Valuations and Busemann–Petty Type Problems

Franz E. Schuster

Institute of Discrete Mathematics and Geometry
Vienna University of Technology
Wiedner Hauptstraße 8–10/104, 1040 Vienna, Austria

Abstract

Projection and intersection bodies define continuous and $\mathrm{GL}(n)$ contravariant valuations. They played a critical role in the solution of the Shephard problem for projections of convex bodies and its dual version for sections, the Busemann–Petty problem. We consider the question whether $\Phi K \subseteq \Phi L$ implies $V(K) \leq V(L)$, where Φ is a homogeneous, continuous operator on convex or star bodies which is an $\mathrm{SO}(n)$ equivariant valuation. Important previous results for projection and intersection bodies are extended to a large class of valuations.

Key words: valuations; Busemann–Petty problem; Shephard problem

1 Introduction

A compact convex set with non-empty interior in n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 3$, is called a convex body. For $u \in S^{n-1}$, let $u^\perp$ denote the $(n-1)$-dimensional subspace orthogonal to u. We use $V_k(M)$ to denote the k-dimensional volume of a k-dimensional compact convex set M. Instead of V_n we usually write V. In [8] Busemann and Petty posed the following problem:

Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Is there the implication

$$V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp), \quad \forall\, u \in S^{n-1} \quad \Longrightarrow \quad V(K) \leq V(L)?$$

A long list of authors contributed to the solution of this famous problem over a period of 40 years, see [3,4,11,12,14–16,25,32,33,36,42,52,68]. The question has a negative answer for $n \geq 5$ and an affirmative answer for $n = 3, 4$. For a detailed account of the interesting history of the Busemann–Petty problem, see the books by Gardner [13, Chapter 8] and Koldobsky [35, Chapter 5].

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In recent years a remarkable duality between results concerning projections and those concerning sections through a fixed point was discovered. Let $K|u^\perp$ denote the orthogonal projection of a convex body K onto $u^\perp$. The dual question to the Busemann–Petty problem was asked by Shephard [64]:

Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Is there the implication

$$V_{n-1}(K|u^\perp) \leq V_{n-1}(L|u^\perp), \quad \forall\, u \in S^{n-1} \quad \Longrightarrow \quad V(K) \leq V(L)?$$

In spite of the link between the Busemann–Petty and the Shephard problem provided by polar duality, their solutions and respective histories are quite different. The Shephard problem was solved, independently, by Petty [53] and Schneider [55] one year after its formulation. The question has a negative answer for every $n \geq 3$.

The crucial idea in the solution of both problems was to define new convex bodies by the given tomographic information and rephrase the questions in terms of geometric properties of these new bodies. In the case of the Shephard problem the convex body determined by the $(n-1)$-dimensional volume of the projections is the projection body ΠK of K, introduced already by Minkowski. Projection bodies of convex bodies are special origin-symmetric convex bodies called zonoids. For their numerous applications in different areas, see [5,6,13,20,37,65] and the surveys [19,61]. In order to define them, let $h(K,u) = \max\{u \cdot x : x \in K\}$, $u \in S^{n-1}$, denote the support function of the convex body K. The *projection body* ΠK of K is defined by

$$h(\Pi K, u) = V_{n-1}(K|u^\perp), \qquad u \in S^{n-1}. \tag{1.1}$$

From (1.1) and the simple fact that convex bodies K and L satisfy $K \subseteq L$ if and only if $h(K,\cdot) \leq h(L,\cdot)$, we see that the Shephard problem can be reformulated in the following way:

Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Is there the implication

$$\Pi K \subseteq \Pi L \quad \Longrightarrow \quad V(K) \leq V(L)? \tag{1.2}$$

Petty and Schneider both showed that the answer to this problem is affirmative if the body L belongs to the class of projection bodies (zonoids). In addition, Schneider showed that if K is sufficiently smooth and has positive curvature but is not a zonoid, then there is an L such that (1.2) does not hold.

While for the Shephard problem the notion of projection bodies was already available, in the case of the Busemann–Petty problem the new notion of intersection bodies had to be introduced. This was done by Lutwak [42] whose work is considered the starting point of the solution of the Busemann–Petty problem in all dimensions. It turned out that sets which are starshaped with respect to the origin form a more appropriate domain for the intersection body operator than convex bodies. Let $\rho(L,u) = \max\{\lambda \geq 0 : \lambda u \in L\}, u \in S^{n-1}$, denote the radial function of a compact set L in $\mathbb{R}^n$ which is starshaped with respect to the origin. If $\rho(L,\cdot)$ is continuous, we call L a star body. The *intersection body* IL of a star body L is defined by

$$\rho(IL,u) = V_{n-1}(L \cap u^{\perp}), \qquad u \in S^{n-1}. \tag{1.3}$$

Note that the intersection body of a convex body need not be convex, but, by Busemann's theorem [7], the intersection body of an origin-symmetric convex body is always convex. Although the notion of intersection bodies is relatively new, the topic has been intensively studied in recent years, see [17,20,27,34,40,51] and the books [13,35,65]. From (1.3) and the fact that star bodies K and L satisfy $K \subseteq L$ if and only if $\rho(K,\cdot) \leq \rho(L,\cdot)$, we see that the Busemann–Petty problem can be rephrased in the following way:

Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Is there the implication

$$IK \subseteq IL \quad \Longrightarrow \quad V(K) \leq V(L)? \tag{1.4}$$

Lutwak established in [42] duals of the results by Petty and Schneider. He showed that the Busemann–Petty problem has an affirmative answer if now the body K is restricted to the class of intersection bodies. In addition, Lutwak proved that if L is a sufficiently smooth origin-symmetric star body with positive radial function which is not an intersection body, then there exists an origin-symmetric star body K such that $IK \subseteq IL$ but $V(K) > V(L)$.

The reformulations (1.2) and (1.4) of the Shephard and the Busemann–Petty problem led not only to their complete solution, but also to several interesting variations of the original questions, where the intersection and projection body operators were replaced by other well-known operators. For example, the centroid body operator was considered by Lutwak in [43]. If $K \in \mathcal{K}^n$ is origin-symmetric, then the centroid body ΓK of K is the convex body whose boundary consists of the locus of the centroids of halves of K formed when K is cut by hyperplanes through the origin. Lutwak showed that if $\Gamma K \subseteq \Gamma L$ and L is the polar body of a projection body, then $V(K) \leq V(L)$. Further Busemann–Petty type questions have been considered in the context of the L_p Brunn–Minkowski Theory, a relatively new branch of convex geometry, cf. [9,10,44–47]. Here, the L_p extensions of the projection, intersection and centroid body operators were studied, see [20,22,54,66]. A property shared by these operators is that all of them are convex or star body valued valuations.

A function Φ defined on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ (or on the space $\mathcal{S}^n$ of star bodies) and taking values in an abelian semigroup is called a *valuation* if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L, \tag{1.5}$$

whenever $K, L, K \cap L, K \cup L \in \mathcal{K}^n$ (or $\mathcal{S}^n$, respectively).

The theory of real valued valuations is at the center of convex geometry. A systematic study was initiated by Blaschke in the 1930s and continued by Hadwiger culminating in his famous classification of continuous, rigid motion invariant valuations on convex bodies. The surveys [49,50] and the book [31] are an excellent source for the classical theory of valuations. For some of the more recent results, see [1,2,29,30,38,41].

First results on convex body valued valuations were obtained by Schneider [57] in the 1970s, where addition of convex bodies in (1.5) is Minkowski addition defined by $K+L=\{x+y : x \in K, y \in L\}$. In recent years the investigations of convex and star body valued valuations gained momentum through a series of articles by Ludwig [37,39,40], see also [23]. She started systematic studies and established complete classifications of convex and star body valued valuations with respect to L_p Minkowski and L_p radial addition which are compatible with the action of the group $\mathrm{GL}(n)$ (see Section 3 for precise definitions). For example, characterizations of the projection and intersection body operators as well as the moment (centroid) body operator and their respective L_p extensions have been established. Ludwig showed that these operators are the only non-trivial valuations (with respect to certain semigroup structures on $\mathcal{K}^n$ and $\mathcal{S}^n$) that are compatible with the action of the group $\mathrm{GL}(n)$.

In light of Ludwig's results and the variations of the Busemann–Petty problem mentioned above, we propose the following unifying question:

Problem 1 *Let Φ be a continuous convex or star body valued valuation which is compatible with the action of some group of transformations of $\mathbb{R}^n$. Is there the implication*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad V(K) \leq V(L)?$$

All the previously investigated variations of the Busemann–Petty problem are now special cases of this general question. Another motivation for considering Problem 1 comes from geometric tomography. Here, convex or star body valued valuations Φ arise naturally, like the projection and intersection body operators, from data about sections or projections of a body. The basic task is to understand what kind of geometric information about a body K can be retrieved from the knowledge of its image ΦK. Or, more specifically, what kind of information can be retrieved from the inclusion relation $\Phi K \subseteq \Phi L$.

The identity and the reflection in the origin are trivial valuations which yield an affirmative answer to Problem 1 in every dimension. Further examples are the translation invariant and n-homogeneous valuations on convex bodies. A theorem of Hadwiger [24, p. 79] implies that these valuations are of the form $\Phi_n K = V(K)M$, for some fixed convex body M. Obviously, these operators also provide positive solutions to Problem 1.

From Ludwig's results, we know that Problem 1 has already been solved for most $\mathrm{GL}(n)$ compatible valuations. Thus we will weaken the strong assumption of $\mathrm{GL}(n)$ compatibility and consider the Euclidean conditions of $\mathrm{SO}(n)$ equivariance or rigid motion compatibility. Previous investigations show that these new properties lead to a large class of valuations, see [28,57,59,62].

In this article we focus on continuous valuations which are $\mathrm{SO}(n)$ equivariant and $(n-1)$-homogeneous. In Sections 3 and 5 we study translation invariant Minkowski valuations Φ, i.e., convex body valued valuations on $\mathcal{K}^n$, where addition in (1.5) is Minkowski addition. A main representative of this class of valuations is the projection body operator. Thus, the special case of Problem 1 for these valuations is close to the original problem of Shephard. Our first aim is to show that, like the Shephard problem, Problem 1 has an affirmative answer in this class of valuations, if the body L belongs to the image of Φ:

Theorem 1 *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous and translation invariant Minkowski valuation which is $(n-1)$-homogeneous and $\mathrm{SO}(n)$ equivariant. If $K \in \mathcal{K}^n$ and $L \in \Phi\mathcal{K}^n$, then*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad V(K) \leq V(L).$$

Petty [53] and Schneider [55] have shown that implication (1.2) does not hold for all K if the body L is not centrally symmetric. Note that Theorem 1, however, also provides a sufficient condition for the comparison of the volume of non-symmetric convex bodies using injective Minkowski valuations different from the projection body operator.

Using Theorem 1, we will reduce Problem 1 for these Minkowski valuations to the question whether every (sufficiently smooth) convex body is contained in their image, generalizing results by Petty [53] and Schneider [55]. Combining this fact with a previously obtained result on the range of these operators, we conclude that Problem 1 for this class of Minkowski valuations has, in general, a negative answer for every $n \geq 3$, in analogy to the Shephard problem.

In Sections 4 and 6 we will consider a (large) subclass of $(n-1)$-homogeneous, $\mathrm{SO}(n)$ equivariant star body valued valuations Ψ, called radial Blaschke–Minkowski homomorphisms. Here, addition is radial addition of star bodies (see next section for the definition). A main representative of this class is the intersection body operator. We will establish a dual result to Theorem 1 which generalizes Lutwak's result on intersection bodies:

Theorem 2 *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke–Minkowski homomorphism. If $K \in \Psi\mathcal{S}^n$ and $L \in \mathcal{S}^n$, then*

$$\Psi K \subseteq \Psi L \quad \Longrightarrow \quad V(K) \leq V(L).$$

As in the case of Minkowski valuations we reduce the special case of Problem 1 for these star body valued valuations to the question whether every (sufficiently smooth) star body is contained in their image. However, a complete solution of Problem 1 for this class of valuations remains an open problem.

Section 2 contains all the basic notation and definitions concerning convex and star bodies and some well-known facts about spherical harmonics. In order to emphasize the duality between the results on Minkowski valuations and those on star body valued operators, as well as the similarity of the proofs, we follow Lutwak's example in the organization of the article and present the corresponding material in parallel sections. In Sections 3 and 4 we discuss basic properties of convex and star body valued valuations needed in the proofs of the main results contained in Sections 5 and 6.

2 Background material

In the following we state necessary background material and develop further notation related to convex and star bodies. For quick reference, we collect basic properties of mixed and dual mixed volumes. Finally, we state some well-known facts about spherical harmonics needed in subsequent sections. As general references for this section we recommend [13, Appendixes A & B] and the article [42]. For the material on convolution and spherical harmonics we refer to [21, Chapter 3] and [62].

Let $\mathcal{C}^n$ denote the set of non-empty compact convex sets in $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff topology, and let $\mathcal{K}^n_e$ denote the subset of $\mathcal{K}^n$ that contains the origin-symmetric bodies. A compact convex set $K \in \mathcal{C}^n$ is determined by the values of its support function, $h(K,\cdot)$, on the unit sphere S^{n-1}. From the definition of $h(K,\cdot)$, it follows immediately that for $\lambda > 0$ and $\vartheta \in \mathrm{SO}(n)$,

$$h(\lambda K, u) = \lambda\, h(K,u) \qquad \text{and} \qquad h(\vartheta K, u) = h(K, \vartheta^{-1}u).$$

It is easy to verify that the Minkowski sum $K + L$ of $K, L \in \mathcal{C}^n$ satisfies

$$h(K+L,\cdot) = h(K,\cdot) + h(L,\cdot).$$

The *Steiner point* $s(K)$ of $K \in \mathcal{C}^n$ is the point in K defined by

$$s(K) = n \int_{S^{n-1}} h(K,u)u\, du, \tag{2.1}$$

where integration is with respect to the rotation invariant probability measure on S^{n-1}. If $K \in \mathcal{K}^n_e$, then its Steiner point coincides with the origin. Let $\mathcal{K}^n_o$ denote the set of convex bodies whose Steiner point is at the origin.

A convex body K is also determined up to translation by its surface area measure $S_{n-1}(K,\cdot)$. For a Borel set $\omega \subseteq S^{n-1}$, $S_{n-1}(K,\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. For $\lambda > 0$ and $\vartheta \in \mathrm{SO}(n)$,

$$S_{n-1}(\lambda K,\cdot) = \lambda^{n-1} S_{n-1}(K,\cdot) \qquad \text{and} \qquad S_{n-1}(\vartheta K,\cdot) = \vartheta S_{n-1}(K,\cdot),$$

where $\vartheta S_{n-1}(K,\cdot)$ is the image measure of $S_{n-1}(K,\cdot)$ under the rotation ϑ. By Minkowski's existence theorem, a non-negative measure μ on S^{n-1} is the surface area measure of a convex body if and only if μ has its center of mass at the origin and is not concentrated on any great subsphere. The Blaschke sum $K \# L$ of $K, L \in \mathcal{K}^n$ is the convex body with

$$S_{n-1}(K \# L,\cdot) = S_{n-1}(K,\cdot) + S_{n-1}(L,\cdot)$$

and, say, the Steiner point at the origin.

For $K, L \in \mathcal{K}^n$, let $V_1(K,L)$ denote the mixed volume defined by

$$nV_1(K,L) = \lim_{\varepsilon \to 0} \frac{V(K+\varepsilon L) - V(K)}{\varepsilon} = \int_{S^{n-1}} h(L,u)\, dS_{n-1}(K,u). \tag{2.2}$$

The functional V_1 is translation invariant and monotone with respect to set inclusion in each component. From (2.2), one easily sees that the diagonal form of V_1 reduces to ordinary volume, i.e., $V_1(K,K) = V(K)$ for $K \in \mathcal{K}^n$.

The Minkowski inequality states that if $K, L \in \mathcal{K}^n$, then

$$V_1(K,L)^n \geq V(K)^{n-1} V(L), \tag{2.3}$$

and there is equality if and only if K and L are homothetic.

Let $\mathcal{S}^n$ denote the space of star bodies in $\mathbb{R}^n$ with the Hausdorff metric, and let $\mathcal{S}^n_e$ denote the subset of $\mathcal{S}^n$ that contains the origin-symmetric bodies. We call a star body *trivial* if it contains only the origin. A star body $L \in \mathcal{S}^n$ is determined by the values of its radial function, $\rho(L,\cdot)$, on S^{n-1}. From the definition of $\rho(L,\cdot)$, it follows immediately that for $\lambda > 0$ and $\vartheta \in \mathrm{SO}(n)$,

$$\rho(\lambda L, u) = \lambda\, \rho(L,u) \qquad \text{and} \qquad \rho(\vartheta L, u) = \rho(L, \vartheta^{-1} u).$$

The radial sum $K \mathbin{\tilde{+}} L$ of $K, L \in \mathcal{S}^n$ is the star body defined by

$$\rho(K \mathbin{\tilde{+}} L,\cdot) = \rho(K,\cdot) + \rho(L,\cdot).$$

The radial Blaschke sum $K \mathbin{\tilde{\#}} L$ of $K, L \in \mathcal{S}^n$ is the star body defined by

$$\rho(K \mathbin{\tilde{\#}} L,\cdot)^{n-1} = \rho(K,\cdot)^{n-1} + \rho(L,\cdot)^{n-1}.$$

For $K, L \in \mathcal{S}^n$, let $\widetilde{V}_1(K, L)$ denote the dual mixed volume defined by

$$n\widetilde{V}_1(K, L) = \lim_{\varepsilon \to 0} \frac{V(K \,\widetilde{+}\, \varepsilon L) - V(K)}{\varepsilon} = \int_{S^{n-1}} \rho(L, u)\rho(K, u)^{n-1} \, dS(u). \quad (2.4)$$

Here, integration is with respect to spherical Lebesgue measure. From (2.4), it follows that the functional $\widetilde{V}_1$ is monotone with respect to set inclusion and its diagonal form reduces to ordinary volume, i.e., $\widetilde{V}_1(L, L) = V(L)$ for $L \in \mathcal{S}^n$.

The dual Minkowski inequality states that if $K, L \in \mathcal{S}^n$, then

$$\widetilde{V}_1(K, L)^n \leq V(K)^{n-1}V(L), \quad (2.5)$$

and, if K and L are non-trivial, there is equality if and only if K and L are dilatations of each other.

In order to state the material on spherical harmonics, we first introduce further basic notions connected to $\mathrm{SO}(n)$ and S^{n-1}. As usual, $\mathrm{SO}(n)$ and S^{n-1} will be equipped with the invariant probability measures. Let $\mathcal{C}(\mathrm{SO}(n)), \mathcal{C}(S^{n-1})$ denote the spaces of continuous functions on $\mathrm{SO}(n)$ and S^{n-1} with the uniform topology and $\mathcal{M}(\mathrm{SO}(n))$, $\mathcal{M}(S^{n-1})$ their dual spaces of signed finite Borel measures with the weak* topology. The group $\mathrm{SO}(n)$ acts on these spaces by left translation, i.e., for $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$, say, we have $\vartheta f(u) = f(\vartheta^{-1}u)$, $\vartheta \in \mathrm{SO}(n)$, and $\vartheta\mu$ is the image measure of μ under the rotation ϑ.

As $\mathrm{SO}(n)$ is a compact Lie group, the space $\mathcal{M}(\mathrm{SO}(n))$ carries a natural convolution structure. If $\mu, \sigma \in \mathcal{M}(\mathrm{SO}(n))$, the convolution $\mu * \nu$ is defined by

$$\int_{\mathrm{SO}(n)} f(\vartheta) \, d(\mu * \sigma)(\vartheta) = \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} f(\eta\tau) \, d\mu(\eta) \, d\sigma(\tau),$$

for every $f \in \mathcal{C}(\mathrm{SO}(n))$. The convolution on $\mathcal{M}(\mathrm{SO}(n))$ induces a convolution on $\mathcal{C}(\mathrm{SO}(n))$, by identifying a continuous function f with the absolutely continuous measure with density f.

In the following, we identify the sphere S^{n-1} with the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, where $\mathrm{SO}(n-1)$ denotes the subgroup of rotations leaving the pole $\widehat{e}$ of S^{n-1} fixed. The projection from $\mathrm{SO}(n)$ onto S^{n-1} is $\vartheta \mapsto \widehat{\vartheta} := \vartheta\widehat{e}$. Functions on S^{n-1} can be identified with right $\mathrm{SO}(n-1)$-invariant functions on $\mathrm{SO}(n)$, by $\check{f}(\vartheta) = f(\widehat{\vartheta}\,)$, for $f \in \mathcal{C}(S^{n-1})$. In fact, it is not difficult to show that $\mathcal{C}(S^{n-1})$ is isomorphic to the subspace of right $\mathrm{SO}(n-1)$-invariant functions in $\mathcal{C}(\mathrm{SO}(n))$ and that this correspondence carries over to an identification of the space $\mathcal{M}(S^{n-1})$ with right $\mathrm{SO}(n-1)$-invariant measures in $\mathcal{M}(\mathrm{SO}(n))$.

Convolution structures on $\mathcal{M}(S^{n-1})$ and $\mathcal{C}(S^{n-1})$ can now be defined via this identification. The Dirac measure $\delta_{\widehat{e}}$ becomes the unique rightneutral element for the convolution on $\mathcal{M}(S^{n-1})$ and also the convolution $\mu * f \in \mathcal{C}(S^{n-1})$ of a measure $\mu \in \mathcal{M}(\mathrm{SO}(n))$ and a function $f \in \mathcal{C}(S^{n-1})$ is now defined:

$$(\mu * f)(u) = \int_{\mathrm{SO}(n)} \vartheta f(u)\, d\mu(\vartheta). \tag{2.6}$$

We will denote the canonical pairing of $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$ by

$$\langle \mu, f\rangle = \langle f, \mu\rangle = \int_{S^{n-1}} f(u)\, d\mu(u).$$

The following property of spherical convolution, see [62, Lemma 2.2], will be very useful: If $\mu, \nu \in \mathcal{M}(S^{n-1})$ and $f \in \mathcal{C}(S^{n-1})$, then

$$\langle \mu * \nu, f\rangle = \langle \mu, f * \nu\rangle. \tag{2.7}$$

An important role is played by convolution operators on $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$, which are generated by $\mathrm{SO}(n-1)$-invariant functions and measures. A function $f \in \mathcal{C}(S^{n-1})$ is called *zonal*, if $\vartheta f = f$ for every $\vartheta \in \mathrm{SO}(n-1)$. Zonal functions depend only on the value $u \cdot \widehat{e}$. The set of continuous zonal functions on S^{n-1} will be denoted by $\mathcal{C}(S^{n-1}, \widehat{e}\,)$ and the definition of $\mathcal{M}(S^{n-1}, \widehat{e}\,)$ is analogous. Define a map $\Lambda : \mathcal{C}[-1,1] \to \mathcal{C}(S^{n-1}, \widehat{e})$ by

$$\Lambda f(u) = f(u \cdot \widehat{e}), \qquad u \in S^{n-1}.$$

It is not difficult to show that the map Λ is an isomorphism between functions on $[-1,1]$ and zonal functions on S^{n-1}.

If $f \in \mathcal{C}(S^{n-1})$, $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$ and $\eta \in \mathrm{SO}(n)$, then we have

$$(f * \mu)(\widehat{\eta}\,) = \int_{S^{n-1}} f(\eta u)\, d\mu(u). \tag{2.8}$$

Note that, if $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$, then, by (2.8), for each $f \in \mathcal{C}(S^{n-1})$ and every $\vartheta \in \mathrm{SO}(n)$,

$$(\vartheta f) * \mu = \vartheta(f * \mu).$$

Thus, the spherical convolution from the right induces an $\mathrm{SO}(n)$ equivariant operator on $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$. It is also easy to check from (2.8) that the convolution of zonal functions and measures is abelian.

Closely related to convolution operators on $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$ are multiplier transformations with respect to spherical harmonics series expansions. We use $\mathcal{H}_k^n$ to denote the finite dimensional vector space of spherical harmonics of dimension n and order k. Let $N(n,k)$ denote the dimension of $\mathcal{H}_k^n$. The space of all finite sums of spherical harmonics of dimension n is denoted by $\mathcal{H}^n$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to the usual inner product on $\mathcal{C}(S^{n-1})$. Clearly, $\mathcal{H}_k^n$ is invariant with respect to rotations.

Let $P_k^n \in \mathcal{C}([-1,1])$ denote the Legendre polynomial of dimension n and order k. The zonal function ΛP_k^n is up to a multiplicative constant the unique zonal spherical harmonic in $\mathcal{H}_k^n$. In each space $\mathcal{H}_k^n$ we choose an orthonormal basis $H_{k1}, \ldots, H_{kN(n,k)}$. The collection $\{H_{k1}, \ldots, H_{kN(n,k)} : k \in \mathbb{N}\}$ forms a complete orthogonal system in $\mathcal{L}^2(S^{n-1})$. In particular, for every $f \in \mathcal{C}(S^{n-1})$, the series

$$f \sim \sum_{k=0}^{\infty} \pi_k f$$

converges to f in the $\mathcal{L}^2$-norm, where $\pi_k f \in \mathcal{H}_k^n$ is the orthogonal projection of f on the space $\mathcal{H}_k^n$. Using well-known properties of the Legendre polynomials, it is not hard to show that

$$\pi_k f = N(n,k)(f * \Lambda P_k^n). \tag{2.9}$$

This leads to the spherical expansion of a measure $\mu \in \mathcal{M}(S^{n-1})$,

$$\mu \sim \sum_{k=0}^{\infty} \pi_k \mu, \tag{2.10}$$

where $\pi_k \mu \in \mathcal{H}_k^n$ is defined by

$$\pi_k \mu = N(n,k)(\mu * \Lambda P_k^n). \tag{2.11}$$

From $P_0^n(t) = 1$, $N(n,0) = 1$ and $P_1^n(t) = t, N(n,1) = n$, we obtain, for $\mu \in \mathcal{M}(S^{n-1})$, the following special cases of (2.11):

$$\pi_0 \mu = \mu(S^{n-1}) \qquad \text{and} \qquad (\pi_1 \mu)(u) = n \int_{S^{n-1}} u \cdot v \, d\mu(v). \tag{2.12}$$

Let κ_n denote the volume of the Euclidean unit ball B. By definition (2.2) and (2.12), for every convex body K in $\mathbb{R}^n$,

$$\kappa_n \pi_0 h(K, \cdot) = V_1(B, K) \qquad \text{and} \qquad \pi_0 S_{n-1}(K, \cdot) = nV_1(K, B), \tag{2.13}$$

and, by definition (2.1) and the fact that the center of mass of a surface area measure is at the origin,

$$\pi_1 h(K, \cdot) = h(\{s(K)\}, \cdot) \qquad \text{and} \qquad \pi_1 S_{n-1}(K, \cdot) = 0. \tag{2.14}$$

From (2.12) and definition (2.4), it follows that, for every star body $L \in \mathcal{S}^n$,

$$\kappa_n \pi_0 \rho(L, \cdot) = \widetilde{V}_1(B, L) \qquad \text{and} \qquad \kappa_n \pi_0 \rho(L, \cdot)^{n-1} = \widetilde{V}_1(L, B). \tag{2.15}$$

A measure $\mu \in \mathcal{M}(S^{n-1})$ is uniquely determined by its series expansion (2.10). Using the fact that ΛP_k^n is (essentially) the unique zonal function in $\mathcal{H}_k^n$, a simple calculation shows that for $\mu \in \mathcal{M}(S^{n-1}, \hat{e}\,)$, formula (2.11) becomes

$$\pi_k \mu = N(n,k) \langle \mu, \Lambda P_k^n \rangle \Lambda P_k^n. \tag{2.16}$$

Thus, a zonal measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$ is determined by its so-called *Legendre coefficients* $\mu_k := \langle\, \mu, \Lambda P_k^n \rangle$. Using $\pi_k H = H$ for every $H \in \mathcal{H}_k^n$ and the fact that spherical convolution of zonal measures is commutative, we obtain the Funk–Hecke Theorem: If $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$ and $H \in \mathcal{H}_k^n$, then $H * \mu = \mu_k H$.

Definition We call a map $\Phi : \mathcal{D} \subseteq \mathcal{M}(S^{n-1}) \rightarrow \mathcal{M}(S^{n-1})$ a *multiplier transformation* if there exist real numbers c_k, the multipliers of Φ, such that, for every $k \in \mathbb{N}$,

$$\pi_k \Phi\mu = c_k \pi_k \mu, \qquad \forall\, \mu \in \mathcal{D}.$$

From the Funk–Hecke Theorem and the fact that the spherical convolution of zonal measures is commutative, it follows that, for $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$, the map $\Phi_\mu : \mathcal{M}(S^{n-1}) \rightarrow \mathcal{M}(S^{n-1})$, defined by $\Phi_\mu(\nu) = \nu * \mu$, is a multiplier transformation. The multipliers of this convolution operator are just the Legendre coefficients of the measure μ.

3 Minkowski valuations and convolutions

In this section we discuss basic properties of Minkowski valuations. We will also prove several auxiliary results needed in the proofs of our main theorems.

A map Φ defined on $\mathcal{C}^n$ (or on a certain subset of $\mathcal{C}^n$) and taking values in $\mathcal{C}^n$ is called a *Minkowski valuation* if

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K \cap L$ and $K \cup L$ are in the domain of Φ. A valuation Φ is called $\mathrm{GL}(n)$ *contravariant* (of weight 1), if for all $A \in \mathrm{GL}(n)$ and all K,

$$\Phi(AK) = |\det A| A^{-\mathrm{T}} \Phi K.$$

Here, $A^{-\mathrm{T}}$ denotes the inverse of the transpose of A. Note that an immediate consequence of $\mathrm{GL}(n)$ contravariance is homogeneity of degree $n-1$, i.e., $\Phi(\lambda K) = \lambda^{n-1} \Phi K$ for $K \in \mathcal{K}^n$ and $\lambda \geq 0$.

The following characterization of the projection body operator was obtained by Ludwig in [37, Corollary 2]:

Theorem *A map $\Phi : \mathcal{C}^n \rightarrow \mathcal{C}^n$ is a continuous, translation invariant and $\mathrm{GL}(n)$ contravariant Minkowski valuation if and only if there exists a constant $c \geq 0$ such that $\Phi = c\,\Pi$.*

It was also shown in [37] that the assumption of continuity can be omitted when $\mathcal{K}^n$ as the domain of Φ is replaced by $\mathcal{P}^n$, the set of convex polytopes in $\mathbb{R}^n$. These results were further generalized in [39].

In the following, we consider continuous and translation invariant Minkowski valuations $\Phi : \mathcal{K}^n \to \mathcal{K}^n$, but we will replace the strong assumption of $\mathrm{GL}(n)$ contravariance by the conditions of $(n-1)$-homogeneity and rotation equivariance, i.e., $\Phi\vartheta K = \vartheta\Phi K$ for $K \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$. The projection body operator is no longer characterized by these properties. The operator $\Theta : \mathcal{K}^n \to \mathcal{K}^n$, defined by

$$h(\Theta K, u) = \int_{-\infty}^{\infty} V_{n-2}(K \cap (u^{\perp} + tu))\, dt, \tag{3.1}$$

where $2V_{n-2}(L)$ is the $(n-2)$-dimensional surface area of an $(n-1)$-dimensional compact convex set L, is a Minkowski valuation with the above properties, see [26,56]. Another example is the (normalized) mean section operator M_2 introduced in [18] and further investigated in [26]: Let $\mathcal{E}_2^n$ be the affine Grassmannian of two-dimensional planes in $\mathbb{R}^n$ and μ_2 its (suitably normalized) motion invariant measure. The support function of $M_2 K$, $K \in \mathcal{K}^n$, is given by

$$h(M_2 K, \cdot) = \int_{\mathcal{E}_2^n} h(K \cap E, \cdot)\, d\mu_2(E) - h(\{z_{n-1}(K)\}, \cdot),$$

where $z_{n-1}(K)$ is the $(n-1)$st intrinsic moment vector of K, see [58, p.304].

In [62,63] the author investigated continuous maps $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ called *Blaschke–Minkowski homomorphisms*, which are $\mathrm{SO}(n)$ equivariant and satisfy

$$\Phi(K \,\#\, L) = \Phi K + \Phi L. \tag{3.2}$$

Since $(K \cup L) \,\#\, (K \cap L) = K \,\#\, L$ whenever $K, L, K \cup L \in \mathcal{K}^n$, every map satisfying (3.2) is a Minkowski valuation. Moreover, a result of McMullen [48] implies (cf. the proof of [62, Theorem 1.2]) that Blaschke–Minkowski homomorphisms are precisely the continuous and translation invariant Minkowski valuations which are homogeneous of degree $n-1$ and $\mathrm{SO}(n)$ equivariant. Thus, in the following we will use the more compact terminology of Blaschke–Minkowski homomorphism for such a valuation.

We say that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is even if $\Phi(K) = \Phi(-K)$. A crucial tool in the proofs of our main results is a representation theorem for Blaschke–Minkowski homomorphisms obtained in [62]:

Proposition 3.1 *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke–Minkowski homomorphism.*

(a) There exists a function $g \in \mathcal{C}(S^{n-1}, \widehat{e}\,)$, the generating function of Φ, which is unique up to addition of a linear function $u \mapsto x \cdot u$, $x \in \mathbb{R}^n$, such that

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * g. \tag{3.3}$$

(b) *If Φ is even, then there is a unique (generating) origin-symmetric compact convex set of revolution $L \in \mathcal{C}^n$ such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot).$$

If Φ is a Blaschke–Minkowski homomorphism, then, by (2.6) and (3.3), the support functions $h(\Phi K, \cdot)$ are weighted rotation means of the generating zonal function g. In particular, every compact convex set of revolution L generates a Blaschke–Minkowski homomorphism, by $h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot)$. In general, there are Blaschke–Minkowski homomorphisms which are not generated by support functions: Define

$$g_2(t) = \arccos(-t)\sqrt{1-t^2}, \qquad t \in [-1, 1].$$

In [18] it was proved that the generating function of M_2 is given by Λg_2. It is easy to very that Λg is not a support function. However, in the case of even maps, the set of generating functions coincides with the set of support functions of origin-symmetric compact convex sets of revolution.

The following simple consequence of Proposition 3.1 and (2.7) will be critical:

Lemma 3.2 *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke–Minkowski homomorphism, then, for $K, L \in \mathcal{K}^n$,*

$$V_1(K, \Phi L) = V_1(L, \Phi K).$$

Proof. Let $g \in \mathcal{C}(S^{n-1}, \hat{e}\,)$ denote the generating function of Φ. From definition (2.2), Proposition 3.1 and (2.7), it follows that

$$\begin{aligned} nV_1(K, \Phi L) &= \langle h(\Phi L, \cdot), S_{n-1}(K, \cdot)\rangle = \langle S_{n-1}(L, \cdot) * g, S_{n-1}(K, \cdot)\rangle \\ &= \langle S_{n-1}(L, \cdot), S_{n-1}(K, \cdot) * g\rangle = nV_1(L, \Phi K). \qquad \square \end{aligned}$$

As another consequence of Proposition 3.1 we obtain that Blaschke–Minkowski homomorphisms are multiplier transformations.

Lemma 3.3 *If Φ is a Blaschke–Minkowski homomorphism with generating function g, then, for every $K \in \mathcal{K}^n$,*

$$\pi_k h(\Phi K, \cdot) = g_k \pi_k S_{n-1}(K, \cdot), \qquad k \in \mathbb{N},$$

where the numbers g_k are the Legendre coefficients of g, i.e., $g_k = \langle\, g, \Lambda P_k^n\rangle$.

Proof. By (2.9) and Proposition 3.1, we have

$$\pi_k h(\Phi K, \cdot) = N(n, k)(S_{n-1}(K, \cdot) * g * \Lambda P_k^n).$$

Since spherical convolution is associative and g is zonal, we obtain from (2.16):

$$\pi_k h(\Phi K, \cdot) = g_k N(n, k)(S_{n-1}(K, \cdot) * \Lambda P_k^n) = g_k \pi_k S_{n-1}(K, \cdot). \qquad \square$$

Lemma 3.3 is of great value for answering injectivity and uniqueness questions arising in geometric tomography.

Definition If Φ is a Blaschke–Minkowski homomorphism with generating function g, then we call the subset $\mathcal{K}^n(\Phi)$ of $\mathcal{K}^n_o$, defined by

$$\mathcal{K}^n(\Phi) = \{K \in \mathcal{K}^n_o : \pi_k S_{n-1}(K, \cdot) = 0 \text{ if } g_k = 0\}, \tag{3.4}$$

the *injectivity set* of Φ.

Clearly, for every Blaschke–Minkowski homomorphism Φ, the set $\mathcal{K}^n(\Phi)$ is a non-empty rotation and dilatation invariant subset of $\mathcal{K}^n_o$ which is closed under Blaschke addition. By Lemma 3.3, a convex body $K \in \mathcal{K}^n(\Phi)$ is uniquely determined by its image ΦK. From (2.13), Lemma 3.2 and Lemma 3.3 on one hand, and (2.14), (3.3) and the fact that spherical convolution of zonal measures is commutative on the other hand, it follows that, for every $K \in \mathcal{K}^n$,

$$\pi_0 h(\Phi K, \cdot) = n g_0 V_1(K, B) > 0 \qquad \text{and} \qquad s(\Phi K) = o. \tag{3.5}$$

Examples The projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ is an even Blaschke–Minkowski homomorphism. Its injectivity set $\mathcal{K}^n(\Pi)$ coincides with the space of origin-symmetric convex bodies $\mathcal{K}^n_e$. Its generating compact convex set of revolution is the segment $\frac{1}{2}[-\widehat{e}\,, \widehat{e}\,]$, i.e., $h(\Pi K, \cdot)$ is a multiple of the cosine transform of $S_{n-1}(K, \cdot)$:

$$h(\Pi K, \cdot) = S_{n-1}(K, \cdot) * h(\tfrac{1}{2}[-\widehat{e}\,, \widehat{e}\,], \cdot).$$

The map $\Theta : \mathcal{K}^n \to \mathcal{K}^n$, defined in (3.1), is also even and its injectivity set $\mathcal{K}^n(\Theta)$ also coincides with $\mathcal{K}^n_e$. The generating compact convex set of Θ is the disk $B \cap \widehat{e}^{\,\perp}$. Finally, the mean section operator M_2 is injective on $\mathcal{K}^n_o$, i.e., $\mathcal{K}^n(M_2) = \mathcal{K}^n_o$, as was proved in [18].

In view of Theorem 1, the size of the range, $\Phi\mathcal{K}^n$, of a Blaschke–Minkowski homomorphism Φ will be of importance. We will first show that the set of convex bodies whose support functions are elements of the vector space

$$\operatorname{span}\{h(\Phi K, \cdot) - h(\Phi L, \cdot) : K, L \in \mathcal{K}^n\} \tag{3.6}$$

is a large subset of $\mathcal{K}^n$, provided the injectivity set $\mathcal{K}^n(\Phi)$ is not too small.

Definition We call a convex body $K \in \mathcal{K}^n$ *polynomial* if $h(K, \cdot) \in \mathcal{H}^n$.

It is well-known that the set of polynomial convex bodies is dense in $\mathcal{K}^n$, see for example [58, p. 160]. Similarly, the set of all origin-symmetric polynomial convex bodies is dense in $\mathcal{K}^n_e$.

Theorem 3.4 *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke–Minkowski homomorphism such that $\mathcal{K}_e^n \subseteq \mathcal{K}^n(\Phi)$, then, for every polynomial convex body $K \in \mathcal{K}_e^n$ there exist origin-symmetric convex bodies $L_1, L_2 \in \mathcal{K}_e^n$ such that*

$$K + \Phi L_1 = \Phi L_2.$$

In particular, the set of convex bodies whose support functions are elements of (3.6) is dense in $\mathcal{K}_e^n$.

Proof. Let $K \in \mathcal{K}_e^n$ be a polynomial convex body and let $g \in \mathcal{C}(S^{n-1}, \widehat{e}\,)$ denote the generating function of Φ. We will first show that there is an even function $f \in \mathcal{H}^n$ such that

$$h(K, \cdot) = f * g. \tag{3.7}$$

Let

$$h(K, \cdot) = \sum_{k=0}^{m} \pi_k h(K, \cdot)$$

and let $g_k = \langle\, g, \Lambda P_k^n \rangle$ be the kth Legendre coefficient of g. Since $K \in \mathcal{K}_e^n$, we have $\pi_k h(K, \cdot) = 0$ for all odd $k \in \mathbb{N}$. From $\mathcal{K}_e^n \subseteq \mathcal{K}^n(\Phi)$ and definition (3.4), it follows that $g_k \neq 0$ for every even $k \in \mathbb{N}$. We define

$$f := \sum_{k=0}^{m} b_k \pi_k h(K, \cdot),$$

where $b_k = 0$ for odd k and $b_k = g_k^{-1}$ if k is even. Clearly, $f \in \mathcal{H}^n$ is even and, since spherical convolution operators are multiplier transformations,

$$f * g = \sum_{k=0}^{m} b_k g_k \pi_k h(K, \cdot) = \sum_{k=0}^{m} \pi_k h(K, \cdot) = h(K, \cdot).$$

Denoting the positive and negative parts of the function f by $f^{\,+}$ and $f^{\,-}$ and using the existence theorem of Minkowski [58, p. 392], it follows that there are convex bodies $L_1, L_2 \in \mathcal{K}_e^n$ such that

$$S_{n-1}(L_1, \cdot) = f^{\,-} \qquad \text{and} \qquad S_{n-1}(L_2, \cdot) = f^{\,+}.$$

By Proposition 3.1, this finishes the proof. $\square$

The case $\Phi = \Pi$ of Theorem 3.4 is well-known. A convex body whose support function is a difference of support functions of zonoids (projection bodies) is called a generalized zonoid. These bodies played a critical role in Schneider's solution [55] of the Shephard problem. For related results in a more general context, see [60].

If Φ is a Blaschke–Minkowski homomorphism, then, as mentioned before, the support function $h(\Phi K, \cdot)$ is a weighted rotation mean of the generating zonal function g for every $K \in \mathcal{K}^n$. This observation led in [62] to the following result, contrasting Theorem 3.4:

Proposition 3.5 *The range of every Blaschke–Minkowski homomorphism is nowhere dense in $\mathcal{K}^n$.*

4 Radial valuations and convolutions

This section contains the background material on valuations with respect to radial addition. We collect the dual results to Proposition 3.1, Lemma 3.2, Lemma 3.3 and Theorem 3.4.

A map Ψ defined on $\mathcal{S}^n$ (or on a certain subset of $\mathcal{S}^n$) and taking values in $\mathcal{S}^n$ is called a *radial valuation* if

$$\Psi K \,\tilde{+}\, \Psi L = \Psi(K \cup L) \,\tilde{+}\, \Psi(K \cap L),$$

whenever $K \cap L$ and $K \cup L$ are in the domain of Ψ.

The following characterization of the intersection body operator follows from arguments employed by Ludwig [40] to deduce a more general result:

Theorem *A map $\Psi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ is a continuous and $\mathrm{GL}(n)$ contravariant radial valuation if and only if there exists a constant $c \geq 0$ such that $\Psi = c\, I$.*

Apart from Ludwig's result little is known about radial valuations. In [29,30] Klain investigated real-valued valuations on the space of L^n-stars, where he called a set in $\mathbb{R}^n$ which is starshaped with respect to the origin an L^p-star, $p > 0$, if its radial function is an $\mathcal{L}^p$ function on S^{n-1}. A special case of one of Klain's results [29, Proposition 4.1] is the following: Any continuous $(n-1)$-homogeneous real-valued valuation ψ on the space of L^n-stars satisfies

$$\psi(K \,\widetilde{\#}\, L) = \psi(K) + \psi(L). \tag{4.1}$$

In light of (4.1) and the results on Blaschke–Minkowski homomorphisms, the author introduced and investigated in [63] continuous maps $\Psi : \mathcal{S}^n \rightarrow \mathcal{S}^n$ called *radial Blaschke–Minkowski homomorphisms.* These operators are $\mathrm{SO}(n)$ equivariant and satisfy

$$\Phi(K \,\widetilde{\#}\, L) = \Phi K \,\tilde{+}\, \Phi L. \tag{4.2}$$

Radial Blaschke–Minkowski homomorphisms turned out to be in many respects dual to Blaschke–Minkowski homomorphisms. A main example is the intersection body operator.

Since $(K \cup L) \,\widetilde{\#}\, (K \cap L) = K \,\widetilde{\#}\, L$ for $K, L \in \mathcal{S}^n$, every map satisfying (4.2) is an $(n-1)$-homogeneous radial valuation. Conversely, Klain's result leads to the following conjecture:

Conjecture *The set of radial Blaschke–Minkowski homomorphisms coincides with the set of continuous radial valuations which are* $\mathrm{SO}(n)$ *equivariant and* $(n-1)$*-homogeneous.*

A dual version of Proposition 3.1 was obtained in [63]. It shows that the set of radial Blaschke–Minkowski homomorphisms is in one-to-one correspondence with the set of non-negative zonal measures on S^{n-1}:

Proposition 4.1 *A map* $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ *is a radial Blaschke–Minkowski homomorphism if and only if there is a unique non-negative measure* $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$*, the generating measure of* Ψ*, such that*

$$\rho(\Psi L, \cdot) = \rho(L, \cdot)^{n-1} * \mu.$$

Proposition 4.1 and (2.7) now lead to a dual version of Lemma 3.2:

Lemma 4.2 *If* $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ *is a radial Blaschke–Minkowski homomorphism, then, for* $K, L \in \mathcal{S}^n$*,*

$$\widetilde{V}_1(K, \Psi L) = \widetilde{V}_1(L, \Psi K).$$

Proof. Let $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$ be the generating measure of Ψ. Using definition (2.4), Proposition 4.1 and (2.7), it follows that

$$\begin{aligned} \widetilde{V}_1(K, \Psi L) &= \kappa_n \langle \rho(\Psi L, \cdot), \rho(K, \cdot)^{n-1} \rangle = \kappa_n \langle \rho(L, \cdot)^{n-1} * \mu, \rho(K, \cdot)^{n-1} \rangle \\ &= \kappa_n \langle \rho(L, \cdot)^{n-1}, \rho(K, \cdot)^{n-1} * \mu \rangle = \widetilde{V}_1(L, \Psi K). \qquad \square \end{aligned}$$

Using Proposition 4.1 and the fact that spherical convolution operators are multiplier transformations, one obtains a dual version of Lemma 3.3:

Lemma 4.3 *If* Ψ *is a radial Blaschke–Minkowski homomorphism which is generated by the zonal measure* μ*, then, for every star body* $L \in \mathcal{S}^n$*,*

$$\pi_k \rho(\Psi L, \cdot) = \mu_k \pi_k \rho(L, \cdot)^{n-1}, \qquad k \in \mathbb{N},$$

where the numbers μ_k *are the Legendre coefficients of* μ*.*

Definition If Ψ is a radial Blaschke–Minkowski homomorphism, generated by the zonal measure μ, then we call the subset $\mathcal{S}^n(\Psi)$ of $\mathcal{S}^n$, defined by

$$\mathcal{S}^n(\Psi) = \{L \in \mathcal{S}^n : \pi_k \rho(L, \cdot)^{n-1} = 0 \text{ if } \mu_k = 0\}, \tag{4.3}$$

the *injectivity set* of Ψ.

It is easy to verify that for every radial Blaschke–Minkowski homomorphism Ψ, the set $\mathcal{S}^n(\Psi)$ is a non-empty rotation and dilatation invariant subset of $\mathcal{S}^n$ which is closed under radial Blaschke addition. By Lemma 4.3, a star body $L \in \mathcal{S}^n(\Psi)$ is uniquely determined by its image ΨL. From Lemma 4.2 and Lemma 4.3, it follows that, for every $L \in \mathcal{S}^n$,

$$\widetilde{V}_1(B, \Psi L) = \widetilde{V}_1(L, \Psi B) = \mu_0 \widetilde{V}_1(L, B).$$

Since $\widetilde{V}_1(B, L) = 0$ if and only if L is trivial and $\widetilde{V}_1(L, B) = 0$ if and only if L is trivial, we obtain $\mu_0 = 0$ if and only if Ψ is the *trivial* radial Blaschke–Minkowski homomorphism, mapping every star body to the origin.

Example The intersection body operator $I : \mathcal{S}^n \to \mathcal{S}^n$ is an even radial Blaschke–Minkowski homomorphism. Its injectivity set $\mathcal{S}^n(I)$ coincides with the space of origin-symmetric star bodies $\mathcal{S}^n_e$. The generating measure of I is the (suitably normalized) invariant measure $\mu_{S_0^{n-2}}$ which is concentrated on $S_0^{n-2} := S^{n-1} \cap \hat{e}^{\perp}$, i.e., $\rho(IL, \cdot)$ is the spherical Radon transform of $\rho(L, \cdot)^{n-1}$:

$$\rho(IL, \cdot) = \rho(L, \cdot)^{n-1} * \mu_{S_0^{n-2}}.$$

Considering their duality with Blaschke–Minkowski homomorphisms and the examples of those maps given in Section 3, one would expect that other radial Blaschke–Minkowski homomorphisms have appeared in the context of geometric tomography. Unfortunately the intersection body operator is the only example (the author is aware of) so far.

The last result of this section is a dual version of Theorem 3.4. It shows that the set of star bodies whose radial functions are elements of the vector space

$$\operatorname{span}\{\rho(\Psi K, \cdot) - \rho(\Psi L, \cdot) : K, L \in \mathcal{S}^n\} \tag{4.4}$$

is a dense subset of $\mathcal{S}^n_e$, provided the injectivity set $\mathcal{S}^n(\Psi)$ contains the set of origin-symmetric star bodies.

Definition We call a star body $L \in \mathcal{S}^n$ *polynomial* if $\rho(L, \cdot) \in \mathcal{H}^n$.

Clearly, the set of polynomial star bodies is dense in $\mathcal{S}^n$ and the set of all origin-symmetric polynomial star bodies is dense in $\mathcal{S}^n_e$.

Theorem 4.4 *If $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke–Minkowski homomorphism such that $\mathcal{S}^n_e \subseteq \mathcal{S}^n(\Psi)$, then, for every polynomial star body $S \in \mathcal{S}^n_e$, there exist origin-symmetric star bodies $K_1, K_2 \in \mathcal{S}^n_e$ such that*

$$L \,\tilde{+}\, \Psi K_1 = \Psi K_2.$$

In particular, the set of star bodies whose radial functions are elements of (4.4) is dense in $\mathcal{S}^n_e$.

Proof. Let $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$ denote the generating measure of Ψ and let $L \in \mathcal{S}_e^n$ be a polynomial star body, say,

$$\rho(L, \cdot) = \sum_{k=0}^{m} \pi_k \rho(L, \cdot).$$

Since $L \in \mathcal{S}_e^n$, we have $\pi_k \rho(L, \cdot) = 0$ for all odd $k \in \mathbb{N}$. Let μ_k denote the Legendre coefficients of μ. From $\mathcal{S}_e^n \subseteq \mathcal{S}^n(\Psi)$ and definition (4.3), it follows that $\mu_k \neq 0$ for every even $k \in \mathbb{N}$. We define

$$f := \sum_{k=0}^{m} c_k \pi_k \rho(L, \cdot),$$

where $c_k = 0$ for odd k and $c_k = \mu_k^{-1}$ if k is even. Clearly, f is an even continuous function on S^{n-1} and, since spherical convolution operators are multiplier transformations,

$$f * \mu = \sum_{k=0}^{m} c_k \mu_k \pi_k \rho(L, \cdot) = \sum_{k=0}^{m} \pi_k \rho(L, \cdot) = \rho(L, \cdot).$$

Denote by $f^{\,+}$ and $f^{\,-}$ the positive and negative parts of f and let K_1 and K_2 be the star bodies such that $\rho(K_1, \cdot)^{n-1} = f^-$ and $\rho(K_2, \cdot)^{n-1} = f^+$. By Proposition 4.1, it follows that $L \,\widetilde{+}\, \Psi K_1 = \Psi K_2$. $\square$

The case $\Psi = I$ of Theorem 4.4 is well-known and closely related to the notion of generalized intersection bodies, see [17,67].

The argument used in the proofs of Theorem 3.4 and Theorem 4.4 implies a more general result: Let $\Upsilon : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ be defined by

$$\Upsilon f = f * \mu, \qquad f \in \mathcal{C}(S^{n-1}),$$

for some zonal measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\,)$. If $\mathcal{H}^n(\Upsilon)$ denotes the subset of $\mathcal{H}^n$ defined by

$$\mathcal{H}^n(\Upsilon) = \{H \in \mathcal{H}^n : \pi_k H = 0 \text{ if } \mu_k = 0\},$$

then the restriction $\Upsilon : \mathcal{H}^n(\Upsilon) \to \mathcal{H}^n(\Upsilon)$ is a bijection.

5 Minkowski valuations and the comparison of volume

Throughout this section, let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ denote a Blaschke–Minkowski homomorphism, i.e., Φ is a continuous and translation invariant Minkowski valuation which is $(n-1)$-homogeneous and $\mathrm{SO}(n)$ equivariant. We consider the following special case of Problem 1:

Problem 5.1 *Let K and L be convex bodies in $\mathbb{R}^n$. Is there the implication*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad V(K) \leq V(L)?$$

Note that Problem 5.1 does not include the corresponding question for the centroid body operator Γ which was investigated by Lutwak (as mentioned in the Introduction). The map Γ is indeed (up to volume normalization) a Minkowski valuation which is $\mathrm{SO}(n)$ equivariant but neither is Γ translation invariant nor homogeneous of degree $n-1$.

Our first result of this section generalizes the Petty–Schneider theorem for projection bodies. It is a stronger version of Theorem 1 of the Introduction.

Theorem 5.1 *If $K \in \mathcal{K}^n$ and a translate of L is contained in $\Phi\mathcal{K}^n$, then*

$$\Phi K \subseteq \Phi L \quad \Longrightarrow \quad V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if K and L are translates of each other.

Proof. Since a translate of L is contained in $\Phi\mathcal{K}^n$, there exist a convex body L_0 and a vector $t \in \mathbb{R}^n$, such that $L = \Phi L_0 + t$. Using Lemma 3.2 and the fact that the mixed volume V_1 is translation invariant and monotone with respect to set inclusion, it follows that

$$V_1(K, L) = V_1(K, \Phi L_0) = V_1(L_0, \Phi K) \leq V_1(L_0, \Phi L) = V_1(L, \Phi L_0) = V(L).$$

From the Minkowski inequality (2.3), we thus obtain

$$V(K) \leq V(L),$$

with equality if and only if K and L are homothetic. The observation that homothetic convex bodies of equal volume must be translates of each other finishes the proof. $\square$

In light of Proposition 3.5 it is a natural question whether in Theorem 5.1 the set $\Phi\mathcal{K}^n$ can be replaced by a larger class of convex bodies. Our next result shows that if the injectivity set $\mathcal{K}^n(\Phi)$ does not exhaust all of $\mathcal{K}^n_o$, the answer to Problem 5.1 is negative, in general.

Theorem 5.2 *If $\mathcal{K}^n(\Phi)$ does not coincide with $\mathcal{K}^n_o$, then there exist convex bodies $K, L \in \mathcal{K}^n_o$, such that*

$$\Phi K \subseteq \Phi L$$

but

$$V(K) > V(L).$$

Proof. Let $g \in \mathcal{C}(S^{n-1}, \hat{e}\,)$ be the generating function of Φ and let g_k denote its Legendre coefficients. Since $\mathcal{K}^n(\Phi) \neq \mathcal{K}^n_o$, it follows from (3.4) and (3.5) that there exists an integer $k \in \mathbb{N}$, $k \geq 2$, such that $g_k = 0$. Choose $\alpha > 0$ such that the function $f(u) = 1 + \alpha P^n_k(u \cdot \hat{e}\,), u \in S^{n-1}$, is non-negative. By (2.14) and Minkowski's existence theorem, there exists a convex body $L \in \mathcal{K}^n_o$ such that $S_{n-1}(L, \cdot) = f$. Clearly, $L \notin \mathcal{K}^n(\Phi)$ and, by (2.13),

$$nV_1(L, B) = \pi_0 S_{n-1}(L, \cdot) = 1. \tag{5.1}$$

Using Lemma 3.3, we see that $\Phi L = \Phi K$, where K denotes the Euclidean ball centered at the origin with surface area $S(K) = 1$. To complete the proof, we use (5.1) and Minkowski's inequality (2.3) to conclude

$$V(K)^{n-1} = \frac{1}{n^n V(B)} > V(L)^{n-1}. \qquad \square$$

An argument related to the one used in the proof of Theorem 5.2 led Shephard in [64] to a restriction of Problem 5.1, in the special case $\Phi = \Pi$, to origin-symmetric convex bodies. We will follow Shephard's example and consider Problem 5.1 for convex bodies contained in $\mathcal{K}^n(\Phi)$. Since $\mathcal{K}^n(\Phi)$ consists in the worst case only of balls centered at the origin, in which case Problem 5.1 becomes of little interest, we will make the further assumption that the set of origin-symmetric convex bodies is contained in $\mathcal{K}^n(\Phi)$.

Theorem 5.3 *Suppose that $\mathcal{K}^n_e \subseteq \mathcal{K}^n(\Phi)$. If $K \in \mathcal{K}^n_e$ is a polynomial convex body and has positive curvature, then if $K \notin \Phi\mathcal{K}^n$, there exists a convex body $L \in \mathcal{K}^n_e$, such that*

$$\Phi K \subseteq \Phi L,$$

but

$$V(K) > V(L).$$

Proof. Let $g \in \mathcal{C}(S^{n-1}, \hat{e}\,)$ denote the generating function of Φ. Since $K \in \mathcal{K}^n_e$ is polynomial, by (3.7), there exists an even function $f \in \mathcal{H}^n$, such that

$$h(K, \cdot) = f * g. \tag{5.2}$$

The function f must assume negative values, otherwise f is the density of a surface area measure by Minkowski's existence theorem and thus, $K \in \Phi\mathcal{K}^n$, by Proposition 3.1. Let $F \in \mathcal{C}(S^{n-1})$ be a non-constant even function, such that

$$F(u) \begin{cases} \geq 0 & \text{when } f(u) < 0, \\ = 0 & \text{when } f(u) \geq 0. \end{cases}$$

By suitable approximation of the function F with spherical harmonics, we can find a non-negative, even function $G \in \mathcal{H}^n$ such that

$$\langle f, G \rangle < 0.$$

Since G is even and $g_k \neq 0$ for even $k \in \mathbb{N}$, we can find another even function $H \in \mathcal{H}^n$ (cf. the remark after Theorem 4.4), such that

$$G = H * g. \tag{5.3}$$

Since K is polynomial and has positive curvature, the surface area measure of K has a positive density $s_{n-1}(K, \cdot)$. Thus, we can choose $\alpha > 0$ such that

$$s_{n-1}(K, \cdot) + \alpha H > 0.$$

By Minkowski's existence theorem, there exists an origin-symmetric convex body $L \in \mathcal{K}_e^n$ such that

$$S_{n-1}(L, \cdot) = s_{n-1}(K, \cdot) + \alpha H. \tag{5.4}$$

From (5.3) and Proposition 3.1, we see that

$$h(\Phi L, \cdot) = h(\Phi K, \cdot) + \alpha G.$$

Since $G \geq 0$, it follows that

$$\Phi K \subseteq \Phi L.$$

Definition (2.2), (5.2) and (5.4), yield

$$n(V_1(L, K) - V(K)) = \langle h(K, \cdot), S_{n-1}(L, \cdot) - S_{n-1}(K, \cdot)\rangle = \alpha\langle f * g, H\rangle.$$

Thus, using (2.7) and (5.3), we obtain

$$n(V_1(L, K) - V(K)) = \alpha\langle f, G\rangle < 0.$$

To finish the proof, we can use now the Minkowski inequality (2.3), to conclude

$$V(K) > V(L). \qquad \square$$

Combining Theorem 5.1, Theorem 5.2 and Theorem 5.3, we finally obtain a generalization of the Petty–Schneider connection between a positive solution to the Shephard problem and the range of Π.

Corollary 5.4 *For origin-symmetric convex bodies in $\mathbb{R}^n$, Problem 5.1 has a positive answer if and only if every polynomial convex body $K \in \mathcal{K}_e^n$ with positive curvature is contained in $\Phi\mathcal{K}^n$.*

Proof. Suppose that K and L are origin-symmetric convex bodies for which Problem 5.1 has a negative answer, i.e., $\Phi K \subseteq \Phi L$ but $V(K) > V(L)$. By (3.5), the convex bodies ΦK and ΦL have their Steiner points at the origin and thus contain the origin as an interior point (cf. [58, p.43]). Since Φ is homogeneous, we can thus dilate K by a suitable factor $\lambda < 1$, so that still $V(\lambda K) > V(L)$, but the inclusion $\Phi(\lambda K) \subseteq \Phi L$ becomes strict. Since the set of origin-symmetric polynomial convex bodies with positive curvature is dense in $\mathcal{K}_e^n$ (cf. [58, p.160]) and Φ is continuous, we can find an origin-symmetric polynomial convex body L' with positive curvature, such that Problem 5.1 for the pair λK and L' has a negative answer. By Theorem 5.1, $L' \notin \Phi\mathcal{K}^n$. The converse follows from Theorem 5.3 if $\mathcal{K}_e^n \subseteq \mathcal{K}^n(\Phi)$ and a construction as in the proof of Theorem 5.2 otherwise. $\square$

The observation that on one hand, by Proposition 3.5, the set $\Phi\mathcal{K}^n$ is nowhere dense in $\mathcal{K}^n$ and on the other hand, by [58, p.160], the set of polynomial convex bodies with positive curvature is dense in $\mathcal{K}^n$, allows us to conclude this section with a complete solution of Problem 5.1:

Corollary 5.5 *For every Blaschke–Minkowski homomorphism, Problem 5.1, even if restricted to origin-symmetric convex bodies, has a negative answer.*

6 Radial valuations and the comparison of volume

Throughout this section, let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ denote a non-trivial radial Blaschke–Minkowski homomorphism, i.e., Ψ is a continuous and $\mathrm{SO}(n)$ equivariant map satisfying $\Psi(K \,\widetilde{\#}\, L) = \Psi K \,\widetilde{+}\, \Psi L$ and Ψ does not map every star body to the origin. We consider the following special case of Problem 1:

Problem 6.1 *Let K and L be star bodies in $\mathbb{R}^n$. Is there the implication*

$$\Psi K \subseteq \Psi L \quad \Longrightarrow \quad V(K) \leq V(L)?$$

The following result generalizes Lutwak's theorem for intersection bodies. It is stronger than Theorem 2 of the Introduction and dual to Theorem 5.1:

Theorem 6.1 *If $K \in \Psi\mathcal{S}^n$ and $L \in \mathcal{S}^n$, then*

$$\Psi K \subseteq \Psi L \quad \Longrightarrow \quad V(K) \leq V(L),$$

and $V(K) = V(L)$ if and only if $K = L$.

Proof. Since $K \in \Psi\mathcal{K}^n$, there exists a star body K_0, such that $K = \Psi K_0$. Using Lemma 4.2 and the fact that the dual mixed volume $\widetilde{V}_1$ is monotone with respect to set inclusion, it follows that

$$\widetilde{V}_1(L, K) = \widetilde{V}_1(L, \Psi K_0) = \widetilde{V}_1(K_0, \Psi L) \geq \widetilde{V}_1(K_0, \Psi K) = \widetilde{V}_1(K, \Psi K_0) = V(K).$$

From the dual Minkowski inequality (2.5), we thus obtain

$$V(K) \leq V(L),$$

with equality if and only if K and L are dilatations of each other. Clearly, star bodies of equal volume which are dilatations of each other must be equal. $\square$

In the following we will see that the question whether in Theorem 6.1 the set $\Psi\mathcal{S}^n$ can be replaced by a larger class of star bodies is again intimately connected to the size of $\mathcal{S}^n(\Psi)$.

Theorem 6.2 *If $\mathcal{S}^n(\Psi)$ does not coincide with $\mathcal{S}^n$, then there exist star bodies $K, L \in \mathcal{S}^n$, such that*

$$\Psi K \subseteq \Psi L$$

but

$$V(K) > V(L).$$

Proof. Let $\mu \in \mathcal{M}(S^{n-1}, \hat{e}\,)$ be the generating measure of Ψ and let μ_k denote its Legendre coefficients. Since $\mathcal{S}^n(\Psi) \neq \mathcal{S}^n$ and Ψ is non-trivial, there exists, by definition (4.3) and the remark after it, an integer $k \in \mathbb{N}$, such that $\mu_k = 0$ and $k \geq 1$. Choose $\beta > 0$ such that the function $f(u) = 1 + \beta P_k^n(u \cdot \hat{e}\,)$, $u \in S^{n-1}$, is positive and let $K \in \mathcal{S}^n$ be the star body with $\rho(K, \cdot)^{n-1} = f$. Clearly, $K \notin \mathcal{S}^n(\Psi)$ and, by (2.15),

$$\widetilde{V}_1(K, B) = V(B). \tag{6.1}$$

Using Lemma 4.3, we see that $\Psi K = \Psi B$. To complete the proof, we use (6.1) and the dual Minkowski inequality (2.5) to conclude

$$V(B) < V(K). \qquad \square$$

The following is a dual version of Theorem 5.3. It provides a generalization of an important result by Lutwak [42, Theorem 12.2] for intersection bodies.

Theorem 6.3 *Suppose that $\mathcal{S}_e^n \subseteq \mathcal{S}^n(\Psi)$. If $L \in \mathcal{S}_e^n$ is a polynomial star body whose radial function is positive, then if $L \notin \Psi\mathcal{S}^n$, there exists a star body $K \in \mathcal{S}_e^n$, such that*

$$\Psi K \subseteq \Psi L$$

but

$$V(K) > V(L).$$

Proof. Let $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$ denote the generating measure of Ψ. Since $L \in \mathcal{S}_e^n$ is polynomial, it follows from the proof of Theorem 4.4 that there exists an even function $f \in \mathcal{H}^n$, such that

$$\rho(L, \cdot) = f * \mu. \tag{6.2}$$

The function f must assume negative values, otherwise, by Proposition 4.1, we have $L = \Psi L_0$, where L_0 is the star body with $\rho(L_0, \cdot)^{n-1} = f$. As in the proof of Theorem 5.3, we can find a non-negative, even function $G \in \mathcal{H}^n$ and an even function $H \in \mathcal{H}^n$ such that

$$\langle f, G \rangle < 0 \qquad \text{and} \qquad G = H * \mu. \tag{6.3}$$

Since L has a positive radial function, there exists a $\beta > 0$ and an origin-symmetric star body K such that

$$\rho(K, \cdot)^{n-1} = \rho(L, \cdot)^{n-1} - \beta H. \tag{6.4}$$

From (6.3) and Proposition 4.1, we see that $\rho(\Psi K, \cdot) = \rho(\Psi L, \cdot) - \beta G$. Since $G \geq 0$, it follows that

$$\Psi K \subseteq \Psi L.$$

Definition (2.4), (6.2), (6.4) and (2.7), yield

$$V(L) - \widetilde{V}_1(K, L) = \kappa_n \beta \langle f * \mu, H \rangle = \kappa_n \beta \langle f, G \rangle < 0.$$

To finish the proof, we can use the dual Minkowski inequality (2.5), to conclude

$$V(K) > V(L). \qquad \square$$

Combining Theorem 6.1, Theorem 6.2 and Theorem 6.3, we obtain the dual version of Corollary 5.4 (and in almost verbally the same way). It provides a generalization of Lutwak's connection between a positive solution to the Busemann–Petty problem and the range of I.

Corollary 6.4 *For origin-symmetric star bodies in $\mathbb{R}^n$, Problem 6.1 has a positive answer if and only if every polynomial star body $L \in \mathcal{S}^n_e$ with positive radial function is contained in $\Psi \mathcal{S}^n$.*

In fact the arguments employed to establish Corollary 6.4 also lead to a generalization of statements stronger than Lutwak's, due independently to Gardner [11, Theorem 3.1] and Zhang [67, Theorem 2.22].

In Section 4, the information provided by Proposition 3.5 led to a complete solution of Problem 5.1. Since a corresponding result for radial Blaschke–Minkowski homomorphisms is not available, Problem 6.1 is still open. However, we want to point out that the radial Blaschke–Minkowski homomorphism generated by the Dirac measure $\delta_{\widehat{e}}$ provides a positive answer to Problem 6.1 in every dimension, which is in contrast to Corollary 5.5. Note also that the answer to Problem 6.1, in the special case $\Psi = I$, is negative for every $n \geq 3$, even for origin-symmetric star bodies (cf. [13, Theorem 8.2.4]). Thus, as in the case of the original Busemann–Petty problem, a restriction of Problem 6.1 to convex bodies might be of interest.

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