Even Minkowski Valuations

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Abstract. A new integral representation of smooth translation invariant and rotation equivariant even Minkowski valuations is established. Explicit formulas relating previously obtained descriptions of such valuations with the new more accessible one are also derived. Moreover, the action of Alesker's Hard Lefschetz operators on these Minkowski valuations is explored in detail.

1. Introduction

A *Minkowski valuation* is a map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ defined on the set $\mathcal{K}^n$ of convex bodies (compact convex sets) in $\mathbb{R}^n$ such that

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K \cup L \in \mathcal{K}^n$ and addition on $\mathcal{K}^n$ is the usual Minkowski addition. While a number of classical (reverse) affine isoperimetric inequalities involve well known Minkowski valuations, like the projection and difference body maps, the underlying reason for the special role of these maps has only been revealed recently, when they were characterized as the unique Minkowski valuations which intertwine linear transformations. This line of research can be traced back to two seminal articles by Ludwig [**41**, **42**] and has become the focus of increased interest in recent years, see [**1**, **30–33**, **43–45**, **62**, **65**].

Due to these recent characterization results a more and more complete picture on linearly intertwining Minkowski valuations could be developed. However, the theory of Minkowski valuations which are merely translation invariant and $\mathrm{SO}(n)$ equivariant is still in its infancy. Here, Schneider [**57**] and Kiderlen [**36**] have obtained first classification theorems of Minkowski valuations homogeneous of degree one. For *even* valuations their results were subsequently generalized by the first author [**61**]: It was shown that smooth translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations of an *arbitrary* degree of homogeneity are generated by convolutions between projection functions and *spherical Crofton measures*. Unfortunately, the rather complicated technical nature of this general result also made it difficult to work with. In particular, uniqueness as well as the problem of finding necessary and sufficient conditions for a measure to be a spherical Crofton measure of a Minkowski valuation remained (essentially) open.

In this article we first show that the spherical Crofton measure of an even Minkowski valuation is uniquely determined. This is done by exploiting tools from harmonic analysis and a different description of such valuations based on an embedding of even translation invariant real valued valuations in continuous functions on the Grassmannian by Klain [**38**]. Our approach has the added advantage that it helps to simplify a previously known necessary condition for a measure to be the spherical Crofton measure of a Minkowski valuation: We show that its spherical cosine transform has to be the support function of a convex body. As our main result we then derive a new more elementary integral representation of smooth translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations. We also give explicit formulas, involving Radon transforms on Grassmannians, which relate the previously known descriptions of such valuations with our new more accessible one.

By recent results of Bernig [**12**] and Parapatits with the first author [**55**] Alesker's Hard Lefschetz operators (originally defined only for translation invariant real valued valuations; see [**4**, **5**, **7**, **15**]) can be extended to translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations. In the final section of this article we study these operators in terms of their action on the different possible representations of such Minkowski valuations.

2. Statement of principal results

We endow the set $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ with the Hausdorff metric and assume throughout that $n \geq 3$. All measures in this article are *signed* finite Borel measures. A convex body K is uniquely determined by its support function $h(K,u) = \max\{u \cdot x : x \in K\}$ for $u \in S^{n-1}$. For $i \in \{1, \ldots, n-1\}$, the ith projection function $\mathrm{vol}_i(K|\cdot\,)$ of $K \in \mathcal{K}^n$ is the continuous function on the Grassmannian $\mathrm{Gr}_{i,n}$ of i-dimensional subspaces of $\mathbb{R}^n$ defined such that $\mathrm{vol}_i(K|E)$, for $E \in \mathrm{Gr}_{i,n}$, is the i-dimensional volume of the orthogonal projection of K onto E.

A function $\phi : \mathcal{K}^n \to \mathcal{A}$ with values in an Abelian semigroup $\mathcal{A}$ is called a *valuation*, or *additive*, if

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L)$$

whenever $K \cup L$ is convex. With its origins in Dehn's solution of Hilbert's Third Problem, the notion of *scalar* valued valuations has long played a central role in convex, discrete, and integral geometry (see, e.g., [**39**] for more information). Minkowski valuations are of newer vintage.

Recent classifications of Minkowski valuations (see [**32**, **41**, **42**, **62**, **65**]) showed that only a small number of such operators, like the projection and the difference body map (see Section 4 for their definitions), intertwine *affine* transformations. In this article, we study the much larger class of continuous Minkowski valuations which are *translation invariant* and $\mathrm{SO}(n)$ *equivariant*.

A map Φ from $\mathcal{K}^n$ to $\mathcal{K}^n$ (or $\mathbb{R}$) is said to have *degree* i if $\Phi(\lambda K) = \lambda^i \Phi K$ for $K \in \mathcal{K}^n$ and $\lambda > 0$. In the case of Minkowski valuations which are translation invariant and $\mathrm{SO}(n)$ equivariant of degree i, Kiderlen [**36**] (for $i = 1$; building on previous results by Schneider [**57**]) and the first author [**60**] (for $i = n - 1$) were the first to obtain representations of these maps by spherical convolution operators. The following extension of their results to all remaining (non-trivial) degree cases (by a result of McMullen [**54**], only integer degrees $0 \leq i \leq n$ can occur) was established in [**61**] for *even* Minkowski valuations, that is, $\Phi(-K) = \Phi K$ for $K \in \mathcal{K}^n$:

Theorem 1 ([61]) *Let $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ be a smooth translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuation of degree $i \in \{1, \ldots, n-1\}$. If Φ_i is even, then there exists a smooth $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant measure μ on S^{n-1} such that for every $K \in \mathcal{K}^n$,*

$$h(\Phi_i K, \cdot) = \mathrm{vol}_i(K|\,\cdot\,) * \mu. \tag{2.1}$$

We note that Theorem 1 was stated in a different, but equivalent, form in [**61**] (cf. the discussion in Section 4 and the Appendix). The convolution in (2.1) is induced from the group $\mathrm{SO}(n)$ by identifying S^{n-1} and $\mathrm{Gr}_{i,n}$ with the homogeneous spaces $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ and $\mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$, respectively (see Section 3 for details).

The notion of *smooth* translation invariant Minkowski valuations which are $\mathrm{SO}(n)$ equivariant was introduced in [**61**] (extending the definition of smooth scalar valued valuations by Alesker [**4**]; see Section 4). Moreover, it was shown in [**61**] that every translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuation which is continuous and even can be approximated uniformly on compact subsets of $\mathcal{K}^n$ by smooth ones.

The invariant signed measure appearing in Theorem 1 is essentially a Crofton measure of a real valued valuation associated with the given even Minkowski valuation (see Section 4). This motivates the following definition.

Definition *Let $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ be an even Minkowski valuation of degree $i \in \{1, \ldots, n-1\}$. We call an $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant measure μ on S^{n-1} a* spherical Crofton measure *for Φ_i if (2.1) holds for every $K \in \mathcal{K}^n$.*

Note that an even Minkowski valuation which admits a spherical Crofton measure is continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant.

While Theorem 1 generalized the previously known representation results of translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations, basic questions concerning spherical Crofton measures, such as their uniqueness, remained open. With our first result we answer some of them:

Theorem 2 *If $\Phi_i : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is an even Minkowski valuation of degree $i \in \{1, \ldots, n-1\}$ which admits a spherical Crofton measure μ, then Φ_i determines μ uniquely. Moreover, there exists an $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant convex body $L \in \mathcal{K}^n$ such that*

$$h(L, u) = \int_{S^{n-1}} |u \cdot v| \, d\mu(v), \qquad u \in S^{n-1}. \tag{2.2}$$

The proof of Theorem 2 relies on tools from harmonic analysis and Klain's [**38**] embedding of even translation invariant real valued valuations in continuous functions on the Grassmannian (see Section 4). In fact, the support function in (2.2) is closely related to the Klain function of the real valued valuation associated with Φ_i.

Note that if a spherical Crofton measure μ exists for an even Minkowski valuation Φ_i, then, by its uniqueness and the injectivity of the spherical cosine transform, also the body L from (2.2) is uniquely determined by Φ_i. In [**61**] it was shown that, while not every *continuous* translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuation Φ_i admits a spherical Crofton measure, it is always possible to associate a convex body with Φ_i that uniquely determines the Minkowski valuation. We call this body the *Klain body* of Φ_i. If Φ_i admits a spherical Crofton measure μ, then its Klain body is (up to a factor) the body L given by (2.2). However, we recall in Section 4 a much simpler way to determine this body from the given Minkowski valuation.

Using Theorems 1 and 2, we establish in Section 5 the following new and much more elementary integral representation for smooth translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations:

Theorem 3 *Let $\Phi_i : \mathcal{K}^n \rightarrow \mathcal{K}^n$ be a smooth translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuation of degree $i \in \{1, \ldots, n-1\}$. If Φ_i is even, then there exists a unique even $g \in C^\infty((-1,1)) \cap C([-1,1])$ such that for every $K \in \mathcal{K}^n$,*

$$h(\Phi_i K, u) = \int_{S^{n-1}} g(u \cdot v) \, dS_i(K, v), \qquad u \in S^{n-1}. \tag{2.3}$$

Here, the measures $S_i(K,\cdot), 1 \le i \le n-1$, are the area measures of order i of $K \in \mathcal{K}^n$ (see Section 4 for their definition).

Now, given three different ways to represent a smooth translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuation (namely, spherical Crofton measures, Klain bodies, and the *generating functions* on $[-1,1]$ from Theorem 3), a natural question to ask is how to convert one into another. A partial answer is already provided by Theorem 2: The Klain body can be obtained from the spherical Crofton measure by the spherical cosine transform. In Section 5, we derive explicit formulas involving Radon transforms on Grassmannians for all the remaining cases.

In the final section we explore the action of Alesker's Hard Lefschetz operators on translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations. These operators, one being a derivation and the other an integration operator, have been introduced only recently and have since played an important role in what is now called algebraic integral geometry (see, e.g., [**4**, **16**]). The Hard Lefschetz Theorem for them (see [**4**, **5**, **7**, **14**]) is a fundamental theorem in the theory of translation invariant *scalar valued* valuations. The authors feel that the Hard Lefschetz operators will play a similarly important role in the theory of convex body valued valuations, in particular, in connection with geometric inequalities (see [**11**, **55**] for first results in this direction). A critical tool in our investigations is a Fourier type transform, introduced for translation invariant scalar valued valuations by Alesker [**7**], that connects the two Hard Lefschetz operators.

3. Intertwining transforms on Grassmannians

In the following we recall the notion of convolution of functions on the homogeneous spaces $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ and $\mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ as well as basic facts about spherical functions on Grassmannians. At the end of this section we establish two critical auxiliary results which are needed in the proofs of our main results. As a reference for this section we recommend the book [**63**] by Takeuchi and the article [**28**] by Grinberg and Zhang.

In order to simplify the exposition we first consider a general compact Lie group G and a closed subgroup H of G (although we only need the cases where $G = \mathrm{SO}(n)$ and H is either $\mathrm{SO}(n-1)$ or $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$).

Let $C(G)$ denote the space of continuous functions on G with the topology of uniform convergence. For $f, g \in C(G)$, the convolution $f * g \in C(G)$ is defined by

$$(f * g)(\eta) = \int_G f(\eta\vartheta^{-1})g(\vartheta)\, d\vartheta = \int_G f(\vartheta)g(\vartheta^{-1}\eta)\, d\vartheta,$$

where integration is with respect to the Haar probability measure on G.

For $f \in C(G)$ and a measure μ on G, the convolutions $f * \mu \in C(G)$ and $\mu * f \in C(G)$ are defined by

$$(f * \mu)(\eta) = \int_G f(\eta\vartheta^{-1})\, d\mu(\vartheta), \qquad (\mu * f)(\eta) = \int_G f(\vartheta^{-1}\eta)\, d\mu(\vartheta). \quad (3.1)$$

This definition continuously extends the convolution of functions. Moreover, it follows from (3.1) that $f * \mu$ and $\mu * f$ are smooth if $f \in C^\infty(G)$.

For $\vartheta \in G$, we denote the left and right translations of $f \in C(G)$ by

$$(l_\vartheta f)(\eta) = f(\vartheta^{-1}\eta), \qquad (r_\vartheta f)(\eta) = f(\eta\vartheta).$$

For a measure μ on G, we define $l_\vartheta\mu$ and $r_\vartheta\mu$ as the image measures of μ under the left and right multiplication by ϑ in G, respectively. We will also often write $\vartheta f := l_\vartheta f$ and $\vartheta\mu := l_\vartheta\mu$ for the left translation of $f \in C(G)$ or a measure μ on G, respectively.

We emphasize that, if $f \in C(G)$ and μ is a measure on G, then, by (3.1),

$$(l_\vartheta f) * \mu = l_\vartheta(f * \mu) \qquad \text{and} \qquad \mu * (r_\vartheta f) = r_\vartheta(\mu * f). \quad (3.2)$$

for every $\vartheta \in G$. Thus, the convolution from the right gives rise to operators on $C(G)$ which intertwine left translations and the convolution from the left gives rise to operators which intertwine right translations.

For $f \in C(G)$, the function $\widehat{f} \in C(G)$ is defined by

$$\widehat{f}(\vartheta) = f(\vartheta^{-1}).$$

For a measure μ on G, we define the measure $\widehat{\mu}$ by

$$\int_G f(\vartheta)\, d\widehat{\mu}(\vartheta) = \int_G f(\vartheta^{-1})\, d\mu(\vartheta), \qquad f \in C(G).$$

From (3.1) it follows that for $f, g \in C(G)$ and a measure σ on G,

$$\int_G f(\vartheta)(g * \sigma)(\vartheta)\, d\vartheta = \int_G (f * \widehat{\sigma})(\vartheta)g(\vartheta)\, d\vartheta. \quad (3.3)$$

Therefore, it is consistent to define the convolution $\mu * \sigma$ of two measures μ, σ on G by

$$\int_G f(\vartheta)\, d(\mu * \sigma)(\vartheta) = \int_G (f * \widehat{\sigma})(\vartheta)\, d\mu(\vartheta) = \int_G (\widehat{\mu} * f)(\vartheta)\, d\sigma(\vartheta)$$

for every $f \in C(G)$. The convolution of functions and measures on G, thus defined, is easily seen to be associative but in general not commutative. If μ, σ are measures on G, then

$$\widehat{\mu * \sigma} = \widehat{\sigma} * \widehat{\mu}. \tag{3.4}$$

We now define convolutions between functions and measures on the homogeneous spaces

$$S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1) \quad \text{and} \quad \mathrm{Gr}_{i,n} = \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i)) \tag{3.5}$$

by identifying the space $C(G/H)$ of continuous functions on the homogeneous space G/H with the closed subspace of $C(G)$ of all functions which are right H-invariant, that is, $r_\vartheta f = f$ for every $\vartheta \in H$. Similarly, we identify measures on G/H with right H-invariant measures on G. For a detailed description of the one-to-one correspondence between functions and measures on G/H and right H-invariant functions and measures on G we refer to [**28**] or [**61**].

If, for example, $f \in C(G/H)$ and μ is a measure on G/H, then, by (3.2), the convolution $f * \mu$ satisfies

$$r_\vartheta(f * \mu) = f * (r_\vartheta \mu) = f * \mu \tag{3.6}$$

for every $\vartheta \in G$. Thus $f * \mu$ can again be identified with a continuous function on G/H. In the same way we can define convolutions between functions and measures on different homogeneous spaces: Let H_1, H_2 be two closed subgroups of G. If, say, $f \in C(G/H_1)$ and $g \in C(G/H_2)$, then, by (3.2), $f * g$ defines a continuous right H_2-invariant function on G and, thus, can be identified with a continuous function on G/H_2.

Let $\pi : G \to G/H$ be the canonical projection and write $\pi(\vartheta) = \bar{\vartheta}$. If $e \in G$ denotes the identity element, then H is the stabilizer in G of $\bar{e} \in G/H$ and we have $\bar{\vartheta} = \vartheta\bar{e}$ for every $\vartheta \in G$.

If $\delta_{\bar{e}}$ denotes the Dirac measure on G/H, then it is not difficult to show that for $f \in C(G)$,

$$f * \delta_{\bar{e}} = \int_H r_\vartheta f\, d\vartheta. \tag{3.7}$$

Thus, $f * \delta_{\bar{e}}$ is right H-invariant for every $f \in C(G)$ and $\delta_{\bar{e}}$ is the unique rightneutral element for the convolution of functions and measures on G/H. We also note that

$$\delta_{\bar{e}} * f = \int_H \vartheta f \, d\vartheta \tag{3.8}$$

is left H-invariant for every $f \in C(G)$.

We call a left H-invariant function or measure on G/H (or, equivalently, an H-biinvariant function or measure on G) *zonal*. Zonal functions (and measures) on G/H play an essential role with respect to convolutions: If $f, g \in C(G/H)$, then, by (3.7) and (3.8),

$$f * g = (f * \delta_{\bar{e}}) * g = f * (\delta_{\bar{e}} * g). \tag{3.9}$$

Hence, for convolutions from the right on G/H, it is sufficient to consider zonal functions and measures. Note that if $f \in C(G)$ is H-biinvariant (or, equivalently, $f \in C(G/H)$ is zonal), then the function $\widehat{f} \in C(G)$ is also H-biinvariant and, thus, zonal.

A different consequence of the identifications (3.5) which we will use frequently is the following: If $h \in C(S^{n-1})$ is $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant, then $\widehat{h} \in C(\mathrm{Gr}_{i,n})$ is $\mathrm{SO}(n-1)$ invariant and, vice versa, if $f \in C(\mathrm{Gr}_{i,n})$ is $\mathrm{SO}(n-1)$ invariant, then $\widehat{f} \in C(S^{n-1})$ is $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant.

The following examples of convolution transforms will play a critical role in the proof of our main theorems.

Examples:

(a) Suppose that $1 \leq i \neq j \leq n-1$ and let $F \in \mathrm{Gr}_{j,n}$. We denote by $\mathrm{Gr}_{i,n}^F$ the submanifold of $\mathrm{Gr}_{i,n}$ which comprises of all $E \in \mathrm{Gr}_{i,n}$ that contain (respectively, are contained in) F. The *Radon transform* $R_{i,j} : L^2(\mathrm{Gr}_{i,n}) \to L^2(\mathrm{Gr}_{j,n})$ is defined by

$$(R_{i,j} f)(F) = \int_{\mathrm{Gr}_{i,n}^F} f(E) \, d\nu_i^F(E),$$

where ν_i^F is the unique invariant probability measure on $\mathrm{Gr}_{i,n}^F$. It is well known that the Radon transform is a continuous linear operator and that $R_{j,i}$ is the adjoint of $R_{i,j}$, that is,

$$\int_{\mathrm{Gr}_{j,n}} (R_{i,j} f)(F) g(F) \, dF = \int_{\mathrm{Gr}_{i,n}} f(E)(R_{j,i} g)(E) \, dE$$

for every $f \in L^2(\mathrm{Gr}_{i,n})$ and $g \in L^2(\mathrm{Gr}_{j,n})$.

Using the last observation, one can extend the Radon transform to the space of measures on $\mathrm{Gr}_{i,n}$ by

$$\int_{\mathrm{Gr}_{j,n}} f(F)\,d(R_{i,j}\mu)(F) = \int_{\mathrm{Gr}_{i,n}} (R_{j,i}f)(E)\,d\mu(E)$$

for $f \in C(\mathrm{Gr}_{j,n})$. For $g \in L^2(\mathrm{Gr}_{i,n})$, we write $g^\perp$ for the function in $L^2(\mathrm{Gr}_{n-i,n})$ defined by $g^\perp(E) = g(E^\perp)$. Then

$$(R_{i,j}f)^\perp = R_{n-i,n-j}f^\perp \tag{3.10}$$

and, for $1 \le i < j < k \le n-1$, we have

$$R_{i,k} = R_{j,k} \circ R_{i,j} \qquad \text{and} \qquad R_{k,i} = R_{j,i} \circ R_{k,j}.$$

If $1 \le i < j \le n-1$ and $\lambda_{i,j}$ denotes the probability measure on $\mathrm{Gr}_{j,n}$ which is uniformly concentrated on the submanifold

$$\{\vartheta\bar{e} \in \mathrm{Gr}_{j,n} : \vartheta \in \mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))\},$$

then (see, e.g., [**28**])

$$R_{i,j}f = f * \lambda_{i,j} \qquad \text{and} \qquad R_{j,i}g = g * \widehat{\lambda}_{i,j} \tag{3.11}$$

for every $f \in L^2(\mathrm{Gr}_{i,n})$ and $g \in L^2(\mathrm{Gr}_{j,n})$. In particular, the Radon transform intertwines the natural group action (by left translation) of $\mathrm{SO}(n)$ and maps smooth functions to smooth ones, that is,

$$R_{i,j} : C^\infty(\mathrm{Gr}_{i,n}) \to C^\infty(\mathrm{Gr}_{j,n}).$$

(b) For two subspaces $E, F \in \mathrm{Gr}_{i,n}$, where $1 \le i \le n-1$, the cosine of the angle between E and F is defined by

$$|\cos(E,F)| = \mathrm{vol}_i(Q|E),$$

where Q is an arbitrary subset of F with $\mathrm{vol}_i(Q) = 1$. (This definition is independent of the choice of $Q \subseteq F$.) It is not difficult to show that

$$|\cos(E,F)| = |\cos(F,E)| \qquad \text{and} \qquad |\cos(E^\perp,F^\perp)| = |\cos(E,F)|.$$

The continuous linear operator $C_i : L^2(\mathrm{Gr}_i) \to L^2(\mathrm{Gr}_i)$ defined by

$$(C_i f)(F) = \int_{\mathrm{Gr}_{i,n}} |\cos(E,F)| f(E)\,dE$$

is called the *cosine transform*. Here integration is with respect to the Haar probability measure on $\mathrm{Gr}_{i,n}$.

It is easy to see that also the cosine transform is a continuous linear operator and that it is self-adjoint, that is,

$$\int_{\mathrm{Gr}_{i,n}} (C_i f)(E) g(E)\, dE = \int_{\mathrm{Gr}_{i,n}} f(E)(C_i g)(E)\, dE \tag{3.12}$$

for all $g, f \in L^2(\mathrm{Gr}_{i,n})$. Based on (3.12), one defines the cosine transform of a measure μ on $\mathrm{Gr}_{i,n}$ by

$$\int_{\mathrm{Gr}_{i,n}} f(E)\, d(C_i\mu)(E) = \int_{\mathrm{Gr}_{i,n}} (C_i f)(E)\, d\mu(E)$$

for $f \in C(\mathrm{Gr}_{i,n})$. For all $f \in L^2(\mathrm{Gr}_{i,n})$, we have

$$(C_i f)^{\perp} = C_{n-i} f^{\perp} \tag{3.13}$$

and in [**25**] it was shown that for all $1 \leq i \neq j \leq n-1$,

$$R_{i,j} \circ C_i = \frac{i!(n-i)!\kappa_i \kappa_{n-i}}{j!(n-j)!\kappa_j \kappa_{n-j}}\, C_j \circ R_{i,j}, \tag{3.14}$$

where κ_i is the i-dimensional volume of the i-dimensional Euclidean unit ball. It is not difficult to show that

$$C_i f = f * |\cos(\bar{e}, \cdot)| \tag{3.15}$$

for every $f \in L^2(\mathrm{Gr}_{i,n})$. Thus, also the cosine transform intertwines the action of $\mathrm{SO}(n)$ and it follows that

$$C_i : C^{\infty}(\mathrm{Gr}_{i,n}) \to C^{\infty}(\mathrm{Gr}_{i,n}).$$

For $1 \leq i \leq n-1$, let $\mathbf{T}_i^{\infty}$ denote the image of $C^{\infty}(\mathrm{Gr}_{i,n})$ under the cosine transform. While C_i is not injective for $2 \leq i \leq n-2$ (see below), it follows from (3.12) that its restriction to $\mathbf{T}_i^{\infty}$ has trivial kernel. Moreover, from an application of the Casselman–Wallach theorem [**18**] to the main result of [**9**], Alesker [**4**, p. 73] deduced that

$$C_i(\mathbf{T}_i^{\infty}) = \mathbf{T}_i^{\infty}.$$

Since both the Radon and the cosine transform intertwine the action of $\mathrm{SO}(n)$, we need some background from harmonic analysis on Grassmannians to discuss their injectivity properties. More precisely, we require information on the decomposition of the space $L^2(\mathrm{Gr}_{i,n})$ into $\mathrm{SO}(n)$ irreducible subspaces.

Since $\mathrm{SO}(n)$ is a compact Lie group, all its irreducible representations are finite dimensional. Equivalence classes of irreducible complex representations of $\mathrm{SO}(n)$ are indexed by their highest weights (see, e.g., [**17**, p. 219]), which can be identified with $\lfloor n/2 \rfloor$-tuples of integers $(\lambda_1, \lambda_2, \dots, \lambda_{\lfloor n/2 \rfloor})$ such that

$$\begin{cases} \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\lfloor n/2 \rfloor} \geq 0 & \text{for odd } n, \\ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n/2-1} \geq |\lambda_{n/2}| & \text{for even } n. \end{cases} \tag{3.16}$$

We use Γ_λ to denote any isomorphic copy of an irreducible representation of $\mathrm{SO}(n)$ with highest weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\lfloor n/2 \rfloor})$.

Examples:

(a) The trivial one-dimensional representation of $\mathrm{SO}(n)$ corresponds to the $\mathrm{SO}(n)$ module $\Gamma_{(0,\dots,0)}$, while the standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ is isomorphic to $\Gamma_{(1,0,\dots,0)}$.

(b) The decomposition of $L^2(S^{n-1})$ into an orthogonal sum of $\mathrm{SO}(n)$ irreducible subspaces is given by

$$L^2(S^{n-1}) = \bigoplus_{k \geq 0} \Gamma_{(k,0,\dots,0)}. \tag{3.17}$$

It is well known that here the spaces $\Gamma_{(k,0,\dots,0)}$ are precisely the spaces of spherical harmonics of degree k in dimension n.

(c) For $1 \leq i \leq n-1$, the space $L^2(\mathrm{Gr}_{i,n})$ is a sum of orthogonal irreducible representations of $\mathrm{SO}(n)$ with highest weights $(\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying the following two additional conditions (see, e.g., [**40**, Theorem 8.49]):

$$\begin{cases} \lambda_j = 0 \ \text{ for all } j > \min\{i, n-i\}, \\ \lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor} \ \text{ are all even.} \end{cases} \tag{3.18}$$

Note that since both S^{n-1} and $\mathrm{Gr}_{i,n}$ are Riemannian symmetric spaces the respective decompositions of $L^2(S^{n-1})$ and $L^2(\mathrm{Gr}_{i,n})$ into $\mathrm{SO}(n)$ irreducible subspaces are *multiplicity free*. In fact, even more can be said. To this end let $\mathrm{GL}(V)$ denote the general linear group of a vector space V.

Definition (see [**63**, p. 14]) *Let G be a compact Lie group and let H be a closed subgroup of G. A representation $\rho : G \to \mathrm{GL}(V)$ is called* spherical *with respect to H if there exists a non-zero $v \in V$ which is invariant under H, that is, $\rho(\vartheta)v = v$ for every $\vartheta \in H$. Let V^H denote the subspace of V consisting of all H-invariant $v \in V$.*

For the following well-known facts we also refer to [**63**, p. 17]:

Theorem 3.1 *Let G be a compact Lie group and $H \subseteq G$ a closed subgroup.*

(i) Every subrepresentation of $L^2(G/H)$ is spherical with respect to H.

(ii) Every irreducible representation which is spherical with respect to H is isomorphic to a subrepresentation of $L^2(G/H)$.

(iii) If G/H is a Riemannian symmetric space, then $\dim V^H = 1$ for every irreducible representation which is spherical with respect to H.

In order to see how we will use Theorem 3.1, consider the subspace $L^2(\mathrm{Gr}_{i,n})^{\mathrm{SO}(n-1)}$ of $\mathrm{SO}(n-1)$ invariant functions in $L^2(\mathrm{Gr}_{i,n})$. By definition, every irreducible subspace of $L^2(\mathrm{Gr}_{i,n})$ which has non-trivial intersection with $L^2(\mathrm{Gr}_{i,n})^{\mathrm{SO}(n-1)}$ corresponds to a spherical subrepresentation with respect to $\mathrm{SO}(n-1)$. Thus, by Theorem 3.1 (ii), (3.17), and (3.18), we conclude that

$$L^2(\mathrm{Gr}_{i,n})^{\mathrm{SO}(n-1)} \subseteq \bigoplus_{k\geq 0} \Gamma_{(2k,0,\ldots,0)}. \tag{3.19}$$

Let G be a compact Lie group and let Γ be a (not necessarily irreducible) finite dimensional complex G module. Recall that the dual representation is defined on the dual space Γ^* by

$$(\vartheta\, u^*)(v) = u^*(\vartheta^{-1}v), \qquad \vartheta \in G, u^* \in \Gamma^*, v \in \Gamma,$$

and that Γ is called *self-dual* if Γ and Γ^* are isomorphic representations. The module Γ is called *real* if there exists a non-degenerate symmetric G invariant bilinear form on Γ. In particular, if Γ is real, then Γ is also self-dual.

Due to (3.4), the following auxiliary result is crucial to our investigations:

Lemma 3.2 *Let H_1 and H_2 be closed subgroups of $\mathrm{SO}(n)$ such that both $\mathrm{SO}(n)/H_1$ and $\mathrm{SO}(n)/H_2$ are Riemannian symmetric spaces and let*

$$L^2(\mathrm{SO}(n)/H_1) = \bigoplus_{\lambda} \Gamma_\lambda^{(1)} \qquad \textit{and} \qquad L^2(\mathrm{SO}(n)/H_2) = \bigoplus_{\xi} \Gamma_\xi^{(2)},$$

where both sums range over suitable equivalence classes of irreducible $\mathrm{SO}(n)$ representations. If $f \in \Gamma_\lambda^{(1)}$ is H_2 invariant and $\Gamma_\lambda^{(1)}$ is real, then $\widehat{f} \in \Gamma_\lambda^{(2)}$ and $\widehat{f}$ is H_1 invariant.

Proof. Let $L^2(\mathrm{SO}(n)) = \bigoplus_\lambda \Gamma(\lambda)$ be the decomposition of $L^2(\mathrm{SO}(n))$ into isotypical components (see, e.g., [**17**, p. 70]). Consider the action of the group $\mathrm{SO}(n) \times \mathrm{SO}(n)$ on $L^2(\mathrm{SO}(n))$ given by

$$((\vartheta, \eta) f)(\zeta) = f(\vartheta^{-1}\zeta\eta).$$

It is well known (see, e.g., [**63**, Theorem 1.1]) that

$$\Gamma(\lambda) \cong \Gamma_\lambda \otimes \Gamma_\lambda^*$$

as $\mathrm{SO}(n) \times \mathrm{SO}(n)$ modules. If Γ_λ is real, then $\Gamma_\lambda \cong \Gamma_\lambda^*$ and the isomorphism $\Theta : \Gamma_\lambda \otimes \Gamma_\lambda \to \Gamma(\lambda)$ is given by

$$\Theta(g \otimes h)(\vartheta) = \langle g, \vartheta h \rangle,$$

where $\langle\ ,\ \rangle$ denotes the non-degenerate, symmetric, and $\mathrm{SO}(n)$ invariant bilinear form on Γ_λ. Since

$$\Theta(g \otimes h)(\vartheta^{-1}) = \Theta(h \otimes g)(\vartheta),$$

it follows that $\widehat{f} \in \Gamma(\lambda)$ for every $f \in \Gamma(\lambda)$. But, since clearly,

$$\Gamma_\lambda^{(1)}, \Gamma_\lambda^{(2)} \subseteq \Gamma(\lambda),$$

we deduce that if $f \in \Gamma_\lambda^{(1)}$ is (left) H_2 invariant, then $\widehat{f}$ is right H_2 invariant and, thus, $\widehat{f} \in \Gamma_\lambda^{(2)}$. ∎

We return now to cosine and Radon transforms. Let $1 \leq i \neq j \leq n-1$ and let $\Gamma_\lambda \subseteq L^2(\mathrm{Gr}_{i,n})$ be an $\mathrm{SO}(n)$ irreducible subspace. Since both transforms $C_i : L^2(\mathrm{Gr}_{i,n}) \to L^2(\mathrm{Gr}_{i,n})$ and $R_{i,j} : L^2(\mathrm{Gr}_{i,n}) \to L^2(\mathrm{Gr}_{j,n})$ intertwine the action of $\mathrm{SO}(n)$, the spaces

$$C_i\Gamma_\lambda \subseteq L^2(\mathrm{Gr}_{i,n}) \qquad \text{and} \qquad R_{i,j}\Gamma_\lambda \subseteq L^2(\mathrm{Gr}_{j,n})$$

are also $\mathrm{SO}(n)$ irreducible subspaces of $L^2(\mathrm{Gr}_{i,n})$ and $L^2(\mathrm{Gr}_{j,n})$, respectively. By Schur's Lemma, these subspaces are either trivial or we must have

$$C_i\Gamma_\lambda = \Gamma_\lambda \qquad \text{and} \qquad R_{i,j}\Gamma_\lambda \cong \Gamma_\lambda.$$

Therefore the cosine transform C_i must act as a multiple of the identity on Γ_λ, that is, there exist *multipliers* $\mathrm{c}_\lambda^i \in \mathbb{R}$ such that for every $f \in \Gamma_\lambda$,

$$C_i f = \mathrm{c}_\lambda^i\, f.$$

Problems concerning injectivity of C_i are now reduced to questions as to which of these multipliers are zero.

By well known classical facts (see, e.g., [**29**]), the *spherical* cosine transform $C_1 = C_{n-1}$ and the *spherical* Radon transform $R_{1,n-1} = R_{n-1,1}$ are injective on $L^2(\mathrm{Gr}_1)$ and $L^2(\mathrm{Gr}_{n-1})$, respectively. In particular,

$$\mathrm{c}^1_{(2k,0,\dots,0)} \neq 0 \tag{3.20}$$

for every $k \in \mathbb{N}$. Moreover, when restricted to smooth functions these spherical transforms are *bijective*.

For general $1 < i < n-1$, the cosine transform C_i is not injective as was first shown by Goodey and Howard [**22**] (see also, [**9**]). However, of particular importance for us is the fact that not only (3.20) holds, but that for all $1 \leq i \leq n-1$ and for every $k \in \mathbb{N}$,

$$\mathrm{c}^i_{(2k,0,\dots,0)} = \frac{n\kappa_i\kappa_{n-i}}{2\kappa_{n-1}}\binom{n}{i}^{-1}\mathrm{c}^1_{(2k,0,\dots,0)} \neq 0. \tag{3.21}$$

This relation was first obtained by Goodey and Zhang [**25**, Lemma 2.1]. Using (3.21) one can show that the restriction of C_i to the subspace of $C^\infty(\mathrm{Gr}_{i,n})$ defined by

$$\mathrm{cl}_{C^\infty}\bigoplus_{k\geq 0}\Gamma_{(2k,0,\dots,0)} \tag{3.22}$$

is *bijective*. Here cl_{C^∞} denotes the closure in the C^∞ topology, .

If $1 \leq i < j \leq n-1$, then $R_{i,j}$ is injective if and only if $i+j \leq n$, whereas if $i > j$, then $R_{i,j}$ is injective if and only if $i+j \geq n$, see Grinberg [**27**]. Moreover, it was shown in [**25**] that for all $1 \leq i \neq j \leq n-1$ also the restriction of the Radon transform $R_{i,j}$ to the subspace defined by (3.22) is bijective.

The following consequences of these facts and Lemma 3.2 will be the key ingredients in the proof of Theorems 1 and 2.

Lemma 3.3 *Suppose that* $1 \leq i \leq n-1$.

(i) If μ *is an* $\mathrm{S}(\mathrm{O}(i)\times\mathrm{O}(n-i))$ *invariant measure on* S^{n-1}*, then*

$$\left(\widehat{C_i\widehat{\mu}}\right)(u) = \frac{n\kappa_i\kappa_{n-i}}{2\kappa_{n-1}}\binom{n}{i}^{-1}\int_{S^{n-1}}|u\cdot v|\,d\mu(v), \qquad u \in S^{n-1}.$$

(ii) If $f \in C^\infty(S^{n-1})$ *is* $\mathrm{S}(\mathrm{O}(i)\times\mathrm{O}(n-i))$ *invariant, then there exists a unique zonal* $g \in C^\infty(S^{n-1})$ *such that*

$$f = \lambda_{i,n-1} * g. \tag{3.23}$$

Proof. In order to prove (i) we may assume that μ is absolutely continuous with respect to the invariant probability measure on S^{n-1}. Let $h \in C(S^{n-1})$ be the $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant density of μ and let $h = \sum_{k\geq 0} h_k$ be the decomposition of h into spherical harmonics, that is, $h_k \in \Gamma_{(k,0,\ldots,0)}$. Since h is $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant, so is each h_k. Clearly, $\widehat{h} \in C(\mathrm{Gr}_{i,n})^{\mathrm{SO}(n-1)}$ and thus, by (3.19) we have in fact, $h = \sum_{k\geq 0} h_{2k}$, that is, h is even.

It is well known (cf. [**17**, p.292]) that each $\Gamma_{(k,0,\ldots,0)}$ is a real $\mathrm{SO}(n)$ module. Therefore, it follows from Lemma 3.2 that $\widehat{h}_{2k} \in \Gamma_{(2k,0,\ldots,0)}$ for every $k \in \mathbb{N}$. Hence, by (3.21) we obtain

$$\widehat{C_i\widehat{h}} = \sum_{k\geq 0} \widehat{C_i\widehat{h}_{2k}} = \frac{n\kappa_i\kappa_{n-i}}{2\kappa_{n-1}}\binom{n}{i}^{-1}\sum_{k\geq 0} C_1 h_{2k} = \frac{n\kappa_i\kappa_{n-i}}{2\kappa_{n-1}}\binom{n}{i}^{-1} C_1 h,$$

which is precisely the claim from (i).

In order to prove (ii), first note that for any zonal $g \in C^\infty(S^{n-1})$ we have $g = \widehat{g}$ (cf. [**60**]). Thus, by (3.4) and (3.11), relation (3.23) is equivalent to

$$\widehat{f} = g * \widehat{\lambda}_{i,n-1} = R_{n-1,i}\, g.$$

Since $\widehat{f} \in C^\infty(\mathrm{Gr}_{i,n})^{\mathrm{SO}(n-1)}$, it follows from (3.19) that $\widehat{f}$ is contained in the subspace of $C^\infty(\mathrm{Gr}_{i,n})$ defined in (3.22). Since the Radon transform $R_{n-1,i}$ is an $\mathrm{SO}(n)$ intertwining bijection from even smooth functions on S^{n-1} to this subspace, statement (ii) follows. ∎

4. Even translation invariant valuations

In this section we collect basic facts about convex bodies (see, e.g., [**58**]) and the background material from the theory of *even* translation invariant scalar and convex body valued valuations. In particular, we recall Klain's embedding of even translation invariant valuations into functions on the Grassmannian, the definitions of Alesker's Hard Lefschetz operators and his Fourier type transform on translation invariant valuations.

The definition of the support function $h(K, u) = \max\{u \cdot x : x \in K\}$, $u \in S^{n-1}$, of a convex body $K \in \mathcal{K}^n$, implies that $h(\vartheta K, u) = h(K, \vartheta^{-1}u)$ for every $u \in S^{n-1}$ and $\vartheta \in \mathrm{O}(n)$. For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$, the support function of the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is given by

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot).$$

The *surface area measure* $S_{n-1}(K,\cdot)$ of a convex body K is defined for Borel sets $\omega \subseteq S^{n-1}$, as the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω. If the body $K \in \mathcal{K}^n$ has non-empty interior, then K is determined up to translations by its surface area measure.

Let B denote the Euclidean unit ball in $\mathbb{R}^n$. The surface area measure of $K \in \mathcal{K}^n$ satisfies the Steiner-type formula

$$S_{n-1}(K+\varepsilon B,\cdot) = \sum_{i=0}^{n-1} \varepsilon^{n-1-i}\binom{n-1}{i} S_i(K,\cdot). \tag{4.1}$$

The measure $S_i(K,\cdot)$, $0 \leq i \leq n-1$, is called the *area measure of order* i of $K \in \mathcal{K}^n$. The relation $S_i(\lambda K,\cdot) = \lambda^i S_i(K,\cdot)$ holds for all $K \in \mathcal{K}^n$ and every $\lambda > 0$. For $\vartheta \in \mathrm{O}(n)$, we have $S_i(\vartheta K,\cdot) = \vartheta S_i(K,\cdot)$.

The area measure $S_i(K,\cdot)$ is (up to normalization) a localization of the ith *intrinsic volume* $V_i(K)$ of a convex body K. More precisely,

$$S_i(K,S^{n-1}) = n\binom{n}{i}^{-1}\kappa_{n-i}V_i(K).$$

For $i \in \{1,\ldots,n-1\}$, a special case of the Cauchy–Kubota formulas implies that

$$R_{i,n-1}\mathrm{vol}_i(K|\,\cdot\,) = \frac{\kappa_i}{4\kappa_{n-1}}\, C_{n-1}\left(S_i(K,\cdot) + S_i(-K,\cdot)\right), \tag{4.2}$$

where we identify the even measure $S_i(K,\cdot) + S_i(-K,\cdot)$ on S^{n-1} with a measure on $\mathrm{Gr}_{n-1,n}$ of the same total mass.

Intrinsic volumes and area measures are both valuations. In the case of intrinsic volumes we have scalar-valued valuations; in the case of area measures we have valuations with values in the set of Borel measures on S^{n-1}.

If G is a group of affine transformations on $\mathbb{R}^n$, a valuation ϕ is called G-invariant if $\phi(gK) = \phi(K)$ for all $K \in \mathcal{K}^n$ and every $g \in G$. We denote the vector space of continuous translation invariant scalar-valued valuations by $\mathbf{Val}$. A seminal result in the structure theory of translation invariant valuations was obtained by McMullen [**54**], who showed that

$$\mathbf{Val} = \bigoplus_{0\leq i\leq n} \mathbf{Val}_i = \bigoplus_{0\leq i\leq n}\left(\mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-\right), \tag{4.3}$$

where $\mathbf{Val}_i^+ \subseteq \mathbf{Val}$ denotes the subspace of *even* valuations (homogeneous) of degree i, $\mathbf{Val}_i^-$ denotes the subspace of *odd* valuations of degree i, and $\mathbf{Val}_i = \mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-$ is the subspace of valuations of degree i.

From (4.3), it follows easily that the space **Val** becomes a Banach space, when endowed with the norm $\|\phi\| = \sup\{|\phi(K)| : K \subseteq B\}$. The general linear group $\mathrm{GL}(n)$ acts on this Banach space in a natural way: For every $A \in \mathrm{GL}(n)$ and every $K \in \mathcal{K}^n$,

$$(A\phi)(K) = \phi(A^{-1}K), \qquad \phi \in \mathbf{Val}.$$

Assume that $1 \leq i \leq n-1$ and define for any finite Borel measure μ on $\mathrm{Gr}_{i,n}$ an even valuation $\mathrm{Cr}_i\mu \in \mathbf{Val}_i^+$ by

$$(\mathrm{Cr}_i\mu)(K) = \int_{\mathrm{Gr}_{i,n}} \mathrm{vol}_i(K|E)\, d\mu(E).$$

It follows from a deep result of Alesker [**3**] that the image of the map Cr_i is dense in $\mathbf{Val}_i^+$. This leads to the following

Definition *A finite Borel measure μ on $\mathrm{Gr}_{i,n}$, $1 \leq i \leq n-1$, is called a* Crofton measure *for the valuation $\phi \in \mathbf{Val}_i^+$ if $\mathrm{Cr}_i\mu = \phi$.*

Using the notion of smooth vectors of a representation (see [**64**, p. 31]) and an imbedding of the space $\mathbf{Val}_i^+$ in $C(\mathrm{Gr}_{i,n})$ by Klain, a more precise description of valuations admitting a Crofton measure is possible.

Definition *A valuation $\phi \in \mathbf{Val}$ is called smooth if the map $\mathrm{GL}(n) \to \mathbf{Val}$ defined by $A \mapsto A\phi$ is infinitely differentiable.*

We write $\mathbf{Val}^\infty$ for the space of smooth translation invariant valuations, and we use $\mathbf{Val}_i^\infty$ and $\mathbf{Val}_i^{\pm,\infty}$ for the subspaces of smooth valuations in $\mathbf{Val}_i$ and $\mathbf{Val}_i^\pm$, respectively. It is well known (cf. [**64**, p. 32]) that the set of smooth valuations $\mathbf{Val}_i^{\pm,\infty}$ is a dense $\mathrm{GL}(n)$ invariant subspace of $\mathbf{Val}_i^\pm$.

For $1 \leq i \leq n-1$, the *Klain map* $\mathrm{Kl}_i : \mathbf{Val}_i^+ \to C(\mathrm{Gr}_{i,n})$, $\phi \mapsto \mathrm{Kl}_i\phi$, is defined as follows: For $\phi \in \mathbf{Val}_i^+$ and every $E \in \mathrm{Gr}_{i,n}$, consider the restriction ϕ_E of ϕ to convex bodies in E. This is a continuous translation invariant valuation of degree i in E. Thus, by a result of Hadwiger [**34**, p. 79], $\phi_E = (\mathrm{Kl}_i\phi)(E)\,\mathrm{vol}_i$, where $(\mathrm{Kl}_i\phi)(E)$ is a constant depending only on E. This gives rise to a continuous function $\mathrm{Kl}_i\phi \in C(\mathrm{Gr}_{i,n})$, called the *Klain function* of the valuation ϕ. By an important result of Klain [**38**], the Klain map Kl_i is injective for every $i \in \{1, \ldots, n-1\}$.

Consider now the restriction of the map Cr_i, $1 \leq i \leq n-1$, to smooth functions:

$$(\mathrm{Cr}_i f)(K) = \int_{\mathrm{Gr}_{i,n}} \mathrm{vol}_i(K|E) f(E)\, dE, \qquad f \in C^\infty(\mathrm{Gr}_{i,n}).$$

It is not difficult to see that the valuation $\mathrm{Cr}_i f$ is smooth, i.e., $\mathrm{Cr}_i f \in \mathbf{Val}_i^{+,\infty}$. Moreover, if $F \in \mathrm{Gr}_{i,n}$, then, for any $f \in C^\infty(\mathrm{Gr}_{i,n})$ and convex body $K \subseteq F$,

$$(\mathrm{Cr}_i f)(K) = \mathrm{vol}_i(K) \int_{\mathrm{Gr}_{i,n}} |\cos(E,F)| f(E)\, dE.$$

Thus, the Klain function of the valuation $\mathrm{Cr}_i f$ is equal to the cosine transform $C_i f$ of f. From an application of the Casselman–Wallach theorem [**18**] to the main result of [**9**], Alesker [**4**, p. 73] deduced the following:

Theorem 4.1 (Alesker and Bernstein [**9**], Alesker [**4**]) *Let $1 \leq i \leq n-1$. The image of the Klain map $\mathrm{Kl}_i : \mathbf{Val}_i^{+,\infty} \to C^\infty(\mathrm{Gr}_{i,n})$ coincides with the image of the cosine transform $C_i : C^\infty(\mathrm{Gr}_{i,n}) \to C^\infty(\mathrm{Gr}_{i,n})$. Moreover, for any valuation $\phi \in \mathbf{Val}_i^{+,\infty}$, there exists a unique smooth measure $\mu \in \mathbf{T}_i^\infty$ such that μ is a Crofton measure for ϕ.*

We turn now to Alesker's Hard Lefschetz operators. It is well known that McMullen's decomposition (4.3) of $\mathbf{Val}$ into subspaces of homogeneous valuations implies that for every $\phi \in \mathbf{Val}$ and all $K \in \mathcal{K}^n$ the Steiner type formula

$$\phi(K + rB) = \sum_{j=0}^{n} r^{n-j} \phi^{(j)}(K), \tag{4.4}$$

where $\phi^{(j)} \in \mathbf{Val}$ for $0 \leq j \leq n$, holds for every $r \geq 0$. Note that $\phi^{(j)}$ is in general not homogeneous.

In turn, (4.4) gives rise to a derivation operator $\Lambda : \mathbf{Val} \to \mathbf{Val}$ defined by

$$(\Lambda\phi)(K) = \left.\frac{d}{dt}\right|_{t=0} \phi(K + tB).$$

Note that Λ commutes with the action of $\mathrm{O}(n)$ and that it preserves parity. Moreover, if $\phi \in \mathbf{Val}_i$, then $\Lambda\phi \in \mathbf{Val}_{i-1}$.

The importance of the operator Λ becomes evident from the following Hard Lefschetz type theorem established for even valuations by Alesker [**4**] and for general valuations by Bernig and Bröcker [**14**]:

Theorem 4.2 (Alesker [**4**], Bernig and Bröcker [**14**]) *Let $1 \leq i \leq n$.*

(i) The operator $\Lambda : \mathbf{Val}_i^\infty \to \mathbf{Val}_{i-1}^\infty$ is injective if $2i-1 \geq n$ and surjective if $2i - 1 \leq n$.

(ii) If $2i \geq n$, then $\Lambda^{2i-n} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{n-i}^\infty$ is an isomorphism.

More recently, a dual version of Theorem 4.2 was established by Alesker, in [**5**] for even valuations and in [**7**] for general ones. There, the derivation operator Λ is replaced by an integral operator $\mathfrak{L} : \mathbf{Val} \to \mathbf{Val}$ defined by

$$(\mathfrak{L}\phi)(K) = \int_{\mathrm{AGr}_{n-1,n}} \phi(K \cap E)\, d\sigma_{n-1}(E). \tag{4.5}$$

Here and in the following $\mathrm{AGr}_{i,n}$, denotes the affine Grassmannian of i planes in $\mathbb{R}^n$ and σ_i is the invariant measure on $\mathrm{AGr}_{i,n}$ normalized such that the set of planes having non-empty intersection with the Euclidean unit ball has measure

$$\begin{bmatrix} n \\ i \end{bmatrix} \kappa_{n-i} := \binom{n}{i} \frac{\kappa_n}{\kappa_i}.$$

The operator $\mathfrak{L}$ was originally introduced by Alesker in a different way in connection with a newly discovered product structure on the space **Val**. Only in [**12**] Bernig showed that the original definition coincides with (4.5). We also note that $\mathfrak{L}$ commutes with the action of $\mathrm{O}(n)$ and that it preserves parity. Moreover, if $\phi \in \mathbf{Val}_i$, then $\mathfrak{L}\phi \in \mathbf{Val}_{i+1}$.

The Hard Lefschetz type theorem for the operator $\mathfrak{L}$ is

Theorem 4.3 (Alesker [**7**]) *Let $0 \leq i \leq n-1$.*

(i) The operator $\mathfrak{L} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{i+1}^\infty$ is injective if $2i+1 \leq n$ and surjective if $2i+1 \geq n$.

(ii) If $2i \leq n$, then $\mathfrak{L}^{n-2i} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{n-i}^\infty$ is an isomorphism.

The dual nature of Theorems 4.2 and 4.3 was the basis not only for the proof of Theorem 4.3 but for the discovery of another fundamental duality transform, now called the Alesker–Fourier transform (which shares various formal similarities with the classical Fourier transform). In fact, it was shown by Bernig and Fu [**15**] for even valuations and recently by Alesker [**7**] for general valuations that both versions of the Hard Lefschetz Theorem are equivalent via the Alesker–Fourier transform.

We will define the Alesker–Fourier transform $\mathbb{F} : \mathbf{Val}^\infty \to \mathbf{Val}^\infty$ only for even valuations and refer to the article [**7**] for the odd case which is much more involved and will not be needed in the following: If $\phi \in \mathbf{Val}_i^{+,\infty}$, $1 \leq i \leq n-1$, then $\mathbb{F}\phi \in \mathbf{Val}_{n-i}^{+,\infty}$ is the valuation whose Klain function is given by

$$\mathrm{Kl}_{n-i}(\mathbb{F}\phi) = (\mathrm{Kl}_i\phi)^\perp. \tag{4.6}$$

In order to see that the linear operator $\mathbb{F}$ is well defined, use (3.13) and Theorem 4.1. Clearly, $\mathbb{F}$ is an involution that commutes with the action of $\mathrm{O}(n)$. The derivation operator Λ and the integral operator $\mathfrak{L}$ are related by

$$\mathbb{F} \circ \Lambda = 2\,\mathfrak{L} \circ \mathbb{F}. \tag{4.7}$$

This was first observed by Bernig and Fu [**15**] for even valuations and proved in general by Alesker in [**7**].

We conclude this section by collecting previously obtained results on translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations needed in the next section. To this end we denote by $\mathbf{MVal}^{(+)}$ the set of continuous translation invariant (even) Minkowski valuations, and we write $\mathbf{MVal}_i^{(+)}$, $0 \leq i \leq n$, for its subset of all (even) Minkowski valuations of degree i.

A Minkowski valuation $\Phi \in \mathbf{MVal}$ is called $\mathrm{SO}(n)$ *equivariant* if for all $K \in \mathcal{K}^n$ and every $\vartheta \in \mathrm{SO}(n)$,

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

For a number of well-known examples of Minkowski valuations that are $\mathrm{SO}(n)$ equivariant we refer to the articles [**36**, **42**, **61**] and the next section.

Recently, Parapatits and the second author [**56**] have shown that, in general, a decomposition of a Minkowski valuation $\Phi \in \mathbf{MVal}$ into a sum of homogeneous Minkowski valuations is not possible (cf. also [**55**]). However, from an application of McMullen's decomposition (4.3) it is possible to deduce the following: If $\Phi \in \mathbf{MVal}^{(+)}$, then there exist convex bodies $L_0, L_n \in \mathcal{K}^n$ and for every $K \in \mathcal{K}^n$, (even) functions $g_i(K, \cdot) \in C(S^{n-1})$ such that

$$h(\Phi K, \cdot) = h(L_0, \cdot) + \sum_{i=1}^{n-1} g_i(K, \cdot) + V(K) h(L_n, \cdot). \tag{4.8}$$

Moreover, for each $i \in \{1, \dots, n-1\}$:

(i) The map $K \mapsto g_i(K, \cdot)$ is a continuous translation invariant valuation of degree i.

(ii) If Φ is $\mathrm{SO}(n)$ equivariant, then L_0 and L_n are Euclidean balls and for every $\vartheta \in \mathrm{SO}(n)$ and $K \in \mathcal{K}^n$ we have $g_i(\vartheta K, u) = g_i(K, \vartheta^{-1} u)$.

In general, the functions $g_i(K, \cdot)$ need not be support functions (see [**56**]). However, it is an important open problem whether for $\mathrm{SO}(n)$ equivariant $\Phi \in \mathbf{MVal}$, the $g_i(K, \cdot)$ are support functions for every $i \in \{1, \dots, n-1\}$.

If $\Phi \in \mathbf{MVal}$ is $\mathrm{SO}(n)$ equivariant, then for $\bar{\vartheta} \in S^{n-1}$ we have

$$h(\Phi K, \bar{\vartheta}) = h(\Phi K, \vartheta \bar{e}) = h(\vartheta^{-1}(\Phi K), \bar{e}) = h(\Phi(\vartheta^{-1} K), \bar{e}).$$

Consequently, the real valued valuation $K \mapsto h(\Phi K, \bar{e})$ uniquely determines the Minkowski valuation Φ. This motivates the following:

Definition *Suppose that* $\Phi \in \mathbf{MVal}^{(+)}$ *is* $\mathrm{SO}(n)$ *equivariant. The* $\mathrm{SO}(n-1)$ *invariant real valued valuation* $\varphi \in \mathbf{Val}^{(+)}$*, defined by*

$$\varphi(K) = h(\Phi K, \bar{e}), \qquad K \in \mathcal{K}^n,$$

is called the associated real valued valuation *of* $\Phi \in \mathbf{MVal}^{(+)}$*. We say that* Φ *is* smooth *if its associated real valued valuation* φ *is smooth.*

The notion of smoothness for translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations was introduced by the first author in [**61**]. There, it was also shown that any *even* $\Phi \in \mathbf{MVal}$ which is $\mathrm{SO}(n)$ equivariant can be approximated uniformly on compact subsets of $\mathcal{K}^n$ by smooth ones.

The following is a reformulation (and slight variation) of Theorem 1 stated in Section 2:

Theorem 4.4 ([61]) *Suppose that* $i \in \{1, \ldots, n-1\}$*. If* $\Phi_i \in \mathbf{MVal}_i^+$ *is* $\mathrm{SO}(n)$ *equivariant and smooth, then there exists an* $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ *invariant* $f \in C^\infty(S^{n-1})$ *such that for every* $K \in \mathcal{K}^n$*,*

$$h(\Phi_i K, \cdot) = \mathrm{vol}_i(K|\cdot) * f.$$

Theorem 4.4 was proved in [**61**] for Minkowski valuations which are $\mathrm{O}(n)$ equivariant. However, in [**10**, Lemma 7.1] it was shown that any $\mathrm{SO}(n)$ equivariant Minkowski valuation is also $\mathrm{O}(n)$ equivariant. Another difference between Theorem 4.4 and the corresponding statement in [**61**] is that the convolution appearing in Theorem 4.4 is induced from $\mathrm{SO}(n)$ while in [**61**] the convolution used was induced from $\mathrm{O}(n)$. However, since the convolution of functions on $\mathrm{Gr}_{i,n}$ with $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant functions on S^{n-1} does not depend on whether it is induced from $\mathrm{SO}(n)$ or $\mathrm{O}(n)$, the result from [**61**] implies Theorem 4.4. Conversely, since every $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant function on S^{n-1} is also $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant (cf. the Appendix), the result from [**61**] follows from Theorem 4.4. In particular, Theorem 2 also implies the uniqueness of (spherical) Crofton measures in the sense of [**61**].

A final result from [**61**] which we will need concerns the Klain function of the real valued valuation associated with a translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuation:

Theorem 4.5 ([**61**]) *Suppose that $i \in \{1, \dots, n-1\}$. If $\Phi_i \in \mathbf{MVal}_i^+$ is $\mathrm{SO}(n)$ equivariant and $\varphi_i \in \mathbf{Val}_i^+$ denotes its associated real valued valuation, then there exists a unique $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant convex body $M \in \mathcal{K}^n$, called the* Klain body *of Φ_i, such that*

$$\widehat{\mathrm{Kl}_i \varphi_i} = h(M, \cdot).$$

We emphasize that the Klain body of Φ_i determines the valuation Φ_i uniquely. It is easy to see (and was proved in [**61**]) that

$$M = \Phi_i K_{\bar{e}},$$

where $K_{\bar{e}}$ is any convex body in $\bar{e} \in \mathrm{Gr}_{i,n}$ such that $\mathrm{vol}_i(K_{\bar{e}}) = 1$.

5. Proof of the main results

After these preparations we are now in a position to prove our main results, Theorem 2 and Theorem 3 from Section 2. At the end of the section we also provide explicit integral transforms which relate the previously known representations of a translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuation with our new one.

We begin with the proof of Theorem 2 from Section 2:

Proof of Theorem 2. Let $\varphi_i \in \mathbf{Val}_i^+$ denote the associated real valued valuation of Φ_i. From

$$h(\Phi_i K, \cdot) = \mathrm{vol}_i(K|\cdot) * \mu,$$

it follows easily (cf. [**61**, p. 19]) that

$$\varphi_i(K) = \int_{\mathrm{Gr}_{i,n}} \mathrm{vol}_i(K|E)\, d\widehat{\mu}(E) \tag{5.1}$$

for every $K \in \mathcal{K}^n$. Note that the measure $\widehat{\mu}$ on $\mathrm{Gr}_{i,n}$ is $\mathrm{SO}(n-1)$ invariant.

From (5.1), the remarks before Theorem 4.1, and Theorem 4.5, we obtain

$$\widehat{\mathrm{Kl}_i \varphi_i} = \widehat{C_i \widehat{\mu}} = h(M, \cdot). \tag{5.2}$$

Here the $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant convex body $M \in \mathcal{K}^n$ is the Klain body of Φ_i which is uniquely determined by Φ_i. If we define

$$L = \frac{2\kappa_{n-1}}{n\kappa_i\kappa_{n-i}} \binom{n}{i} M,$$

then it follows from (5.2) and Lemma 3.3 (i) that

$$h(L,u) = \int_{S^{n-1}} |u \cdot v| \, d\mu(v), \qquad u \in S^{n-1}.$$

Thus, by the injectivity of the spherical cosine transform, it follows that the spherical Crofton measure μ is uniquely determined by Φ_i. ■

By Theorem 2, the spherical cosine transform of the spherical Crofton measure of an even Minkowski valuation is necessarily a support function. It is an important open problem whether this condition is also sufficient for an $\mathrm{O}(i) \times \mathrm{O}(n-i)$ invariant measure on S^{n-1} to be the spherical Crofton measure of an even Minkowski valuation of degree i. (If $i = n-1$, then this is the case, see [**60**].)

Using the uniqueness of spherical Crofton measures, we can now deduce Theorem 3 from Theorem 4.4 and Lemma 3.3 (ii):

Proof of Theorem 3. First note that $C([-1,1])$ is in one to one correspondence with the subspace of zonal functions in $C(S^{n-1})$ via the map $g \mapsto g(\bar{e} \cdot \,.\,)$. Therefore, for any even zonal function $\breve{g} \in C^\infty(S^{n-1})$, there exists an even function $g \in C^\infty((-1,1)) \cap C([-1,1])$ such that $\breve{g} = g(\bar{e} \cdot \,.\,)$. Moreover, for every measure τ on S^{n-1} and every $u \in S^{n-1}$,

$$\int_{S^{n-1}} g(u \cdot v) \, d\tau(v) = (\tau * \breve{g})(u).$$

Thus, in order to prove Theorem 3, it is sufficient to show that there exists a unique even zonal function $\breve{g} \in C^\infty(S^{n-1})$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * \breve{g}. \tag{5.3}$$

By Theorem 4.4, there exists an $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant function $f \in C^\infty(S^{n-1})$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_i K, \cdot) = \mathrm{vol}_i(K| \cdot) * f. \tag{5.4}$$

Since any $\mathrm{S}(\mathrm{O}(i)\times\mathrm{O}(n-i))$ invariant function is also $\mathrm{O}(i)\times\mathrm{O}(n-i)$ invariant (cf. Appendix), we can view f as the (smooth) density of a spherical Crofton measure for Φ_i. By Theorem 2, f is uniquely determined by Φ_i.

From Lemma 3.3 (ii), it follows that there exists a unique even zonal $\tilde{g} \in C^\infty(S^{n-1})$ such that

$$f = \lambda_{i,n-1} * \tilde{g}. \tag{5.5}$$

Hence, plugging (5.5) into (5.4) and using (3.11), we obtain

$$h(\Phi_i K, \cdot) = \mathrm{vol}_i(K|\cdot) * \lambda_{i,n-1} * \tilde{g} = R_{i,n-1}\mathrm{vol}_i(K|\cdot) * \tilde{g}.$$

Thus, if we define

$$\breve{g}(u) = \frac{\kappa_i}{2\kappa_{n-1}}\, C_{n-1}\, \tilde{g},$$

then, using (4.2), (3.15) and the fact that the spherical convolution of zonal functions is commutative, we arrive at

$$h(\Phi_i K, \cdot) = \frac{\kappa_i}{2\kappa_{n-1}} S_i(K, \cdot) * |\cos(\bar{e}, \cdot)| * \tilde{g} = S_i(K, \cdot) * \breve{g}.$$

■

Theorem 3 gives rise to the following:

Definition *Let $i \in \{1, \ldots, n-1\}$. We call a zonal function $\breve{g} \in C(S^{n-1})$ (or its associated function $g \in C([-1, 1])$) a* generating function *for $\Phi_i \in \mathbf{MVal}_i$ if (5.3) holds for every $K \in \mathcal{K}^n$.*

We collect the relations between spherical Crofton measures, generating functions, and Klain bodies of translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations (established in the proofs of Theorems 2 and 3) in:

Corollary 5.1 *Let $i \in \{1, \ldots, n-1\}$ and let $\Phi_i \in \mathbf{MVal}_i^+$ be $\mathrm{SO}(n)$ equivariant and smooth. If μ denotes the (smooth) spherical Crofton measure of Φ_i, $\breve{g} \in C^\infty(S^{n-1})$ is the generating function of Φ_i, and $M \in \mathcal{K}^n$ denotes the Klain body of Φ_i, then*

$$h(M, u) = \frac{n\kappa_i\kappa_{n-i}}{2\kappa_{n-1}} \binom{n}{i}^{-1} \int_{S^{n-1}} |u \cdot v|\, d\mu(v), \qquad u \in S^{n-1}, \tag{5.6}$$

and

$$\widehat{\mu} = \frac{2\kappa_{n-1}}{\kappa_i} R_{n-1,i}\, C_{n-1}^{-1}\, \breve{g}.$$

Examples:

(a) Kiderlen [**36**] proved (in a more general form) that for any $\Phi_1 \in \mathbf{MVal}_1^+$ which is $\mathrm{SO}(n)$ equivariant and smooth there exists a unique even zonal function $f \in C^\infty(S^{n-1})$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_1 K, \cdot) = h(K, \cdot) * f. \tag{5.7}$$

Using $h(K,u) + h(-K,u) = \mathrm{vol}_1(K|\cdot\,)$, we can rewrite (5.7) as

$$h(\Phi_1 K, \cdot) = \mathrm{vol}_1(K|\cdot\,) * f.$$

Thus, the function f is (a smooth density of) the spherical Crofton measure of Φ_1. In order to relate (5.7) with the new representation provided by Theorem 3, we recall that (in the sense of distributions)

$$S_1(K, \cdot) = \Box_n h(K, \cdot), \tag{5.8}$$

where $\Box_n = \frac{1}{n-1}\Delta_S + 1$ and Δ_S denotes the Laplace–Beltrami operator on S^{n-1} (see, e.g., [**58**, p. 119]). Since $\Box_n$ is a bijection on even functions in $C^\infty(S^{n-1})$, there exists an even zonal $\breve{g} \in C^\infty(S^{n-1})$ with $\Box_n \breve{g} = f$. Thus, the $\mathrm{SO}(n)$ equivariance of $\Box_n$ implies (cf. [**28**, p. 86]) that

$$h(\Phi_1 K, \cdot) = h(K, \cdot) * f = h(K, \cdot) * \Box_n \breve{g} = \Box_n h(K, \cdot) * \breve{g} = S_1(K, \cdot) * \breve{g}.$$

Finally, the Klain body of Φ_1 can be determined from f by using (5.6).

(b) The first author [**60**] proved that $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^+$ is $\mathrm{SO}(n)$ equivariant and smooth if and only if there exists an o-symmetric smooth convex body of revolution $L \in \mathcal{K}^n$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_{n-1} K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot). \tag{5.9}$$

Consequently, the generating functions of $\mathrm{SO}(n)$ equivariant and smooth Minkowski valuations in $\mathbf{MVal}_{n-1}^+$ are precisely the support functions of o-symmetric smooth convex bodies of revolution. It also follows directly from (5.9) that the Klain body of Φ_{n-1} is given by $2L$. From Cauchy's projection formula and the fact that the convolution of zonal functions is Abelian, it follows that (5.9) is equivalent to

$$h(\Phi_{n-1} K, \cdot) = \mathrm{vol}_{n-1}(K|\cdot\,) * f_L,$$

where $f_L \in C^\infty(S^{n-1})$ is the uniquely determined even zonal function such that

$$h(L, u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| f_L(v)\, dv.$$

(c) For $i \in \{1, \dots, n-1\}$, let $\Pi_i \in \mathbf{MVal}_i^+$ denote the projection body map of order i, defined by

$$h(\Pi_i K, u) = V_i(K|u^\perp) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v|\, dS_i(K, v), \qquad u \in S^{n-1}.$$

Note that each Π_i is $\mathrm{SO}(n)$ equivariant but *not* smooth. Their (merely) continuous generating function is given by $g(t) = \frac{1}{2}|t|$. Moreover, it is well known (see, e.g., [**23**, p. 428]) that

$$h(\Pi_i K, \cdot) = \frac{\kappa_{n-1}}{\kappa_i} R_{n-i,1}\mathrm{vol}_i^\perp(K|\cdot\,) = \frac{\kappa_{n-1}}{\kappa_i}\, \mathrm{vol}_i(K|\cdot\,) * \widehat{\lambda}_{n-i,1}^\perp.$$

It follows that the Klain body of Π_i is a multiple of the Euclidean ball contained in $\bar{e}^\perp$, where $\bar{e} \in \mathrm{Gr}_{i,n}$ is the stabilizer of $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$.

(d) For $i \in \{2, \dots, n\}$, let $\mathrm{M}_i \in \mathbf{MVal}_{n+1-i}$ denote the *normalized* mean section operator of order i, introduced by Goodey and Weil [**23**, **24**] and given by

$$h(\mathrm{M}_i K, \cdot) = \int_{\mathrm{AGr}_{i,n}} h(\mathrm{J}(K \cap E), \cdot)\, d\mu_i(E). \tag{5.10}$$

Here $\mathrm{J} \in \mathbf{MVal}_1$ is defined by $\mathrm{J}K = K - s(K)$, where $s : \mathcal{K}^n \to \mathbb{R}^n$ is the Steiner point map (see, e.g., [**58**, p. 50]) and μ_i is the invariant measure on $\mathrm{AGr}_{i,n}$ normalized such that the set of planes having non-empty intersection with the Euclidean unit ball has measure κ_{n-i}. Note that M_i is *not* even. However, it was proved in [**23**] that

$$h(\mathrm{M}_i K, \cdot) + h(\mathrm{M}_i(-K), \cdot) = \frac{i\kappa_i\kappa_{n-1}}{n\kappa_{i-1}\kappa_n}\, R_{n+1-i,1}\mathrm{vol}_{n+1-i}(K|\cdot\,).$$

Thus, a multiple of $\widehat{\lambda}_{n+1-i,1}$ is the spherical Crofton measure for the even part of M_i and its Klain body is a multiple of the Euclidean ball contained in the subspace $\bar{e} \in \mathrm{Gr}_{n+1-i,n}$.

Goodey and Weil [**24**] also determined the family of generating functions for the mean section operators. In order to explain their result, recall that in Berg's solution of the Christoffel–Minkowski problem (see, e.g., [**26**, **58**]) he proved the following: For every $n \geq 2$ there exists a uniquely determined C^∞ function ζ_n on $(-1, 1)$ such that the associated zonal function $\breve{\zeta}_n \in L^1(S^{n-1})$ is orthogonal to the restriction of all linear functions to S^{n-1} and satisfies, for every $K \in \mathcal{K}^n$,

$$h(\mathrm{J}K, \cdot) = S_1(K, \cdot) * \breve{\zeta}_n. \tag{5.11}$$

Goodey and Weil [**24**, Theorem 4.4] proved that

$$h(\mathrm{M}_i K, \cdot) = q_{n,i}\, S_{n+1-i}(K, \cdot) * \breve{\zeta}_i, \tag{5.12}$$

where

$$q_{n,i} = \frac{i-1}{2\pi(n+1-i)}\, \frac{\kappa_{i-1}\kappa_{i-2}\kappa_{n-i}}{\kappa_{i-3}\kappa_{n-2}}. \tag{5.13}$$

Let $C_{\mathrm{o}}^{\infty}(S^{n-1})$ denote the subspace of smooth functions on S^{n-1} which are orthogonal to the restriction of all linear functions. It is well known that the linear differential operator $\Box_n : C_{\mathrm{o}}^{\infty}(S^{n-1}) \to C_{\mathrm{o}}^{\infty}(S^{n-1})$ is an isomorphism (see, e.g., [**29**]). Moreover, since every twice continuously differentiable function on S^{n-1} is a difference of support functions (see, e.g., [**58**, p. 49]), it follows from (5.8) and (5.11) that

$$f = (\Box_n f) * \breve{\zeta}_n \tag{5.14}$$

for every $f \in C_{\mathrm{o}}^{\infty}(S^{n-1})$. In the next section we need the following more general fact.

Proposition 5.2 *For every $n \geq 2$ and $2 \leq j \leq n$, the integral transform*

$$\mathrm{F}_{\zeta_j} : C_{\mathrm{o}}^{\infty}(S^{n-1}) \to C_{\mathrm{o}}^{\infty}(S^{n-1}), \quad f \mapsto f * \breve{\zeta}_j,$$

is an isomorphism.

Proposition 5.2 can be deduced from a result of Goodey and Weil [**24**, Theorem 4.3]. A different and more elementary proof was given very recently in [**11**]. Proposition 5.2 and (5.14) give rise to the following:

Definition *For $2 \leq j \leq n$, let $\Box_j : C_{\mathrm{o}}^{\infty}(S^{n-1}) \to C_{\mathrm{o}}^{\infty}(S^{n-1})$ denote the linear operator which is inverse to the integral transform F_{ζ_j}.*

6. The Hard Lefschetz operators

In this final section we determine the action of Alesker's Hard Lefschetz operators on translation invariant and $\mathrm{SO}(n)$ equivariant even Minkowski valuations in terms of corresponding integral transforms of the generating function, the spherical Crofton measure, and the support function of the Klain body of the respective Minkowski valuation. Our investigations are motivated by recent applications of the derivation operator on Minkowski valuations in the theory of geometric inequalities (see [**55**]).

In the recent article [**55**], Parapatits and the first author showed that for any $\Phi \in \mathbf{MVal}^{(+)}$ there exist $\Phi^{(j)} \in \mathbf{MVal}^{(+)}$, where $0 \leq j \leq n$, such that

$$\Phi(K + rB) = \sum_{j=0}^{n} r^{n-j} \Phi^{(j)}(K)$$

for every $K \in \mathcal{K}^n$ and $r \geq 0$. Moreover, if Φ is $\mathrm{SO}(n)$ equivariant and smooth then so is each $\Phi^{(j)}$. This Steiner-type formula, in turn, gives rise to the definition of a derivation operator $\Lambda : \mathbf{MVal} \to \mathbf{MVal}$.

Definition *For $\Phi \in \mathbf{MVal}$, define $\Lambda\Phi \in \mathbf{MVal}$ by*

$$h((\Lambda\Phi)(K), u) = \left.\frac{d}{dt}\right|_{t=0} h(\Phi(K + tB), u), \qquad u \in S^{n-1}.$$

Note that Λ commutes with the action of $\mathrm{SO}(n)$ and that Λ preserves parity. Moreover, if $\Phi_i \in \mathbf{MVal}_i^{(+)}$ is $\mathrm{SO}(n)$ equivariant and smooth, then so is $\Lambda\Phi_i \in \mathbf{MVal}_{i-1}^{(+)}$. We also emphasize that if $\varphi_i \in \mathbf{Val}_i^{(+)}$ is the real valued valuation associated with Φ_i, then $\Lambda\varphi_i \in \mathbf{Val}_{i-1}^{(+)}$ is associated with $\Lambda\Phi_i$.

Example:
By definition of the projection body maps of order i and (4.1), we have

$$\Lambda^{n-1-i}\Pi_{n-1} = \frac{(n-1)!}{i!}\,\Pi_i.$$

Theorem 6.1 *Suppose that $i \in \{2, \ldots, n-1\}$ and let $\Phi_i \in \mathbf{MVal}_i^+$ be $\mathrm{SO}(n)$ equivariant and smooth.*

(i) If $\breve{g} \in C^\infty(S^{n-1})$ is the generating function of Φ_i, then the generating function of $\Lambda\Phi_i$ is given by $i\,\breve{g}$.

(ii) If μ is the (smooth) spherical Crofton measure of Φ_i, then the spherical Crofton measure ν of $\Lambda\Phi_i$ is determined by

$$\widehat{\nu} = \frac{i\kappa_i}{\kappa_{i-1}}\, R_{i,i-1}\,\widehat{\mu}.$$

(iii) If $M \in \mathcal{K}^n$ is the Klain body of Φ_i, then the Klain body $N \in \mathcal{K}^n$ of $\Lambda\Phi_i$ is determined by

$$\widehat{h(N,\cdot)} = \frac{(n-i+1)\kappa_{n-i+1}}{\kappa_{n-i}}\, R_{i,i-1}\,\widehat{h(M,\cdot)}.$$

Proof. Since (4.1) can be generalized to arbitrary area measures of order i, more precisely,

$$S_i(K+tB,\cdot)=\sum_{j=0}^{i} t^{i-j}\binom{i}{j} S_j(K,\cdot),$$

we have

$$h(\Phi_i(K+tB,\cdot)=S_i(K+tB,\cdot)*\breve{g}=\sum_{j=0}^{i} t^{i-j}\binom{i}{j} S_j(K,\cdot)*\breve{g}$$

and, thus, $h((\Lambda\Phi_i)K,\cdot)=S_{i-1}(K,\cdot)*i\,\breve{g}$, which proves (i).

By Corollary 5.1, on one hand

$$\breve{g}=\frac{\kappa_i}{2\kappa_{n-1}}\,C_{n-1}\,R_{n-1,i}^{-1}\,\widehat{\mu}$$

and, by (i) and Corollary 5.1 again, on the other hand

$$\widehat{\nu}=\frac{2i\kappa_{n-1}}{\kappa_{i-1}}\,R_{n-1,i-1}\,C_{n-1}^{-1}\,\breve{g}$$

which together with $R_{n-1,i-1}=R_{i,i-1}\circ R_{n-1,i}$ prove claim (ii).

In order to prove (iii), we use (5.2) and (ii) to arrive at

$$\widehat{h(N,\cdot)}=C_{i-1}\widehat{\nu}=\frac{i\kappa_i}{\kappa_{i-1}}\,C_{i-1}\,R_{i,i-1}\,\widehat{\mu}.$$

Thus, from another application of (5.2), we obtain

$$\widehat{h(N,\cdot)}=\frac{i\kappa_i}{\kappa_{i-1}}\,C_{i-1}\,R_{i,i-1}\,C_i^{-1}\widehat{h(M,\cdot)}.$$

Using now (3.14) gives the desired result. ■

Theorem 6.1 yields a set of necessary conditions for a function or a measure to be the generating function or the spherical Crofton measure, respectively, of a Minkowski valuation. For example, for the generating function $\breve{g}$ of a Minkowski valuation $\Phi_i\in\mathbf{MVal}_i$, all the functions $S_j(K,\cdot)*\breve{g}$, $1\le j\le i$, must be support functions for every $K\in\mathcal{K}^n$.

Next, we turn to the integration operator. Formula (4.5) of Bernig [**12**] (which we used as a definition of $\mathfrak{L}$ on **Val**) motivates the following:

Definition *For* $\Phi \in \mathbf{MVal}$, *define* $\mathfrak{L}\Phi \in \mathbf{MVal}$ *by*

$$h((\mathfrak{L}\Phi)(K),u) = \int_{\mathrm{AGr}_{n-1,n}} h(\Phi(K \cap E),u)\, d\sigma_{n-1}(E), \qquad u \in S^{n-1}.$$

Note that $\mathfrak{L}$ commutes with the action of $\mathrm{SO}(n)$ and that $\mathfrak{L}$ preserves parity. Moreover, if $\Phi_i \in \mathbf{MVal}_i^{(+)}$ is $\mathrm{SO}(n)$ equivariant and smooth, then so is $\mathfrak{L}\Phi_i \in \mathbf{MVal}_{i+1}^{(+)}$. We also emphasize that if $\varphi_i \in \mathbf{Val}_i^{(+)}$ is the real valued valuation associated with Φ_i, then $\mathfrak{L}\varphi_i \in \mathbf{Val}_{i+1}^{(+)}$ is associated with $\mathfrak{L}\Phi_i$.

Example:

For $k \in \{1,\ldots,n\}$, it follows by induction on k from a well known formula of Crofton (see, e.g., [**39**, p. 124]) that

$$\int_{\mathrm{AGr}_{n-1,n}} \cdots \int_{\mathrm{AGr}_{n-1,n}} f(E_1 \cap \cdots \cap E_k)\, d\sigma_{n-1}(E_1) \cdots d\sigma_{n-1}(E_k) = \frac{k!\kappa_k}{2^k} \int_{\mathrm{AGr}_{n-k,n}} f(F)\, d\sigma_{n-k}(F)$$

for every $f \in L^1(\mathrm{AGr}_{n-k,n})$. Consequently, for $\Phi \in \mathbf{MVal}$,

$$h((\mathfrak{L}^k\Phi)(K),u) = \frac{k!\kappa_k}{2^k} \int_{\mathrm{AGr}_{n-k,n}} h(\Phi(K \cap F),u)\, d\sigma_{n-k}(F), \qquad u \in S^{n-1}.$$

By definition (5.10) of the mean section operator of order i and the fact that

$$\sigma_{n-i} = \left[\begin{array}{c} n \\ i \end{array} \right] \mu_{n-i},$$

we conclude that for $i \in \{0,\ldots,n-2\}$,

$$\mathfrak{L}^i \mathrm{J} = \frac{i!\kappa_i}{2^i} \left[\begin{array}{c} n \\ i \end{array} \right] \mathrm{M}_{n-i} = \frac{n!\kappa_n}{2^i(n-i)!\kappa_{n-i}} \mathrm{M}_{n-i}. \tag{6.1}$$

Theorem 6.2 *Suppose that* $i \in \{1,\ldots,n-2\}$ *and let* $\Phi_i \in \mathbf{MVal}_i^+$ *be* $\mathrm{SO}(n)$ *equivariant and smooth.*

(i) If $\breve{g} \in C^\infty(S^{n-1})$ *is the generating function of* Φ_i*, then the generating function* $\breve{f} \in C^\infty(S^{n-1})$ *of* $\mathfrak{L}\Phi_i$ *is given by*

$$\breve{f} = \frac{(n-i)\kappa_{i+1}\kappa_{n-i}}{2\kappa_i\kappa_{n-i-1}} R_{n-1,i+1}^{-1} R_{i,i+1} R_{n-1,i}\, \breve{g}.$$

(ii) If μ is the (smooth) spherical Crofton measure of Φ_i, then the spherical Crofton measure ν of $\mathfrak{L}\Phi_i$ is determined by

$$\widehat{\nu} = \frac{(n-i)\kappa_{n-i}}{2\kappa_{n-i-1}}\, R_{i,i+1}\, \widehat{\mu}.$$

(iii) If $M \in \mathcal{K}^n$ is the Klain body of Φ_i, then the Klain body $N \in \mathcal{K}^n$ of $\mathfrak{L}\Phi_i$ is determined by

$$\widehat{h(N,\cdot)} = \frac{(i+1)\kappa_{i+1}}{2\kappa_i}\, R_{i,i+1}\, \widehat{h(M,\cdot)}.$$

Proof. We begin with the proof of (iii). To this end let $\varphi_i \in \mathbf{Val}_i^+$ denote the associated real valued valuation of Φ_i. First note that, by (4.7),

$$\mathfrak{L}\varphi_i = \frac{1}{2}\mathbb{F}\Lambda\mathbb{F}\varphi_i \tag{6.2}$$

is the associated real valued valuation of $\mathfrak{L}\Phi_i$. Moreover, by (5.2),

$$\mathrm{Kl}_i\varphi_i = \widehat{h(M,\cdot)} \qquad \text{and} \qquad \mathrm{Kl}_i(\mathfrak{L}\varphi_i) = \widehat{h(N,\cdot)}.$$

Thus, (6.2), (4.6), Theorem 6.1 (iii), and (3.10) yield

$$\widehat{h(N,\cdot)} = \frac{1}{2}\,(\mathrm{Kl}_{n-i}(\Lambda\mathbb{F}\varphi_i))^{\perp} = \frac{(i+1)\kappa_{i+1}}{2\kappa_i} R_{i,i+1}\, \widehat{h(M,\cdot)}.$$

In order to prove (ii), we use Corollary 5.1, Lemma 3.3 (i), and part (iii) which we just proved, to obtain

$$\widehat{\nu} = \frac{(i+1)\kappa_{i+1}}{2\kappa_i}\, C_{i+1}^{-1}\, R_{i,i+1}\, C_i\, \widehat{\mu}.$$

An application of (3.14) gives the desired result.

Finally, by Corollary 5.1, we have

$$\widehat{\mu} = \frac{\kappa_{n-1}}{\kappa_i}\, R_{n-1,i}\, C_{n-1}^{-1}\, \breve{g} \qquad \text{and} \qquad \breve{f} = \frac{\kappa_{i+1}}{\kappa_{n-1}}\, C_{n-1}\, R_{n-1,i+1}^{-1}\widehat{\nu}.$$

Thus, using (ii), we arrive at

$$\breve{f} = \frac{(n-i)\kappa_{i+1}\kappa_{n-i}}{2\kappa_i\kappa_{n-i-1}}\, C_{n-1}\, R_{n-1,i+1}^{-1}\, R_{i,i+1}\, R_{n-1,i}\, C_{n-1}^{-1}\, \breve{g}.$$

Using now (3.14) three times gives (i). ■

We note that it is an open problem whether the Alesker–Fourier transform $\mathbb{F}$ is well defined for even Minkowski valuations in **MVal** that are $\mathrm{SO}(n)$ equivariant and smooth. More precisely, if μ is the spherical Crofton measure of a smooth and $\mathrm{SO}(n)$ equivariant Minkowski valuation $\Phi_i \in \mathbf{MVal}_i^+$, then it is not known in general whether $\mu^\perp$ is the spherical Crofton measure of a smooth and $\mathrm{SO}(n)$ equivariant Minkowski valuation $\mathbb{F}\Phi_i \in \mathbf{MVal}_{n-i}^+$.

In the last part of this section, we use relation (6.1) and Proposition 5.2 to deduce a more explicit expression for the generating function of $\mathfrak{L}\Phi_i$ in terms of the generating function of an $\mathrm{SO}(n)$ equivariant and smooth $\Phi_i \in \mathbf{MVal}_i$. Since we want to prove this result for Minkowski valuations which are *not necessarily even*, we need the following:

Definition *Suppose that* $i \in \{1, \ldots, n-1\}$ *and let* $\Phi_i \in \mathbf{MVal}_i$ *be* $\mathrm{SO}(n)$ *equivariant and smooth. We call a zonal function* $\breve{g} \in C_{\mathrm{o}}^\infty(S^{n-1})$ *a* generating function *for* Φ_i *if*

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * \breve{g}$$

holds for every $K \in \mathcal{K}^n$.

Note that if $\Phi_i \in \mathbf{MVal}_i$ admits a generating function $\breve{g} \in C_{\mathrm{o}}^\infty(S^{n-1})$, then it follows from well known density properties of area measures of convex bodies that $\breve{g}$ is uniquely determined by Φ_i. Moreover, by Theorem 3, every smooth $\mathrm{SO}(n)$ equivariant *even* $\Phi_i \in \mathbf{MVal}_i$ admits a generating function.

Theorem 6.3 *Suppose that* $i \in \{1, \ldots, n-2\}$ *and let* $\Phi_i \in \mathbf{MVal}_i$ *be* $\mathrm{SO}(n)$ *equivariant and smooth. If* Φ_i *admits a generating function* $\breve{g} \in C^\infty(S^{n-1})$, *then* $\mathfrak{L}\Phi_i$ *also admits a generating function* $\breve{f} \in C^\infty(S^{n-1})$ *which is given by*

$$\breve{f} = c_{n,i} \square_{n-i+1} \breve{g} * \breve{\zeta}_{n-i},$$

where

$$c_{n,i} = \frac{i(n-i-1)(n-i+1)\kappa_{n-i-2}^2 \kappa_{n-i+1} \kappa_i}{2(n-i)(i+1)\kappa_{n-i-3}\kappa_{n-i}^2 \kappa_{i-1}}.$$

Proof. Consider the subspace of $\mathbf{Val}_i^\infty$ spanned by valuations of the form

$$\psi_i(K) = \int_{S^{n-1}} h(u)\, dS_i(K,u), \qquad K \in \mathcal{K}^n,$$

where $h \in C_{\mathrm{o}}^\infty(S^{n-1})$. Note that h is uniquely determined by ψ_i. By the $\mathrm{SO}(n)$ equivariance and linearity of $\mathfrak{L} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{i+1}^\infty$, it follows that there exists a linear operator $\mathrm{T}_i : C^\infty_{\mathrm{o}}(S^{n-1}) \to C^\infty_{\mathrm{o}}(S^{n-1})$ which is $\mathrm{SO}(n)$ equivariant, such that

$$(\mathfrak{L}\psi_i)(K) = \int_{S^{n-1}} (\mathrm{T}_i h)(u)\; dS_{i+1}(K,u), \qquad K \in \mathcal{K}^n.$$

Note that if $\varphi_i \in \mathbf{Val}_i^\infty$ denotes the real valued associated valuation of Φ_i, then

$$\varphi_i(K) = \int_{S^{n-1}} \breve{g}(u)\, dS_i(K,u), \qquad K \in \mathcal{K}^n.$$

Therefore, we have to show that for every $h \in C^\infty_{\mathrm{o}}(S^{n-1})$,

$$\mathrm{T}_i h = c_{n,i}\,\Box_{n-i+1} h * \breve{\zeta}_{n-i} = c_{n,i}\,\mathrm{F}_{\zeta_{n-i}}(\Box_{n-i+1} h). \tag{6.3}$$

To this end, let $h \sim \sum_{k\geq 0} H_k$ be the series expansion of h into spherical harmonics. By the linearity and $\mathrm{SO}(n)$ equivariance of T_i, there exists a sequence $a_k^n[T_i] \in \mathbb{R}$ of multipliers such that the spherical harmonics expansion of $\mathrm{T}_i h$ is given by $\mathrm{T}_i h \sim \sum_{k\geq 0} a_k^n[T_i]\, H_k$. In particular, the operator T_i is uniquely determined by the sequence $a_k^n[T_i]$. Moreover, the sequence $a_k^n[T_i]$ is determined by the T_i-image of any function being the sum of nonzero harmonics of all orders different from 1. Hence, since, by the Funk–Hecke Theorem (see, e.g., [**29**, p. 98]), $\mathrm{F}_{\zeta_{n-i}}$ and $\Box_{n-i+1}$ are also multiplier transformations, it suffices to prove (6.3) for one such function.

By Proposition 5.2, each of Berg's functions $\breve{\zeta}_j$, $2 \leq j \leq n$, is such a sum of nonzero harmonics of all orders different from 1. However, $\breve{\zeta}_j$ is not smooth but merely in $L^1_{\mathrm{o}}(S^{n-1})$, the subspace of $L^1(S^{n-1})$ consisting of functions which are orthogonal to all spherical harmonics of degree 1. This is not a problem because, as multiplier transformations, T_i as well as $\Box_j$ are selfadjoint and thus can be extended in the sense of distributions to $L^1_{\mathrm{o}}(S^{n-1})$. Therefore, on one hand

$$f = (\Box_j f) * \breve{\zeta}_j \tag{6.4}$$

holds in the sense of distributions for every $f \in L^1_{\mathrm{o}}(S^{n-1})$ and $j \in \{2,\ldots,n\}$. On the other hand, since, by (6.1),

$$\mathfrak{L}\mathrm{M}_{n+1-i} = \frac{(n-i+1)\kappa_{n-i+1}}{2\kappa_{n-i}}\, \mathrm{M}_{n-i},$$

we obtain from (5.12) that in the sense of distributions

$$\mathrm{T}_i(q_{n,n-i+1}\breve{\zeta}_{n-i+1}) = \frac{(n-i+1)\kappa_{n-i+1}}{2\kappa_{n-i}}\, q_{n,n-i}\,\breve{\zeta}_{n-i}.$$

Using now (6.4) with $f = \breve{\zeta}_{n-i}$ and $j = n-i+1$ and the fact that multiplier transformations commute, this is equivalent to

$$\mathrm{T}_i \breve{\zeta}_{n-i+1} = \frac{(n-i+1)\kappa_{n-i+1} q_{n,n-i}}{2\kappa_{n-i} q_{n,n-i+1}} (\Box_{n-i+1} \breve{\zeta}_{n-i+1}) * \breve{\zeta}_{n-i}.$$

Plugging in the explicit values of $q_{n,m}$ given in (5.13) completes the proof. ■

Appendix

The purpose of this appendix is the proof of the proposition below, which (together with Theorem 2) shows that the notion of spherical Crofton measure of a Minkowski valuation is independent from the identification of S^{n-1} with $\mathrm{O}(n)/\mathrm{O}(n-1)$ or $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, respectively.

Proposition A *If* $f \in C(S^{n-1})$ *is* $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ *invariant, then* f *is also* $\mathrm{O}(i) \times \mathrm{O}(n-i)$ *invariant.*

Proof. Let $H_1 = \mathrm{O}(i) \times \mathrm{O}(n-i)$, $H_2 = \mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ and let Γ be an arbitrary $\mathrm{O}(n)$ irreducible subspace of $L^2(S^{n-1})$. Recall that we denote by Γ^{H_i} the subspace of H_i invariant elements of Γ. Since $L^2(S^{n-1})$ is an orthogonal direct sum of $\mathrm{O}(n)$ irreducible subspaces, it will be sufficient to prove that $\Gamma^{H_1} = \Gamma^{H_2}$.

Since $H_2 \subset H_1$, we obviously have $\Gamma^{H_1} \subset \Gamma^{H_2}$. Moreover, it follows from the Frobenius reciprocity theorem (see, e.g., [**40**, Theorem 9.9]) that

$$\dim \Gamma^{H_1} = \dim \mathrm{Hom}_{\mathrm{O}(n)}(\Gamma, C(\mathrm{Gr}_{i,n}))$$

and

$$\dim \Gamma^{H_2} = \mathrm{Hom}_{\mathrm{SO}(n)}(\mathrm{Res}^{\mathrm{O}(n)}_{\mathrm{SO}(n)} \Gamma, C(\mathrm{Gr}_{i,n})). \tag{A.1}$$

Here, Hom_G denotes the space of linear G-equivariant maps and $\mathrm{Res}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}$ denotes the restriction of an $\mathrm{O}(n)$ representation to $\mathrm{SO}(n)$.

From the description of the irreducible representations of $\mathrm{O}(n)$ in terms of the irreducible representations of $\mathrm{SO}(n)$ (see, e.g., [**10**, Lemma 3.1]) and (3.17), we obtain

$$\mathrm{Res}^{\mathrm{O}(n)}_{\mathrm{SO}(n)} \Gamma = \Gamma_{(k,0,\dots,0)}$$

for some $k \in \mathbb{N}$. Thus, by (A.1) and (3.18), we have $\dim \Gamma^{H_2} = 0$ if k is odd and $\dim \Gamma^{H_2} = 1$ if k is even. Hence, the proposition will be proved if we can show that

$$\dim \mathrm{Hom}_{\mathrm{O}(n)}(\Gamma, C(\mathrm{Gr}_{i,n})) \geq 1$$

whenever $\mathrm{Res}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}\Gamma = \Gamma_{(2k,0,\ldots,0)}$. But if we identify even functions on S^{n-1} with functions on $\mathrm{Gr}_{1,n}$, then the restriction of the Radon transform $R_{1,i} : C(\mathrm{Gr}_{1,n}) \to C(\mathrm{Gr}_{i,n})$ is a non-trivial $\mathrm{O}(n)$ equivariant map from Γ to $C(\mathrm{Gr}_{i,n})$. ■

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