. A phenomenon arising in this example will turn out to be important in the further investigation:

$$\omega_0 = 0,\ \omega_1 = 0(0/1)0,\ \omega_2 = 0(0, 0/1)(0/1)(1, 0/1)(0/1)(0, 0/1)0, \dots .$$

In Figure 6 we denote the vertices by the corresponding dyadic points P_{ni} instead of the index (n, i) we usually use.

By a branch B we mean a directed path in G which cannot be extended. As description for B we use the sequence of vertices contained in B, i.e. $B = (n, i_n)_{n\geq n_0}$ where $P = P_{n,i_n}$ for all $n \geq n_0$, is a point of order n_0. We say that branch B corresponds to the dyadic point P.

The set $\mathcal{B}$ of all branches in G carries a natural total order $\leq$: Let $B_1 = (n, i_n)_{n\geq n_1}$, $B_2 = (n, j_n)_{n\geq n_2}$ be two branches then we define $B_1 < B_2$ if and only if there exists $n \geq \max\{n_1, n_2\}$ such that $i_n < j_n$. Consequently we then have $i_m < j_m$ for all $m > n$, and $i_m \leq j_m$ for all m with $\max\{n_1, n_2\} \leq m < n$ which reflects the property that

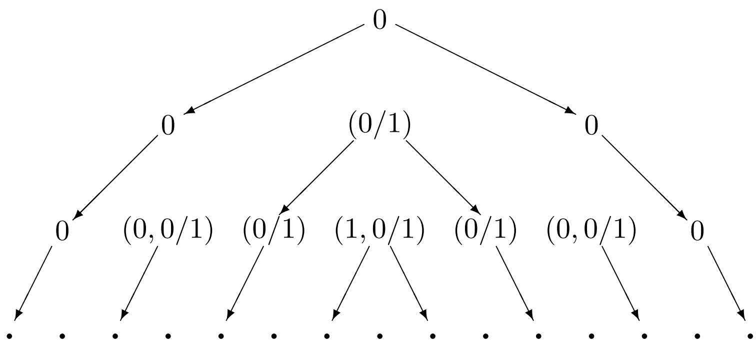

FIGURE 6

branches do not cross in G if we display the vertices in every row n in the order they appear in ω_n. It is straightforward to check that $\leq$ is a total order on $\mathcal{B}$. For instance the property that $B_1 \leq B_2$ and $B_2 \leq B_1$ implies $B_1 = B_2$ is satisfied since we only consider paths which cannot be extended as branches.

The order $\leq$ on $\mathcal{B}$ is dense: Let $B_1 < B_2$ be defined as before. Then $j_{n+1} - i_{n+1} \geq 2$ since the points corresponding to B_1 and B_2 are of order $\leq n$ and thus $P_{n+1,i_{n+1}} \not\sim_{n+1} P_{n+1,j_{n+1}}$. Hence any branch B starting at vertex $(n+1, i_{n+1}+1)$ has the property $B_1 < B < B_2$.

In the following we will consider Dedekind cuts in $(\mathcal{B}, \leq)$: A cut $(\mathcal{B}_1, \mathcal{B}_2)$ is a partition of $\mathcal{B}$ into two (nonempty) subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $B \in \mathcal{B}_1$, $\bar{B} < B$ implies $\bar{B} \in \mathcal{B}_1$, and $B \in \mathcal{B}_2$, $\bar{B} > B$ implies $\bar{B} \in \mathcal{B}_2$. The cut $(\mathcal{B}_1, \mathcal{B}_2)$ is called rational if either $\mathcal{B}_1$ has a largest element or $\mathcal{B}_2$ has a least element. In the remaining case $(\mathcal{B}_1, \mathcal{B}_2)$ is called irrational.

Every cut $(\mathcal{B}_1, \mathcal{B}_2)$ converges to a uniquely defined element of $\triangle$ in the following sense: For all $n \geq 0$ put

$$l_n = \max\{i \mid \exists\, B \in \mathcal{B}_1 : B \text{ contains } (n,i)\}$$
$$r_n = \min\{j \mid \exists\, B \in \mathcal{B}_2 : B \text{ contains } (n,j)\}$$

Obviously we have $1 \leq l_n \leq r_n \leq k_n$ for all $n \geq 0$.

Lemma 3.2. *For the cut* $(\mathcal{B}_1, \mathcal{B}_2)$ *we have* $\lim_{n\to\infty} P_{n,l_n} = \lim_{n\to\infty} P_{n,r_n}$.

Proof. By construction of l_n and r_n we have either $l_n = r_n$ and thus $P_{n,l_n} = P_{n,r_n}$ or $r_n = l_n + 1$ and thus $P_{n,l_n} \sim_n P_{n,r_n}$. Hence it is sufficient to prove the existence of $\lim_{n\to\infty} P_{n,l_n}$.

We prove now for all $n \geq 0$ that $P_{n+1,l_{n+1}}$ lies in the same subtriangle T_n of $\triangle_n$ as P_{n,l_n}: We suppose $P_{n,l_n} \sim_n P_{n,r_n}$, the other case $P_{n,l_n} = P_{n,r_n}$ is proved similarly. Let $B_1 = (\ldots, (n, l_n), (n+1, i), \ldots)$ be a branch in $\mathcal{B}_1$ such that i is a large as possible. Further, let $B_2 = (\ldots, (n, r_n), (n+1, j), \ldots)$ be a branch in $\mathcal{B}_2$ such that j is a small as possible. Note that $P_{n+1,i} = P_{n,l_n}$, $P_{n+1,j} = P_{n,r_n}$ and $l_{n+1} \geq i$. Evidently, all points $P_{n+1,k}$ with $i < k < j$ are of order $n+1$ and lie in the same subtriangle T_n of $\triangle_n$ as P_{n,l_n} and P_{n,r_n}, and it is clear by construction that $P_{n+1,l_{n+1}}$ is one of the points $P_{n+1,k}$ or coincides with P_{n,l_n}.

Thus we obtain a sequence of subtriangles $(T_n)_{n\geq 0}$ with $T_n \supset T_{n+1}$, $\operatorname{diam}(T_n) = 2^{-n}$, $P_{n,l_n} \in T_n$, and hence $\lim\limits_{n\to\infty} P_{n,l_n}$ exists. □

The limit of the cut $(\mathcal{B}_1, \mathcal{B}_2)$ is defined to be the point $\lim\limits_{n\to\infty} P_{n,l_n} = \lim\limits_{n\to\infty} P_{n,r_n}$ in $\triangle$. As the proof of Lemma 3.2 shows, a rational cut has a dyadic limit point, namely the point corresponding to the largest branch in $\mathcal{B}_1$ or the smallest branch in $\mathcal{B}_2$, respectively. An irrational cut may converge to a dyadic or to a generic point. We call $(\omega_n)_{n\geq 0}$ *complete* if every irrational cut in the set of branches $\mathcal{B}$ associated to $(\omega_n)_{n\geq 0}$ converges to a generic point.

Coming back to Example 3.1 we see that $(\omega_n)_{n\geq 0}$ defined there is not complete: Let $\mathcal{B}_1$ consist of all branches which turn left when following them downwards, $\mathcal{B}_2$ all that turn right. Then obviously this cut is irrational and converges to the dyadic point 1.

Next we prove that completeness is a necessary condition for $(\omega_n)_{n\geq 0}$ to be an element of $\sigma(S(\triangle))$.

Proposition 3.3. *For all $f \in S(\triangle)$ the representation $\sigma(f)$ in $S(\triangle)$ is complete.*

Proof. Put $(\omega_n)_{n\geq 0} = (P_{n1}P_{n2}\dots P_{n,k_n})_{n\geq 0} = (\sigma_n(f))_{n\geq 0}$. Let $B = (n, i_n)_{n\geq 0}$ be a branch in the graph G which is associated to $(\omega_n)_{n\geq 0}$.

We will assign to B an interval $[s_B, t_B] \subseteq [0,1]$: Firstly, as we did in the beginning of the proof of Proposition 2.1, for every $n \geq 0$ we can associate to P_{n,i_n} an interval $[s_n, t_n]$ such that $f([s_n,t_n]) \cap D_n = \{P_{n,i_n}\}$. The definition of the edges in the graph G yields $[s_{n+1}, t_{n+1}] \subseteq [s_n, t_n]$, and so we obtain a nonempty interval $[s_B, t_B] = \bigcap\limits_{n\geq 0} [s_n, t_n]$ such that f is constant on $[s_B, t_B]$ with the dyadic point corresponding to B as the constant value.

We list some properties of this relationship between branches and intervals. The order on the branches is preserved by this construction, i.e. if $B_1 = (n, i_n^{(1)})_{n\geq 0}$, $B_2 = (n, i_n^{(2)})_{n\geq 0}$ are two branches then $B_1 < B_2$ implies $t_{B_1} < s_{B_2}$: $B_1 < B_2$ means that there is n such that $i_n^{(1)} < i_n^{(2)}$ and thus for the intervals $[s_{n,k}, t_{n,k}]$ associated to $P_{n,i_n^{(k)}}$, $k = 1,2$, we have $t_{n,1} < s_{n,2}$. Hence $t_{B_1} = \inf_{n\geq 0} t_{n,1} < \sup_{n\geq 0} s_{n,2} = s_{B_2}$.

Utilizing a similar argument we can show that different branches lead to disjoint intervals. Further, it is evident by the construction that for every $u \in [0,1]$ such that $f(u)$ is a dyadic point there exists a branch B with $u \in [s_B, t_B]$.

To sum up, the family $\{[s_B, t_B] \mid B \in \mathcal{B}\}$ forms a partition of $f^{-1}(\bigcup_{n\geq 0} D_n)$ which inherits the order on the set of all branches $\mathcal{B}$ in the sense explained above.

Now we are in position to prove that every irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}$ converges to a generic point in $\triangle$: The irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ corresponds to an irrational cut in $\{[s_B, t_B] \mid B \in \mathcal{B}\}$. Put $s = \sup_{B\in\mathcal{B}_1} s_B$ and $t = \inf_{B\in\mathcal{B}_2} s_B$. Since the cut is irrational it is irrelevant if we take s_B or t_B when forming the inf and the sup, and moreover we have $s > s_{B_1}$ and $t < t_{B_2}$ for all $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$.

Obviously $s \leq t$. We claim that f is constant in the interval $[s, t]$ and the constant value is a generic point: Suppose there exists $u \in [s, t]$ such that $f(u)$ is a dyadic point. Then there is a branch $\bar{B}$ with $u \in [s_{\bar{B}}, t_{\bar{B}}]$. However, due to the definition of $s = \sup_{B\in\mathcal{B}_1} s_B$ all intervals corresponding to branches of $\mathcal{B}_1$ are strictly below s and thus cannot contain u. The same applies to all branches of $\mathcal{B}_2$ since their intervals lie above t. Hence $\bar{B}$ is not in $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}$ which is a contradiction. So f does not assume a dyadic point as value on the interval $[s, t]$. If f would not be constant on $[s, t]$ then $f([s, t])$ would be a connected subset of $\triangle$ containing at least two points and therefore would also contain a dyadic point.

Finally we show that the cut $(\mathcal{B}_1, \mathcal{B}_2)$ converges to the generic point $f(s)$. Put $l_n = \max\{i \mid \exists B \in \mathcal{B}_1 : B \text{ contains } (n, i)\}$. Thus for every $n \geq 0$ there exists a branch $B_n = (m, i_m^{(n)})_{m\geq m_0^{(n)}} \in \mathcal{B}_1$ such that $(n, l_n) = (n, i_n^{(n)})$ and thus $P_{n,l_n} = P_{n,i_n^{(n)}}$. As a consequence $f(s_{B_n}) = P_{n,l_n}$ where as usual $[s_{B_n}, t_{B_n}]$ is the interval corresponding to B_n.

Since $\mathcal{B}_1$ has no largest element for every $B = (n, i_n)_{n\geq n_0} \in \mathcal{B}_1$ there exists $\bar{B} = (n, j_n)_{n\geq \bar{n}_0} \in \mathcal{B}_1$ with $\bar{B} > B$, i.e. there is $n \in \mathbb{N}$ such that $i_n < j_n \leq l_n = i_n^{(n)}$. This means that for all $B \in \mathcal{B}_1$ there is $n \in \mathbb{N}$ such that $s_B < s_{B_n}$. So we infer $\lim_{n\to\infty} s_{B_n} = s$, and using the continuity of f we obtain

$$\lim_{n\to\infty} P_{n,l_n} = \lim_{n\to\infty} f(s_{B_n}) = f(s)$$

and we are done. □

We have already seen that non-complete elements in $\varprojlim S_n$ exist (see Example 3.1). Proposition 3.3 thus shows that $\sigma : S(\triangle)) \to \varprojlim S_n$ is not surjective.

The next proposition aims at finding f in $S(\triangle)$ such that $\sigma(f)$ approximates a given $(\omega_n)_{n\geq 0}$ best possible.

Proposition 3.4. *For every* $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ *there exists* $f \in S(\triangle)$ *such that* $\mathrm{Red}(\sigma(f)) = \mathrm{Red}((\omega_n)_{n\geq 0})$. *This implies for the ranges* $\mathrm{rg}(\mathrm{Red}) = \mathrm{rg}(\mathrm{Red} \circ \sigma)$.

Moreover, if $(\omega_n)_{n\geq 0}$ *is complete then* $\sigma(f) = (\omega_n)_{n\geq 0}$.

Proof. Let $(\omega_n)_{n\geq 0} = (P_{n1}P_{n2}\dots P_{n,k_n})_{n\geq 0}$ be a fixed element of $\varprojlim S_n$. We will define a sequence of functions $(f_n)_{n\geq 0}$ by induction on n such that f_n is piecewise linear with range in $\triangle^n$ and $\sigma_k(f_n) = \omega_k$ for all $k \leq n$.

We start with $n = 0$, $\omega_0 = P_{01}P_{02}\dots P_{0,k_0}$. Divide $[0,1]$ into $2k_0 - 1$ subintervals of equal length by the points

$$0 = s_{01} < t_{01} < s_{02} < t_{02} < \dots < s_{0,k_0} < t_{0,k_0} = 1.$$

Define $f_0(t) = P_{0i}$ for $t \in [s_{0i}, t_{0i}]$, $1 \leq i \leq k_0$, and f_0 to be the linear connection of P_{0i} and $P_{0,i+1}$ in the interval $[t_{0i}, s_{0,i+1}]$, $1 \leq i < k_0$. Obviously $\sigma_0(f_0) = \omega_0$.

Suppose f_n is already defined: $f_n(t) = P_{ni}$ for $t \in [s_{ni}, t_{ni}]$, $1 \leq i \leq k_n$, and f_n is the linear connection of P_{ni} and $P_{n,i+1}$ in the interval $[t_{ni}, s_{n,i+1}]$, $1 \leq i < k_n$. Thus $\sigma_k(f_n) = \omega_k$ for all $k \leq n$. We explain in detail how to define $f_{n+1}(t)$ for $t \in [s_{n1}, t_{n1}]$ and $t \in [t_{n1}, s_{n2}]$. For all other subintervals at level n it works analogously. In the equality $\gamma_{n+1}(\omega_{n+1}) = \omega_n$ we analyze the action of γ_{n+1} on the individual letters of ω_{n+1}: Figure 7 is part of the graph G we associated to $(\omega_n)_{n\geq 0}$ in the

$$\begin{array}{ccccccccccc}
 & P_{n1} & & & & & & P_{n2} & & \cdots \\
\swarrow & & \searrow & & & & \swarrow & & \searrow & \cdots \\
P_{n+1,1} & \dots & P_{n+1,i_1} & P_{n+1,i_1+1} & \dots & P_{n+1,i_2} & P_{n+1,i_2+1} & \dots & P_{n+1,i_3} & \dots
\end{array}$$

FIGURE 7

beginning of this section and should be interpreted as follows: $P_{n+1,1}$ respectively P_{n+1,i_1} is the first respectively last letter in ω_{n+1} that is projected to P_{n1} by γ_{n+1}; P_{n+1,i_1+1} up to P_{n+1,i_2} are all of order $n+1$ and disappear by applying γ_{n+1}, and so on.

Now we define $f_{n+1}(t)$ for $t \in [s_{n1}, t_{n1}]$ analogously as we did for f_0 in $[0,1]$: divide $[s_{n1}, t_{n1}]$ into $2i_1 - 1$ subintervals of equal length and define f_{n+1} in these subintervals alternately to be constant $P_{n+1,i}$, $1 \leq i \leq i_1$, and to connect $P_{n+1,i}$ with $P_{n+1,i+1}$ linearly, $1 \leq i \leq i_1 - 1$.

Next, the interval $[t_{n1}, s_{n2}]$ is divided into $2(i_2 - i_1) + 1$ subintervals. Here f_{n+1} alternately connects $P_{n+1,i}$ with $P_{n+1,i+1}$ linearly, $i_1 \leq i \leq i_2$, and is constant $P_{n+1,i}$, $i_1 + 1 \leq i \leq i_2$.

In the same manner we proceed for the rest of the intervals and obtain f_{n+1} satisfying our requirements.

We compare f_n with f_{n+1} (see Figure 8). For $1 \leq i \leq k_n$:

$$t \in [s_{ni}, t_{ni}] \; : \; \begin{cases} f_n(t) & \dots \text{ constant } P_{ni} \\ f_{n+1}(t) & \dots \text{ stays in the two subtriangles } T_1 \text{ and} \\ & \quad\;\; T_2 \text{ of } \triangle_n \text{ that intersect in } P_{ni}, \end{cases}$$

and for $1 \leq i \leq k_n - 1$

$$t \in [t_{ni}, s_{n,i+1}] : \begin{cases} f_n(t) & \ldots \text{ connects } P_{ni} \text{ and } P_{n,i+1} \text{ linearly} \\ f_{n+1}(t) & \ldots \text{ stays in the subtriangle } T_2 \text{ of } \triangle_n \text{ to} \\ & \quad\;\; \text{which } P_{ni} \text{ and } P_{n,i+1} \text{ belong.} \end{cases}$$

Summing up we obtain $\|f_n - f_{n+1}\|_\infty \leq 2^{-n}$ where $\|.\|_\infty$ denotes the maximum norm for $t \in [0,1]$. Consequently f_n converges for $n \to \infty$ uniformly to a continuous $f : [0,1] \to \triangle$.

By construction we have $f_m(s_{ni}) = P_{ni}$, $1 \leq i \leq k_n$, for all $m \geq n$ and thus also $f(s_{ni}) = P_{ni}$, $1 \leq i \leq k_n$. This means that $\sigma_n(f)$ contains at least all letters appearing in the word ω_n in the proper order, but it may happen that $\sigma_n(f)$ in between the P_{ni} contains further dyadic points of order $\leq n$ and some of the P_{ni} appear in multiplied form. To illustrate this we consider the interval $[s_{ni}, s_{n,i+1}]$:

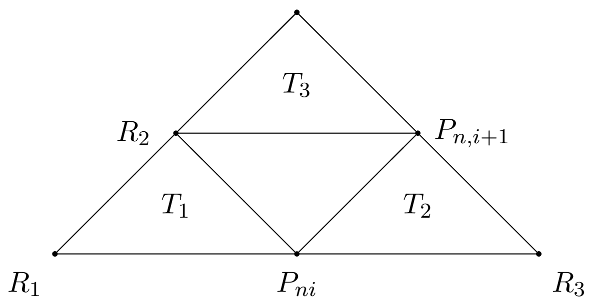

FIGURE 8

f_{n+1} and all f_m with $m \geq n+1$ stay for t in the open interval $(s_{ni}, s_{n,i+1})$ in the interior of the union of the two subtriangles $\text{int}(T_1 \cup T_2)$ of $\triangle_n$ (interior as a subset of $\triangle_n$). This implies that $f = \lim_{m\to\infty} f_m$ stays in the union of the (closed) subtriangles $T_1 \cup T_2$. Hence $\sigma_n(f \restriction [s_{ni}, s_{n,i+1}]) = P_{ni}Q_1Q_2 \ldots Q_sP_{n,i+1}$, $s \geq 0$, where $Q_i \in \{R_1, R_2, R_3, P_{ni}, P_{n,i+1}\}$. However, since $f([s_{ni}, s_{n,i+1}]) \cap (T_3 \setminus \{R_2, P_{n,i+1}\}) = \emptyset$, the two letters R_2 and $P_{n,i+1}$ can never occur in immediate succession in $P_{ni}Q_1Q_2 \ldots Q_sP_{n,i+1}$. This implies that $\text{Red}_n(\sigma_n(f \restriction [s_{ni}, s_{n,i+1}])) = P_{ni}P_{n,i+1}$ and hence on the whole $\text{Red}_n(\sigma_n(f)) = \omega_n$.

Of course, other configurations for P_{ni} and $P_{n,i+1}$ as displayed in Figure 8 are possible. However, as can be checked easily the consequences concerning the respective subtriangles T_1, T_2 and T_3 are always the same.

The first part of the proposition is proved. Now we have to show that $\sigma_n(f) = \omega_n$ for all $n \geq 0$ if $(\omega_n)_{n\geq 0}$ is complete.

We have two sets of branches: The set $\mathcal{B}_f$ corresponding to $\sigma(f)$ and $\mathcal{B}_\omega$ corresponding to $(\omega_n)_{n\geq 0}$. As pointed out above the vertices of the graph G_ω associated to $(\omega_n)_{n\geq 0}$ form a subset of the vertices of the graph G_f associated to $\sigma(f)$. In order to distinguish between these two graphs we use the following notation: Let $\sigma_n(f) = (Q_{n1} \dots Q_{n,\bar{k}_n})$, $n \geq 0$, and $V_f = \{(n,j)^{(f)} \mid n \geq 0, 1 \leq j \leq \bar{k}_n\}$ the vertices in G_f.

Next it will be outlined that in a canonical way to every branch $B = (n, i_n)_{n \geq n_0}$ in $\mathcal{B}_\omega$ a branch in $\mathcal{B}_f$ is associated. Two cases may occur:

(1) The interval $[s,t]$ corresponding to B is a singleton. Recall that when constructing f_n we assigned to every P_{ni} an interval $[s_{ni}, t_{ni}]$ on which f_n has constant value P_{ni}. So $[s,t] = \bigcap_{n \geq n_0} [s_{n,i_n}, t_{n,i_n}]$. The property $s = t$ is equivalent to the feature that in G_ω for an infinite number of n the vertex (n, i_n) has more than on successor: if there is more than one successor of (n, i_n) then $[s_{n+1,i_{n+1}}, t_{n+1,i_{n+1}}]$ has length less than 1/3 of $[s_{n,i_n}, t_{n,i_n}]$. Let P be the point corresponding to the branch B then in this case $f(s) = P$ and in every neighborhood of s, f has infinitely many different dyadic points as values. Anyway, turning to the graph G_f we see that there is a unique branch $\bar{B} = (n, j_n)^{(f)}_{n \geq n_0}$ in $\mathcal{B}_f$ such that Q_{n,j_n} corresponds to the interval $[u_{n,j_n}, v_{n,j_n}]$ (in the sense utilized in the proof of Proposition 2.1) with $s \in [s_{n,j_n}, t_{n,j_n}]$ for all $n \geq n_0$.

(2) The interval $[s,t]$ corresponding to B satisfies $s < t$. This means that there exists an index n_1 such that for all $n \geq n_1$ the interval $[s_{n,i_n}, t_{n,i_n}] = [s,t]$. In this case f_n has constant value P on $[s,t]$ and hence f, as well. Again, there exists a unique branch $\bar{B} = (n, j_n)^{(f)}_{n \geq n_0}$ in $\mathcal{B}_f$ such that Q_{n,j_n} corresponds to the interval $[u_{n,j_n}, v_{n,j_n}]$ with $[s,t] \subseteq [s_{n,j_n}, t_{n,j_n}]$.

In the following we will identify B with the respective $\bar{B}$ from (1) or (2) and thus we may consider $\mathcal{B}_\omega$ as a subset of $\mathcal{B}_f$.

We have already proved in Proposition 3.3 that $\mathcal{B}_f$ is complete. Now we show that $\mathcal{B}_\omega$ is dense in $\mathcal{B}_f$, i.e. for all $B_1, B_2 \in \mathcal{B}_f$ with $B_1 < B_2$ there exists $B \in \mathcal{B}_\omega$ such that $B_1 < B < B_2$: First of all, it is sufficient to prove this for $B_1, B_2 \in \mathcal{B}_f \setminus \mathcal{B}_\omega$:

– if $B_1, B_2 \in \mathcal{B}_\omega$ then there exists an according B since $\mathcal{B}_\omega$ is dense,
– if $B_1 \in \mathcal{B}_\omega$, $B_2 \in \mathcal{B}_f \setminus \mathcal{B}_\omega$, then, since $\mathcal{B}_f$ is dense, there exists $B_3 \in \mathcal{B}_f$ with $B_1 < B_3 < B_2$; if $B_3 \in \mathcal{B}_\omega$ we are done and if $B_3 \in \mathcal{B}_f \setminus \mathcal{B}_\omega$ then the problem is reduced to $B_3 < B_2$ we will deal with.

Let B_i correspond to the interval $[u_i, v_i]$, $f(u_i) = Q_i$, $i = 1, 2$. As $B_1 < B_2$ we have $v_1 < u_2$. Since $\mathcal{B}_f$ is dense there exist $B_3 \in \mathcal{B}_f$ with $B_1 < B_3 < B_2$ and since f cannot be constant on $[v_1, u_2]$ we can choose B_3 such that the point Q_3 corresponding to B_3 satisfies $Q_1 \neq Q_3 \neq Q_2$. Consequently there is $u_3 \in (v_1, u_2)$ with $f(u_3) = Q_3$. We fix some

$k \geq 0$ such that the distance $d(Q_3, Q_i)$ between Q_3 and Q_i is larger than 2^{-k+2}, $i = 1, 2$. Since $(f_m)_{m\geq 0}$ converges uniformly to f we have $\|f - f_m\|_\infty < 2^{-k}$ for all $m \geq m_k$ with appropriate m_k. So for $m \geq m_k$ we have

$$d(Q_1, f_m(v_1)) < 2^{-k}, \quad d(Q_3, f_m(u_3)) < 2^{-k}.$$

Hence $f_m(t)$ must pass from the 2^{-k}-neighborhood of Q_1 for $t = v_1$ to the 2^{-k}-neighborhood of Q_3 for $t = u_3$ and since f_m is alternately constant/linear f_m assumes a dyadic point P (of order $\leq m$) as constant value for some interval between (u_1, u_3). Since $\sigma_m(f_m) = \omega_m$ there is a branch $B \in \mathcal{B}_\omega$ corresponding to P and this branch satisfies $B_1 < B < B_3 < B_2$.

Finally we show $\sigma(f) \neq (\omega_n)_{n\geq 0}$ (which is equivalent to $\mathcal{B}_f \setminus \mathcal{B}_\omega \neq \emptyset$) implies that $(\omega_n)_{n\geq 0}$ is not complete: Let $\bar{B} = (n, i_n)^{(f)}_{n\geq n_0} \in \mathcal{B}_f \setminus \mathcal{B}_\omega$ such that starting from some level n_1 all vertices $(n, i_n)^{(f)}$ in $\bar{B}$ have smallest possible i_n. For instance this is possible if $(n_1 - 1, i_{n_1-1})^{(f)}$ is a vertex not in G_ω. We consider the following cut in $\mathcal{B}_\omega$:

$$\mathcal{B}_1 = \{B \in \mathcal{B}_\omega \mid B < \bar{B}\}, \quad \mathcal{B}_2 = \{B \in \mathcal{B}_\omega \mid B > \bar{B}\}.$$

First we show that $(\mathcal{B}_1, \mathcal{B}_2)$ is irrational: for $B_1 \in \mathcal{B}_1$ we have $B_1 < \bar{B}$ and since $\mathcal{B}_\omega$ is dense in $\mathcal{B}_f$ there is $B \in \mathcal{B}_\omega$ such that $B_1 < B < \bar{B}$ showing that $\mathcal{B}_1$ has no largest element. Analogously one learns that $\mathcal{B}_2$ has no least element.

Now we prove that $(\mathcal{B}_1, \mathcal{B}_2)$ converges to the point $\bar{Q}$ corresponding to $\bar{B}$. Let $(\mathcal{B}_1^f, \mathcal{B}_2^f)$ be the cut in $\mathcal{B}_f$ with smallest element $\bar{B}$ in $\mathcal{B}_2^f$ and

$$\begin{aligned} l_n^f &= \max\{j \mid \exists B_1 \in \mathcal{B}_1^f : B_1 \text{ contains } (n,j)^f\}, \\ l_n &= \max\{j \mid \exists B_1 \in \mathcal{B}_1 : B_1 \text{ contains } (n,j)^f\}. \end{aligned}$$

Due to our choice of $\bar{B}$ we have for all $n \geq n_1$ that $l_n^f = i_n - 1$ and $Q_{n,l_n^f} \sim_n \bar{Q}$. Further let $B_n^f \in \mathcal{B}_f$ the largest branch containing $(n, l_n^f)^{(f)}$ (starting from Q_{n,l_n^f} taking always the rightmost vertex as successor). As a consequence all branches B with $B_n^f < B < \bar{B}$ correspond to a dyadic point in the subtriangle T_n of $\triangle_n$ that contains $\bar{Q}$ and Q_{n,l_n^f}. Since $\mathcal{B}_\omega$ is dense in $\mathcal{B}_f$ there exists $B_n \in \mathcal{B}_\omega$ such that $B_n^f < B_n < \bar{B}$. Hence the points P_n corresponding to B_n must lie in the subtriangle T_n and if P_n is of order r_n then also Q_{k,l_k} lies in T_n for all $k \geq r_n$. So we have proved

$$\lim_{n\to\infty} Q_{n,l_n^f} = \lim_{k\to\infty} Q_{k,l_k} = \bar{Q}.$$

Summing up this means that the irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}_\omega$ converges to the dyadic point $\bar{Q}$ and hence $(\omega_n)_{n\geq 0}$ is not complete. $\square$

We now have precise information on the range of σ. In order to get an idea what the sub-semigroup $\sigma(S(\triangle)) \cong S(\triangle)/\ker(\sigma)$ of $\varprojlim S_n$ describes we have to investigate the kernel of σ.

A first observation in this direction is that $\ker(\sigma)$ is a sub-relation of the homotopy relation of elements $f, g \in S(\triangle)$: $\sigma(f) = \sigma(g)$ implies

$$\varphi([f]) = \mathrm{Red}(\sigma(f)) = \mathrm{Red}(\sigma(g)) = \varphi([g]),$$

and since φ is injective we obtain $[f] = [g]$.

It is palpable that $\ker(\sigma)$ will have a connection with the re-parameterization of loops. Therefore we define for two loops $f, g \in S(\triangle)$: $f \approx g$ if and only if there exist functions $\alpha, \beta : [0,1] \to [0,1]$ which are monotonously increasing and surjective (and hence continuous) such that $f \circ \alpha = g \circ \beta$.

Proposition 3.5. *If $f \approx g$ then $\sigma(f) = \sigma(g)$.*

Proof. First we show that $\sigma_n(f) = \sigma_n(f \circ \alpha)$ for all $n \geq 0$ where $f \circ \alpha = g \circ \beta$ with properties as defined above. We recall that $\sigma_n(f)$ is the sequence of points in D_n that arises when we raster the separated set $f^{-1}(D_n)$ with appropriate small intervals and list the corresponding points. For a letter P appearing in $\sigma_n(f)$ let again $[s,t]$ be the maximal interval such that $f(s) = f(t) = P$ and $f([s,t]) \cap D_n = \{P\}$. Since α is surjective P appears also in $\sigma_n(f \circ \alpha)$ and the monotonicity of α preserves the order of points in $\sigma_n(f)$, in particular $[\varphi^{-1}(s), \varphi^{-1}(t)]$ is the interval corresponding to letter P with respect to the loop $f \circ \alpha$.

The rest is obvious: $\sigma_n(f) = \sigma_n(f \circ \alpha) = \sigma_n(g \circ \beta) = \sigma_n(g)$. □

The converse of Proposition 3.5 is established in the following.

Proposition 3.6. *If $\sigma(f) = \sigma(g)$ then $f \approx g$.*

Proof. For $n \geq 0$ let $\omega_n = \sigma_n(f) = \sigma_n(g) = P_{n1}P_{n2}\dots P_{n,k_n}$. As usual we assign to $(\omega_n)_{n\geq 0}$ the graph G with vertices (n,i), $n \geq 0$, $1 \leq i \leq k_n$, and an edge connecting (n,i) with $(n+1,j)$ if the letter $P_{n+1,j}$ in ω_{n+1} is projected to P_{ni} when performing $\gamma_{n+1}(\omega_{n+1}) = \omega_n$.

In the first step we will introduce an appropriate parametrization $f_n : [0,1] \to \triangle$ of the piecewise linear loop corresponding to $\sigma_n(f)$ such that the sequence $(f_n(t))_{n\geq 0}$ converges uniformly to $f(t)$ for $t \in [0,1]$.

Let n be fixed. As usual we associate to every (n,i) the maximal interval $[s_{ni}, t_{ni}]$ such that $f(s_{ni}) = f(t_{ni}) = P_{ni}$, $D_n \cap f([s_{ni}, t_{ni}]) = \{P_i\}$ and $0 = s_{n1} \leq t_{n1} < s_{n2} \leq t_{n2} < \dots < s_{n,k_n} \leq t_{n,k_n} = 1$. We parameterize the piecewise linear loop corresponding to $\sigma_n(f)$ by f_n such that f_n is constant P_{ni} in the interval $[s_{ni}, t_{ni}]$, $1 \leq i \leq k_n$, and connects P_{ni} and $P_{n,i+1}$ linearly in the interval $[t_{ni}, s_{n,i+1}]$, $1 \leq i \leq k_n - 1$. For $t \in [s_{ni}, t_{ni}]$ the loop $f(t)$ is contained in one of the (at most) two subtriangles of $\triangle_n$ to which P_{ni} belongs, and for $t \in [t_{ni}, s_{n,i+1}]$ the loop $f(t)$ is contained in the subtriangle T_i of $\triangle_n$ to which P_{ni} and $P_{n,i+1}$ belong. Thus we infer that the maximum norm $\|f_n - f\|_\infty \leq \mathrm{diam}(T_i) = 2^{-n}$ and (f_n) converges uniformly to f.

What was done for f can be realized mutatis mutandis with g where the piecewise linear approximations will be denoted by g_n, and $[u_{ni}, v_{ni}]$

is the generic notation for the interval corresponding to the vertex (n, i) with respect to g.

In the following we will need another correlation, namely we associate to the vertex (n, i) also the interval

$$[a_{ni}, b_{ni}] = [(s_{ni} + u_{ni})/2, (t_{ni} + v_{ni})/2].$$

With this concept we now consider $\alpha_n, \beta_n : [0, 1] \to [0, 1]$ such that

$$\alpha_n(a_{ni}) = s_{ni}, \quad \alpha_n(b_{ni}) = t_{ni},$$
$$\beta_n(a_{ni}) = u_{ni}, \quad \beta_n(b_{ni}) = v_{ni},$$

and α_n, β_n are piecewise linear between these points. Evidently, we then have

$$f_n \circ \alpha_n = g_n \circ \beta_n.$$

We recall what was accomplished in Proposition 3.4: Starting from an arbitrary $(\omega_n)_{n\geq 0} \in \varprojlim S_n$ a sequence f_n of loops was constructed converging uniformly to some $f \in S(\triangle)$. Moreover, it was shown that $\sigma(f) = (\omega_n)_{n\geq 0}$ provided $(\omega_n)_{n\geq 0}$ is complete. Now we perform the same starting with $(\omega_n)_{n\geq 0} = \sigma(f) = \sigma(g)$ which is complete by Proposition 3.3. Instead of using subintervals of equal length as in the proof of Proposition 3.4, we here employ the given family $[a_{ni}, b_{ni}]$, $n \geq 0$, $1 \leq i \leq k_n$. However, this difference does not influence the validity of the rest of the proof at all. What we obtain is the sequence $h_n = f_n \circ \alpha_n = g_n \circ \beta_n$ converging uniformly to some $h \in S(\triangle)$ with $\sigma(h) = \sigma(f) = \sigma(g)$. Moreover, one can show with the methods utilized in the proof of Proposition 3.4 that the interval $[x_{ni}, y_{ni}]$ associated to the vertex (n, i) with respect to h in the usual way, i.e. $[x_{ni}, y_{ni}]$ is the maximal interval with the properties $h(x_{ni}) = h(y_{ni}) = P_{ni}$, $h([x_{ni}, y_{ni}]) \cap D_n = \{P_{in}\}$, must coincide with $[a_{ni}, b_{ni}]$: $[a_{ni}, b_{ni}] \subseteq [x_{ni}, y_{ni}]$ is obvious and the assumption $x_{ni} < a_{ni}$ or $y_{ni} > b_{ni}$ leads immediately to a contradiction to the completeness of $(\omega_n)_{n\geq 0} = \sigma(h)$.

Let again $\mathcal{B}$ denote the set of branches in G. To every branch $B = (n, i_n)_{n\geq n_0}$ we assign the interval $[s_B, t_B] = \bigcap_{n\geq n_0} [s_{n,i_n}, t_{n,i_n}]$, and the intervals $[u_B, v_B]$, $[a_B, b_B]$ accordingly, depending on which function f, g or h is considered at the moment.

In the next step we will elaborate that the sequences $(\alpha_n(x))_{n\geq 0}$ and $(\beta_n(x))_{n\geq 0}$ converges pointwise for a good deal of x. First we consider $x \in [0, 2]$ such that there exists $B = (n, i_n)_{n\geq n_0} \in \mathcal{B}$ with $x \in [a_B, b_B] = \bigcap_{n\geq n_0} [a_{n,i_n}, b_{n,i_n}]$. (In the following we will refer to this case by (I).) This implies $x \in [a_{n,i_n}, b_{n,i_n}] = [(s_{n,i_n} + u_{n,i_n})/2, (t_{n,i_n} + v_{n,i_n})/2]$ for all $n \geq n_0$. Recall that

$$\lim_{n\to\infty} s_{n,i_n} = s_B, \ \lim_{n\to\infty} t_{n,i_n} = t_B, \ \lim_{n\to\infty} u_{n,i_n} = u_B, \ \lim_{n\to\infty} v_{n,i_n} = v_B,$$

and that

$$\alpha_n(x) = s_{n,i_n} + \frac{t_{n,i_n} - s_{n,i_n}}{b_{n,i_n} - a_{n,i_n}}(x - a_{n,i_n})$$

if $b_{n,i_n} > a_{n,i_n}$, and $\alpha_n(x) = s_{n,i_n} = t_{n,i_n}$ otherwise. In general we have $\alpha_n(x) \in [s_{n,i_n}, t_{n,i_n}]$. Therefore, if $t_B = s_B$ we infer $\lim_{n\to\infty} \alpha_n(x) = s_B$, and if $t_B > s_B$ we obtain $\lim_{n\to\infty} \alpha_n(x) = s_B + \frac{t_B - s_B}{b_B - a_B}(x - a_B)$. In any case the limit exists and we define $\alpha(x) = \lim_{n\to\infty} \alpha_n(x)$. Analogously we can proceed with $\beta_n(x)$ and define $\beta(x) = \lim_{n\to\infty} \beta_n(x)$.

Now we deal with the case that $x \notin [a_B, b_B]$ for all $B \in \mathcal{B}$ (case (II)). Then x defines a cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}$ by putting $\mathcal{B}_1 = \{B \in \mathcal{B} \mid x > b_B\}$ and $\mathcal{B}_2 = \{B \in \mathcal{B} \mid x < a_B\}$. We recapitulate what was shown in the proof of Proposition 3.3: The cut $(\mathcal{B}_1, \mathcal{B}_2)$ is irrational and if we define $a = \sup_{B\in\mathcal{B}_1} a_B = \sup_{B\in\mathcal{B}_1} b_B$ and $b = \inf_{B\in\mathcal{B}_2} a_B = \inf_{B\in\mathcal{B}_2} b_B$ then $x \in [a, b]$ and h is constant in the interval $[a, b]$ with a generic point Q which is the limit of the cut $(\mathcal{B}_1, \mathcal{B}_2)$ as constant value. With s, t and u, v defined accordingly, $a = (s+u)/2$, $b = (t+v)/2$, we further obtain $f([s, t]) = g([u, v]) = \{Q\}$. For $\tilde{x} \in [a, b]$ we define

$$\alpha(\tilde{x}) = \begin{cases} s = t & \text{if } a = b, \\ s + \frac{t-s}{b-a}(\tilde{x} - a) & \text{otherwise,} \end{cases}$$

$$\beta(\tilde{x}) = \begin{cases} u = v & \text{if } a = b, \\ u + \frac{v-u}{b-a}(\tilde{x} - a) & \text{otherwise.} \end{cases}$$

In order to justify this definition some warning is indicated here. One can easily construct an example of a loop f such that $\lim_{n\to\infty} \alpha_n(x)$ does not exist for some x. However, one always has $s \le \liminf \alpha_n(x) \le \limsup \alpha_n(x) \le t$ and since f is constant in $[s, t]$ this causes no problem.

Now we have to show that α and β comply with the intention they were constructed with.

$(f \circ \alpha)(x) = (g \circ \beta)(x)$ for all $x \in [0, 1]$: In case (I) $x \in [a_B, b_B]$ for some branch $B \in \mathcal{B}$ and we have

$$\|f(\alpha(x)) - f_n(\alpha_n(x))\| \le \|f(\alpha(x)) - f(\alpha_n(x))\| + \|f(\alpha_n(x)) - f_n(\alpha_n(x))\|.$$

The first part on the right hand side can be made arbitrarily small since f is continuous and $\alpha_n(x)$ converges to $\alpha(x)$ and the second part does so since f_n converges to f uniformly. The same applies to g and β. So we arrive at

$$f(\alpha(x)) = \lim_{n\to\infty} f_n(\alpha_n(x)) = \lim_{n\to\infty} g_n(\beta_n(x)) = g(\beta(x)).$$

In case (II) $x \notin [a_B, b_B]$ for any branch B we have with notations as before $\alpha(x) \in [s, t]$ and $\beta(x) \in [u, v]$ and hence $f(\alpha(x)) = Q = g(\beta(x))$. Just as a further remark we mention here that $h = f \circ \alpha$.

α and β are monotonously increasing functions: Let $x_1 < x_2$. Depending on whether case (I) or (II) apply to x_1 and x_2 four cases occur. We only work out the mixed case in detail, the other can be treated similarly. So let $x_1 \in [a_B, b_B]$ for some branch B and let $x_2 \in [a, b]$ where $[a, b]$ is the interval corresponding to an irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ with respect to h. The relation $x_1 < x_2$ just means that $B \in \mathcal{B}_1$ and so we deduce

$$\alpha(x_1) \leq t_B < \sup_{B_1 \in \mathcal{B}_1} t_{B_1} = s = \alpha(a) \leq \alpha(x_2).$$

The proof for the monotonicity of β works analogously.

α and β are surjective and thus continuous: From case (I) we see that

$$\mathrm{rg}(\alpha) \supseteq \bigcup_{B \in \mathcal{B}} [s_B, t_B] = f^{-1}(\bigcup_{n \geq 0} D_n) = D_f,$$

and for all components $[s, t]$ of the complement of D_f which correspond to an irrational cut $(\mathcal{B}_1, \mathcal{B}_2)$ in $\mathcal{B}$, in (II) we tailored α such that the interval $[a, b]$ corresponding to $(\mathcal{B}_1, \mathcal{B}_2)$ with respect to h satisfies $\alpha([a, b]) = [s, t]$. Hence α is surjective, and with the respective proof for g, β is surjective, as well. □

We formulate the last results in a joint statement.

Theorem 3.7. (i) *For f and g in $S(\triangle)$ we have $\sigma(f) = \sigma(g)$ if and only if f and g have a common re-parametrization, i.e. there exist $\alpha, \beta : [0, 1] \to [0, 1]$ monotonously increasing and surjective such that $f \circ \alpha = g \circ \beta$.*

(ii) *An element $(\omega_n)_{n \geq 0}$ in $\varprojlim S_n$ is a representation for a loop f in $S(\triangle)$, i.e. $(\omega_n)_{n \geq 0} = \sigma(f)$, if and only if $(\omega_n)_{n \geq 0}$ is complete.*

In other words, the complete elements of $\varprojlim S_n$ represent the elements of $S(\triangle)$ modulo re-parametrization.

3.2. **A description of the elements in the fundamental group $\boldsymbol{\pi(\triangle)}$.** We have proved in Theorem 2.12 that $\varphi([f]) = \mathrm{Red}(\sigma(f))$ for all continuous loops f in $\triangle$. Since φ is an injection the fundamental group $\pi(\triangle)$ can be considered as a subgroup of $\varprojlim G_n$ and in this subsection we will characterize the elements of this subgroup.

In the following denote by γ_{nk} the projection $\gamma_{k+1} \circ \gamma_{k+2} \circ \ldots \circ \gamma_n : S_n \to S_k$, and analogously δ_{nk} denotes the composition of the corresponding δ_i's.

Before we state the main result we need some preliminaries. Let $P_1 P_2 \ldots P_m$, $Q_1 Q_2 \ldots Q_k$ be two words over some alphabet. We define $P_1 P_2 \ldots P_m \preceq Q_1 Q_2 \ldots Q_k$ if and only if there exists $\alpha : \{1, \ldots, m\} \to \{1, \ldots k\}$, α injective and order preserving, such that $P_i = Q_{\alpha(i)}$ for all $i \in \{1, \ldots, m\}$. This means that the first word is kind of a subword of the second in an other sense than we have used before (cf. elementary moves (2.3)).

Lemma 3.8. *Let $\omega_n, \bar{\omega}_n \in S_n$. Then*

(i) $\mathrm{Red}_n(\omega_n) \preceq \omega_n$,

(ii) $\omega_n \preceq \bar{\omega}_n$ *implies* $\gamma_{nk}(\omega_n) \preceq \gamma_{nk}(\bar{\omega}_n)$ *for all* $k \leq n$,
(iii) *if* $(\omega_k)_{k\geq 0} \in \varprojlim G_n$ *then* $\gamma_{nk}(\omega_n) \preceq \gamma_{n+1,k}(\omega_{n+1})$ *for all* $k \leq n$.

Proof. (i) is evident since Red_n eliminates just some letters from the word.

(ii) γ_{nk} filters out the points of order $\leq k$ from the words over D_n. So, if α testifies $\omega_n \preceq \bar{\omega}_n$ then α restricted to those indices with points of order $\leq k$ testifies the claimed relation.

(iii) We have $\gamma_{nk}(\omega_n) = \gamma_{nk}(\delta_{n+1}(\omega_{n+1})) = \gamma_{nk}(\mathrm{Red}_n(\gamma_{n+1}(\omega_{n+1}))) \preceq \gamma_{nk}(\gamma_{n+1}(\omega_{n+1})) = \gamma_{n+1,k}(\omega_{n+1})$, where we used (i) and (ii) as $\preceq$ came in. □

Theorem 3.9. *An element* $(\omega_n)_{n\geq 0}$ *of* $\varprojlim G_n$ *is in* $\varphi(\pi(\triangle))$ *if and only if for all* $k \geq 0$ *the sequence* $(\gamma_{nk}(\omega_n))_{n\geq k}$ *stabilizes.*

Proof. We fix the element $(\omega_n)_{n\geq 0}$ in $\varprojlim G_n$. First we prove the necessity of the condition. Let $(\bar{\omega}_n)_{n\geq 0}$ be in $\varprojlim S_n$ such that $\mathrm{Red}((\bar{\omega}_n)_{n\geq 0}) = (\omega_n)_{n\geq 0}$. Then for all $k \geq 0$ and all $n \geq k$ we have $\bar{\omega}_k = \gamma_{nk}(\bar{\omega}_n) \succeq \gamma_{nk}(\mathrm{Red}_n(\bar{\omega}_n)) = \gamma_{nk}(\omega_n)$ where we used (i) and (ii) of Lemma 3.8. By (iii) of Lemma 3.8 we get

$$\gamma_{nk}(\omega_n) \preceq \gamma_{n+1,k}(\omega_{n+1}) \preceq \ldots \preceq \bar{\omega}_k,$$

hence $(\gamma_{nk}(\omega_n))_{n\geq k}$ stabilizes.

Now we prove the sufficiency of the condition. Put $\bar{\omega}_k = \gamma_{nk}(\omega_n)$ which holds true for $n \geq n_k$, $k \geq 0$. We show that $(\bar{\omega}_k)_{k\geq 0}$ is in $\varprojlim S_n$ and $\mathrm{Red}(\bar{\omega}_k)_{k\geq 0} = (\omega_n)_{n\geq 0}$. For $k \geq 1$ and $n \geq \max\{n_k, n_{k-1}\}$ we obtain $\gamma_k(\bar{\omega}_k) = \gamma_k(\gamma_{nk}(\omega_n)) = \gamma_{n,k-1}(\omega_n) = \bar{\omega}_{k-1}$. This shows the first assertion.

Before we come to the second part we prove $\delta_{nk} = \mathrm{Red}_k \circ \gamma_{nk}$: In Lemma 2.6 we showed $\mathrm{Red}_{i-1} \circ \gamma_i \circ \mathrm{Red}_i = \mathrm{Red}_{i-1} \circ \gamma_i$ for all $i \geq 1$. Obeying $\delta_i = \mathrm{Red}_{i-1} \circ \gamma_i$, iterated use of this identity leads immediately to the claimed relation.

Finally, for $k \geq 0$ and $n \geq n_k$ we infer $\mathrm{Red}_k(\bar{\omega}_k) = \mathrm{Red}_k(\gamma_{nk}(\omega_n)) = \delta_{nk}(\omega_n) = \omega_k$. Due to the fact $\mathrm{rg}(\mathrm{Red}) = \mathrm{rg}(\mathrm{Red} \circ \sigma)$ from Proposition 3.4 we can find $f \in S(\triangle)$ such that $\mathrm{Red}(\sigma(f)) = \mathrm{Red}(\bar{\omega}_k)_{k\geq 0} = (\omega_n)_{n\geq 0}$ and thus

$$(\omega_n)_{n\geq 0} = \mathrm{Red}(\sigma(f)) = \varphi([f]).$$

This completes the proof. □

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(S. A.) Department of Mathematics, Faculty of Science Niigata University, Ikarashi 2-8050, Niigata 950-2181, JAPAN
E-mail address: akiyama@math.sc.niigata-u.ac.jp

(G. D. & R. W.) Institute for Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstr. 8-10/104, 1040 Vienna, AUSTRIA
E-mail address: g.dorfer@tuwien.ac.at
E-mail address: reinhard.winkler@tuwien.ac.at

(J. M. T.) Chair of Mathematics and Statistics, Department Mathematics and Information Technology, University of Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA
E-mail address: joerg.thuswaldner@unileoben.ac.at