Binary Operations in Spherical Convex Geometry

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Abstract. Characterizations of binary operations between convex bodies on the Euclidean unit sphere are established. The main result shows that the convex hull is essentially the only non-trivial projection covariant operation between pairs of convex bodies contained in open hemispheres. Moreover, it is proved that any continuous and projection covariant binary operation between all proper spherical convex bodies must be trivial.

1. Introduction

In recent years it has been explained why a number of fundamental notions from convex geometric analysis really do have a special place in the theory. For example, Blaschke's classical affine and centro-affine surface areas were given characterizations by Ludwig and Reitzner [**27**] and Haberl and Parapatits [**22**] as unique valuations satisfying certain invariance properties; polar duality and the Legendre transform were characterized by Böröczky and Schneider [**7**] and Artstein-Avidan and Milman [**3**], respectively. These and other results of the same nature (see also, e.g., [**21**, **25**, **26**, **42**, **43**, **45**]) not only show that the notions under consideration are characterized by a surprisingly small number of basic properties but also led to the discovery of seminal new notions.

Gardner, Hug, and Weil [**15**] initiated a new line of research whose goal is to enhance our understanding of the fundamental characteristics of known *binary* operations between sets in Euclidean geometry (see also [**17**]). Their main focus is on operations which are projection covariant, that is, the operation can take place before or after projection onto linear subspaces, with the same effect. One impressive example of the results obtained in [**15**] is a characterization of the classical Minkowski addition between convex bodies (compact convex sets) in $\mathbb{R}^n$ as the only projection covariant operation which also satisfies the identity property. In fact, a characterization of all projection covariant operations between origin-symmetric convex bodies was established in [**15**], by proving that such operations are precisely those given by so-called M-addition (see Section 3 for precise definitions). This little-known addition was later shown in [**16**] to be intimately related to Orlicz addition, a recent important generalization of Minkowski addition.

The Brunn–Minkowski theory, which arises from combining volume *and* Minkowski addition, lies at the very core of classical Euclidean convexity and provides a unifying framework for various extremal and uniqueness problems for convex bodies in $\mathbb{R}^n$ (see, e.g., [**14**, **20**, **40**]). In contrast, the geometry of spherical convex sets is much less well understood. Although certain aspects, like the integral geometry of spherical convex sets (see [**1**, **2**, **5**, **19**, **23**, **39**]), have witnessed considerable progress, contributions to spherical convexity are rather scattered (see [**4**, **8**, **10**, **12**, **13**, **18**, **34**, **37**, **38**, **41**, **49**]). The reason for this might be that so far no natural analogue of Minkowski addition is available on the sphere. (For an attempt to remedy this see [**24**].)

In this article we start a systematic investigation of binary operations between convex bodies (that is, closed convex sets) on the Euclidean unit sphere with a focus on operations which are covariant under projections onto great subspheres. We prove that all continuous such operations between proper spherical convex bodies are trivial. More importantly, our main result shows that the convex hull is essentially the only non-trivial projection covariant operation between pairs of convex bodies contained in open hemispheres. The picture changes drastically when operations between convex bodies in a *fixed* open hemisphere are considered. In this case, we establish a one-to-one correspondence between binary operations on spherical convex bodies that are projection covariant with respect to the center of the hemisphere, and projection covariant operations on convex bodies in $\mathbb{R}^n$.

2. Statement of principal results

Let $\mathbb{S}^n$ denote the n-dimensional Euclidean unit sphere. Throughout the article we assume that $n \geq 2$. The usual *spherical distance* between points on $\mathbb{S}^n$ is given by $d(u,v) = \arccos(u \cdot v)$, $u, v \in \mathbb{S}^n$. For $\lambda > 0$ and $A \subseteq \mathbb{S}^n$, we write A_λ for the set of all points with distance at most λ from A. The *Hausdorff distance* between closed sets $A, B \subseteq \mathbb{S}^n$ is then given by

$$\delta_s(A,B) = \min\left\{0 \leq \lambda \leq \pi : A \subseteq B_\lambda \text{ and } B \subseteq A_\lambda\right\}.$$

A set $A \subseteq \mathbb{S}^n$ is called *(spherical) convex* if

$$\operatorname{rad} A = \{\lambda x : \lambda \geq 0, x \in A\} \subseteq \mathbb{R}^{n+1}$$

is convex. We say $K \subseteq \mathbb{S}^n$ is a *convex body* if K is closed and convex. Let $\mathcal{K}(\mathbb{S}^n)$ denote the space of convex bodies in $\mathbb{S}^n$ with the Hausdorff distance.

We call $K \in \mathcal{K}(\mathbb{S}^n)$ a *proper* convex body if K is contained in an open hemisphere and we write $\mathcal{K}^p(\mathbb{S}^n)$ for the subspace of $\mathcal{K}(\mathbb{S}^n)$ of all proper convex bodies. For fixed $u \in \mathbb{S}^n$ we denote by $\mathcal{K}^p_u(\mathbb{S}^n)$ the subspace of (proper) convex bodies that are contained in the open hemisphere centered at u. Then

$$\mathcal{K}^p(\mathbb{S}^n) = \bigcup_{u \in \mathbb{S}^n} \mathcal{K}^p_u(\mathbb{S}^n).$$

The *convex hull* of $A \subseteq \mathbb{S}^n$ is the intersection of all convex sets in $\mathbb{S}^n$ that contain A. Note that, for $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$, we have $\mathrm{conv}(K \cup L) \in \mathcal{K}^p_u(\mathbb{S}^n)$.

For $0 \leq k \leq n$, a *k-sphere* S is a k-dimensional great sub-sphere of $\mathbb{S}^n$, that is, the intersection of a $(k+1)$-dimensional linear subspace $V \subseteq \mathbb{R}^{n+1}$ with $\mathbb{S}^n$. Clearly, every k-sphere S is convex. For $K \in \mathcal{K}(\mathbb{S}^n)$, the *spherical projection* $K|S$ is defined by

$$K|S = \mathrm{conv}\,(K \cup S^\circ) \cap S = \mathrm{cl}\,(\mathrm{rad}(K)|V) \cap \mathbb{S}^n,$$

where $S = V \cap \mathbb{S}^n$, S° is the $(n-k-1)$-sphere orthogonal to S, that is, $S^\circ = V^\perp \cap \mathbb{S}^n$, and cl denotes the topological closure.

For fixed $u \in \mathbb{S}^n$ we call a binary operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ *u-projection covariant* if for all k-spheres S, $0 \leq k \leq n-1$, with $u \in S$ and for all K, $L \in \mathcal{K}^p_u(\mathbb{S}^n)$, we have

$$(K|S) * (L|S) = (K * L)|S.$$

We call $*$ *projection covariant* if $*$ is u-projection covariant for all $u \in \mathbb{S}^n$.

The main objective of this article is to characterize projection covariant operations between spherical convex bodies. Our first result shows that such operations between *all* proper convex bodies in $\mathbb{S}^n$ are of a very simple form.

Theorem 1 *An operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ between proper convex bodies is projection covariant and continuous with respect to the Hausdorff metric if and only if either $K * L = K$, or $K * L = -K$, or $K * L = L$, or $K * L = -L$ for all $K, L \in \mathcal{K}^p(\mathbb{S}^n)$.*

We call the binary operations from Theorem 1 *trivial*. As the following example shows, the continuity assumption in Theorem 1 cannot be omitted.

Example:

Consider the set $\mathcal{C} \subset \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ of all pairs (K, L) such that both K and L are contained in some open hemisphere, that is,

$$\mathcal{C} = \bigcup_{u \in \mathbb{S}^n} \left(\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \right).$$

Define an operation $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ by

$$K * L = \begin{cases} K & \text{if } (K, L) \in \mathcal{C}, \\ L & \text{if } (K, L) \notin \mathcal{C}. \end{cases}$$

Clearly, $*$ is not continuous but by our definition it is projection covariant.

The proof of Theorem 1 relies on ideas of Gardner, Hug, and Weil. The critical tool to transfer their techniques to the sphere is the gnomonic projection (see Section 4) which establishes the following correspondence between projection covariant operations on $\mathcal{K}(\mathbb{R}^n)$, the space of compact, convex sets in $\mathbb{R}^n$, and u-projection covariant operations on $\mathcal{K}^p_u(\mathbb{S}^n)$:

Theorem 2 *For every fixed $u \in \mathbb{S}^n$, there is a one-to-one correspondence between u-projection covariant operations $*\colon \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ and projection covariant operations $\overline{*}\colon \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$. Moreover, every such u-projection covariant operation $*$ is continuous in the Hausdorff metric.*

Note that by Theorem 2 every projection covariant operation $*$ on $\mathcal{C}$ is also automatically continuous.

Finally, as our main result, we prove that the only *non-trivial* projection covariant operation on the set $\mathcal{C}$ is essentially the spherical convex hull.

Theorem 3 *An operation $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ is non-trivial and projection covariant if and only if either $K * L = \operatorname{conv}(K \cup L)$ or $K * L = -\operatorname{conv}(K \cup L)$ for all $(K, L) \in \mathcal{C}$.*

After briefly recalling the background material on convex bodies in $\mathbb{R}^n$ in Section 3, we discuss the geometry of spherical convex sets in Section 4 and use the gnomonic projection to prove Theorem 2. Sections 5 and 6 contain the proofs of Theorems 1 and 3. Motivated by investigations of Gardner, Hug, and Weil [**15**] in $\mathbb{R}^n$, we discuss *section* covariant operations between spherical *star sets* in the concluding section of the article.

3. Background material from Euclidean convexity

In this section we collect basic material about convex bodies in $\mathbb{R}^n$. As a general reference for these facts we recommend [**40**]. We also recall the definition of the L_p Minkowski addition and, more generally, the M-addition of convex bodies as well as their characterizing properties established in [**15**].

The standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Otherwise, we usually denote the coordinates of $x \in \mathbb{R}^n$ by $x_1, \ldots, x_n$. We write B^n for the Euclidean unit ball in $\mathbb{R}^n$. We call a subset of $\mathbb{R}^n$ 1-*unconditional* if it is symmetric with respect to each coordinate hyperplane.

Let $\mathcal{K}_e(\mathbb{R}^n)$ be the set of origin symmetric convex bodies and let $\mathcal{K}_{\mathrm{o}}(\mathbb{R}^n)$ denote the set of convex bodies containing the origin.

A convex body $K \in \mathcal{K}(\mathbb{R}^n)$ is uniquely determined by its *support function* defined by

$$h(K, x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n.$$

We will sometimes also use h_K to denote the support function of $K \in \mathcal{K}(\mathbb{R}^n)$. Support functions are 1-*homogeneous*, that is, $h(K, \lambda x) = \lambda h(K, x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$, and are therefore often regarded as functions on $\mathbb{S}^{n-1}$. They are also *subadditive*, that is, $h(K, x + y) \le h(K, x) + h(K, y)$ for all $x, y \in \mathbb{R}^n$. Conversely, every 1-homogeneous and subadditive function on $\mathbb{R}^n$ is the support function of a convex body. Clearly, $K \in \mathcal{K}_e(\mathbb{R}^n)$ if and only if $h(K, \cdot)$ is even.

The *Minkowski sum* of subsets X and Y of $\mathbb{R}^n$ is defined by

$$X + Y = \{x + y : x \in X, y \in Y\}.$$

If $K, L \in \mathcal{K}(\mathbb{R}^n)$, then $K + L$ can be equivalently defined as the convex body such that

$$h(K + L, \cdot) = h(K, \cdot) + h(L, \cdot).$$

The *Hausdorff distance* $\delta(X, Y)$ between compact subsets X and Y of $\mathbb{R}^n$ is defined by

$$\delta(X, Y) = \min\{\lambda \ge 0 : X \subseteq Y + \lambda\, B^n \text{ and } Y \subseteq X + \lambda\, B^n\}.$$

If $K, L \in \mathcal{K}(\mathbb{R}^n)$, then $\delta(K, L)$ can be alternatively defined by

$$\delta(K, L) = \|h(K, \cdot) - h(L, \cdot)\|_\infty, \tag{3.1}$$

where $\|\cdot\|_\infty$ denotes the L_∞ norm on $\mathbb{S}^{n-1}$.

For $1 < p \leq \infty$, the L_p *Minkowski sum* of convex bodies $K, L \in \mathcal{K}_o(\mathbb{R}^n)$ was first defined by Firey [**11**] by

$$h(K +_p L, \cdot)^p = h(K, \cdot)^p + h(L, \cdot)^p,$$

for $p < \infty$, and by

$$h(K +_\infty L, \cdot) = \max\{h(K, \cdot), h(L, \cdot)\}.$$

Note that $K +_\infty L$ is just the usual convex hull in $\mathbb{R}^n$ of K and L.

Lutwak [**28**, **29**] showed that the L_p Minkowski addition leads to a very powerful extension of the classical Brunn–Minkowski theory. Since the 1990's this L_p Brunn–Minkowski theory has provided new tools for attacks on major unsolved problems and consolidated connections between convex geometry and other fields (see, e.g., [**6**, **25**, **30–32**, **36**, **44**, **46–48**] and the references therein). An extension of the L_p Minkowski addition to arbitrary sets in $\mathbb{R}^n$ was given only recently in [**33**].

An even more general way of combining two subsets of $\mathbb{R}^n$ is the still more recent *M-addition*: If M is an arbitrary subset of $\mathbb{R}^2$, then the *M-sum* of $X, Y \subseteq \mathbb{R}^n$ is defined by

$$X \oplus_M Y = \bigcup_{(a,b) \in M} a\,X + b\,Y = \{ax + by : (a, b) \in M, x \in X, y \in Y\}\,. \quad (3.2)$$

Protasov [**35**] first introduced M-addition for centrally symmetric convex bodies and a 1-unconditional convex body M in $\mathbb{R}^2$. He also proved that $\oplus_M : \mathcal{K}_e(\mathbb{R}^n) \times \mathcal{K}_e(\mathbb{R}^n) \to \mathcal{K}_e(\mathbb{R}^n)$ for such M.

Gardner, Hug and Weil [**15**] rediscovered M-addition in the more general form (3.2) in their investigation of projection covariant binary operations between convex bodies in $\mathbb{R}^n$. Among several results on this seminal operation, they proved the following:

Theorem 3.1 ([**15**]) *Let $M \subseteq \mathbb{R}^2$. Then $\oplus_M : \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ if and only if $M \in \mathcal{K}(\mathbb{R}^2)$ and M is contained in one of the 4 quadrants of $\mathbb{R}^2$. In this case, let $\varepsilon_i = \pm 1$, $i = 1, 2$, denote the sign of the ith coordinate of a point in the interior of this quadrant and let*

$$M^+ = \{(\varepsilon_1 a, \varepsilon_2 b) : (a, b) \in M\}$$

be the reflection of M contained in $[0, \infty)^2$. If $K, L \in \mathcal{K}(\mathbb{R}^n)$, then

$$h_{K \oplus_M L}(x) = h_{M^+}(h_{\varepsilon_1 K}(x), h_{\varepsilon_2 L}(x)), \qquad x \in \mathbb{R}^n. \quad (3.3)$$

Example:
For some $1 \leq p \leq \infty$, let

$$M = \{(a,b) \in [0,1]^2 : a^{p'} + b^{p'} \leq 1\},$$

where $1/p + 1/p' = 1$. Then $\oplus_M = +_p$ is L_p Minkowski addition on $\mathcal{K}_{\mathrm{o}}(\mathbb{R}^n)$.

The following basic properties of M-addition are of particular interest for us. They are immediate consequences of either definition (3.2) or (3.3).

Proposition 3.2 *Suppose that $M \in \mathcal{K}(\mathbb{R}^2)$ is contained in $[0,\infty)^2$. Then $\oplus_M : \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ has the following properties:*

- Continuity
  $K_i \to K$, $L_i \to L$ implies $K_i \oplus_M L_i \to K \oplus_M L$ as $i \to \infty$ in the Hausdorff metric;
- $\mathrm{GL}(n)$ covariance
  $(AK) \oplus_M (AL) = A(K \oplus_M L)$ for all $A \in \mathrm{GL}(n)$;
- Projection covariance
  $(K|V) \oplus_M (L|V) = (K \oplus_M L)|V$ for every linear subspace V of $\mathbb{R}^n$.

It is easy to show that continuity and $\mathrm{GL}(n)$ covariance imply projection covariance. That the converse statement is also true, follows from a deep result of Gardner, Hug, and Weil which states the following:

Theorem 3.3 ([**15**]) *An operation $* : \mathcal{K}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ is projection covariant if and only if there exists a nonempty closed convex set $\overline{M}$ in $\mathbb{R}^4$ such that, for all K, $L \in \mathcal{K}(\mathbb{R}^n)$,*

$$h_{K*L}(x) = h_{\overline{M}}(h_{-K}(x), h_K(x), h_{-L}(x), h_L(x)), \qquad x \in \mathbb{R}^n. \tag{3.4}$$

Consequently, every such operation is continuous and $\mathrm{GL}(n)$ covariant.

Note that it is an open problem whether the binary operation on $\mathcal{K}(\mathbb{R}^n)$ defined by (3.4) is M-addition for some (convex) subset M of $\mathbb{R}^2$. However, Gardner, Hug, and Weil [**15**] proved that an operation between *o-symmetric* convex bodies is projection covariant if and only if it is M-addition for some 1-unconditional convex body in $\mathbb{R}^2$.

4. The gnomonic projection

In the following we discuss basic facts about spherical convex sets. In particular, we recall the definition of spherical support functions of proper convex bodies in $\mathbb{S}^n$. The second part of this section is devoted to the gnomonic projection. After establishing the basic properties of this critical tool, we conclude this section with the proof of Theorem 2.

For the following alternative definitions of proper convex bodies in $\mathbb{S}^n$, we refer to [**9**].

Proposition 4.1 *The following statements about a closed set $K \subseteq \mathbb{S}^n$ are equivalent:*

- *(a) The set K is a proper convex body.*
- *(b) The set K is an intersection of open hemispheres.*
- *(c) There are no antipodal points in K and for every two points $u, v \in K$, the minimal geodesic connecting u and v is contained in K.*

Although we will make no use of this fact, we remark, that a set $K \subseteq \mathbb{S}^n$ is a convex body if and only if K is the intersection of *closed* hemispheres.

Next we introduce spherical support functions of proper convex sets contained in a fixed hemisphere (cf. [**24**] for a related construction). To this end, for $u \in \mathbb{S}^n$, let $\mathbb{S}_u^+$ denote the open hemisphere with center in u and let $\mathbb{S}_u$ be the boundary of $\mathbb{S}_u^+$, that is,

$$\mathbb{S}_u^+ = \{v \in \mathbb{S}^n : u \cdot v > 0\} \qquad \text{and} \qquad \mathbb{S}_u = \{v \in \mathbb{S}^n : u \cdot v = 0\}.$$

For $u, v \in \mathbb{S}^n$ such that $u \neq \pm v$, we write $S_{u,v}$ for the unique great circle containing u and v.

Definition For $u \in \mathbb{S}^n$ and a proper convex body $K \in \mathcal{K}_u^p(\mathbb{S}^n)$, the *spherical support function* $h_u(K, \cdot) : \mathbb{S}_u \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of K is defined by

$$h_u(K, v) = \max\{\operatorname{sgn}(v \cdot w)\, d(u, w|S_{u,v}) : w \in K\}.$$

Recall that the (Euclidean) support function of a convex body L in $\mathbb{R}^n$ encodes the signed distances of the supporting planes to L from the origin. In other words, we have for every $v \in \mathbb{S}^{n-1}$,

$$L|\operatorname{span}\{v\} = \{tv : t \in [-h(K, -v), h(K, v)]\}.$$

The intuitive meaning of the spherical support function of a proper convex body $K \in \mathcal{K}^p_u(\mathbb{S}^n)$ is similar. It yields the oriented angle between u and the supporting $(n-1)$-spheres to K. More precisely, we have for every $v \in \mathbb{S}_u$,

$$K|S_{u,v} = \{u \cos\alpha + v \sin\alpha : \alpha \in [-h_u(K,-v), h_u(K,v)]\}.$$

In particular, for $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$, $K \subseteq L$ if and only if $h_u(K,\cdot) \leq h_u(L,\cdot)$.

In the following we denote by $\mathbb{R}^n_u$ (instead of $u^\perp$) the hyperplane in $\mathbb{R}^{n+1}$ orthogonal to $u \in \mathbb{S}^n$.

Definition For $u \in \mathbb{S}^n$, the *gnomonic projection* $g_u : \mathbb{S}^+_u \to \mathbb{R}^n_u$ is defined by

$$g_u(v) = \frac{v}{u \cdot v} - u.$$

In the literature, the gnomonic projection is often considered as a map to the tangent plane at u. However, for our purposes it is more convenient if the range of g_u contains the origin.

In the following lemma we collect a number of well-known properties of the gnomonic projection which are immediate consequences of its definition.

Lemma 4.2 *For $u \in \mathbb{S}^n$, the following statements hold:*

(a) The gnomonic projection $g_u : \mathbb{S}^+_u \to \mathbb{R}^n_u$ is a bijection with inverse

$$g_u^{-1}(x) = \frac{x+u}{\|x+u\|}, \qquad x \in \mathbb{R}^n_u.$$

(b) If $S \subseteq \mathbb{S}^n$ is a k-sphere, $0 \leq k \leq n-1$, such that $S \cap \mathbb{S}^+_u$ is non-empty, then $g_u(S \cap \mathbb{S}^+_u)$ is a k-dimensional affine subspace of $\mathbb{R}^n_u$. Conversely, g_u^{-1} maps k-dimensional affine subspaces of $\mathbb{R}^n_u$ to k-hemispheres in $\mathbb{S}^+_u$.

(c) The gnomonic projection maps $\mathcal{K}^p_u(\mathbb{S}^n)$ bijectively to $\mathcal{K}(\mathbb{R}^n_u)$.

If $K \in \mathcal{K}^p_u(\mathbb{S}^n)$, then, by Lemma 4.2 (c), the set $g_u(K)$ is a convex body in $\mathcal{K}(\mathbb{R}^n_u)$. The next lemma relates the (Euclidean) support function of $g_u(K)$ with the spherical support function of K.

Lemma 4.3 *For $u \in \mathbb{S}^n$ and every $K \in \mathcal{K}^p_u(\mathbb{S}^n)$, we have*

$$h(g_u(K), v) = \tan h_u(K,v), \qquad v \in \mathbb{S}_u.$$

In particular, K is uniquely determined by $h_u(K,\cdot)$.

Proof. For $v \in \mathbb{S}_u$ and $w \in \mathbb{S}_u^+$, an elementary calculation shows that

$$\frac{v \cdot w}{u \cdot w} = \tan(\operatorname{sgn}(v \cdot w)\, d(u, w|S_{u,v})).$$

Therefore, the definition of g_u and the monotonicity of the tangent yield

$$h(g_u(K), v) = \max_{x \in g_u(K)} \{v \cdot x\} = \max_{w \in K} \left\{ \frac{v \cdot w}{u \cdot w} \right\} = \tan h_u(K, v).$$

■

By Lemma 4.3, a function $h : \mathbb{S}_u \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the spherical support function of a convex body $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ if and only if the 1-homogeneous extension of $\tan h$ to $\mathbb{R}_u^n$ is the support function of a convex body in $\mathbb{R}_u^n$.

Using spherical support functions, we define a metric γ_u on $\mathcal{K}_u^p(\mathbb{S}^n)$ by

$$\gamma_u(K, L) = \max_{v \in \mathbb{S}_u} |h_u(K, v) - h_u(L, v)|.$$

Since for $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $\varepsilon > 0$, the set K_ε of all points with distance at most ε from K is *not* necessarily convex, it is not difficult to see that the restriction of δ_s to $\mathcal{K}_u^p(\mathbb{S}^n)$ does *not* coincide with γ_u (in contrast to the Euclidean setting). However, our next result shows that γ_u and δ_s induce the same topology on $\mathcal{K}_u^p(\mathbb{S}^n)$. Since we could not find a reference for this basic result, we include a proof for the readers convenience.

Proposition 4.4 *For $u \in \mathbb{S}^n$, the metrics γ_u and δ_s induce the same topology on $\mathcal{K}_u^p(\mathbb{S}^n)$.*

Proof. Let $\alpha \in (0, \frac{\pi}{2})$ and denote by $C_\alpha(u)$ the open geodesic ball in u of radius α, that is, $C_\alpha(u) = \{v \in \mathbb{S}^n : d(u, v) \leq \alpha\}$. We only need to show that for $K, L \in \mathcal{K}_u^p(\mathbb{S}^n)$, such that $K, L \subseteq C_\alpha(u)$, we have

$$\cos(\alpha)\gamma_u(K, L) \leq \delta_s(K, L) \leq \frac{\pi}{2\cos(\alpha)^2}\gamma_u(K, L).$$

To this end, let $p, q \in C_\alpha(u)$, then by the definition of g_u we have

$$\|g_u(p) - g_u(q)\| = \left\| \frac{p}{p \cdot u} - \frac{q}{q \cdot u} \right\|,$$

and since $\|p-q\| = \sqrt{2-2\cos(d(p,q))}$, we conclude that

$$\frac{2d(p,q)}{\pi} \le \|p-q\| \le \|g_u(p)-g_u(q)\| \le \frac{\|p-q\|}{\cos(\alpha)} \le \frac{d(p,q)}{\cos(\alpha)}.$$

Consequently,

$$\frac{2\delta_s(K,L)}{\pi} \le \delta(g_u(K),g_u(L)) \le \frac{\delta_s(K,L)}{\cos(\alpha)}. \tag{4.1}$$

Now, for $-\alpha < s,t < \alpha$, we have $|s-t| \le |\tan(s)-\tan(t)| \le \frac{|s-t|}{\cos(\alpha)^2}$. Therefore, using Lemma 4.3 and (3.1), we conclude that

$$\begin{aligned}\gamma_u(K,L) &= \max_{v\in\mathbb{S}_u} |h_u(K,v)-h_u(L,v)| \\ &\le \max_{v\in\mathbb{S}_u} \left|h_{g_u(K)}(v)-h_{g_u(L)}(v)\right| = \delta(g_u(K),g_u(L)) \le \frac{\gamma_u(K,L)}{\cos(\alpha)^2}.\end{aligned}$$

Combining this with (4.1) completes the proof. ■

Thus, from Proposition 4.4 and the continuity of the tangent we obtain the following.

Corollary 4.5 *The gnomonic projection is a homeomorphism between $(\mathcal{K}^p_u(\mathbb{S}^n),\delta_s)$ and $(\mathcal{K}(\mathbb{R}^n_u),\delta)$. Moreover, it is uniformly continuous when restricted to convex bodies contained in an open geodesic ball with center in u of a fixed radius less than $\frac{\pi}{2}$.*

Using Corollary 4.5 and other basic properties of the gnomonic projection, we can now prove the following refinement of Theorem 2.

Theorem 4.6 *For every fixed $u \in \mathbb{S}^n$, the gnomonic projection g_u induces a one-to-one correspondence between operations $* : \mathcal{K}^p_u(\mathbb{S}^n)\times\mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ which are u-projection covariant and operations $\overline{*} : \mathcal{K}(\mathbb{R}^n_u)\times\mathcal{K}(\mathbb{R}^n_u)\to\mathcal{K}(\mathbb{R}^n_u)$ which are projection covariant. Moreover, every such u-projection covariant operation $*$ is continuous.*

Proof. First assume that $*$ is u-projection covariant and define an operation $\overline{*} : \mathcal{K}(\mathbb{R}^n_u)\times\mathcal{K}(\mathbb{R}^n_u)\to\mathcal{K}(\mathbb{R}^n_u)$ by

$$K \,\overline{*}\, L = g_u(g_u^{-1}(K) * g_u^{-1}(L))$$

for $K, L \in \mathcal{K}(\mathbb{R}^n_u)$. Since for every k-sphere S containing u, there exists a linear subspace V in $\mathbb{R}^n_u$ such that

$$g_u^{-1}(K|V) = g_u^{-1}(K)|S$$

for all $K \in \mathcal{K}(\mathbb{R}^n_u)$, we obtain

$$\begin{aligned}(K|V) \,\overline{*}\, (L|V) &= g_u(g_u^{-1}(K|V) * g_u^{-1}(L|V)) = g_u((g_u^{-1}(K)|S) * (g_u^{-1}(L)|S)) \\ &= g_u((g_u^{-1}(K) * g_u^{-1}(L))|S) = g_u(g_u^{-1}(K) * g_u^{-1}(L))|V \\ &= (K \,\overline{*}\, L)|V.\end{aligned}$$

for all $K, L \in \mathcal{K}(\mathbb{R}^n_u)$. Thus, $\overline{*}$ is projection covariant.

Now, let $\overline{*} : \mathcal{K}(\mathbb{R}^n_u) \times \mathcal{K}(\mathbb{R}^n_u) \to \mathcal{K}(\mathbb{R}^n_u)$ be projection covariant and define $* : \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ by

$$K * L = g_u^{-1}(g_u(K) \,\overline{*}\, g_u(L))$$

for $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$. Using a similar argument as before, it is easy to show that $*$ is u-projection covariant.

The continuity of an operation $* : \mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n) \to \mathcal{K}^p_u(\mathbb{S}^n)$ which is u-projection covariant is now a direct consequence of Theorem 3.3 and Proposition 4.5. ∎

Recall that the set $\mathcal{C} \subset \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ was defined by

$$\mathcal{C} = \bigcup_{u \in \mathbb{S}^n} \left(\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)\right).$$

By Theorem 4.6, the restriction of an operation $* : \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ which is projection covariant to convex bodies contained in a fixed open hemisphere is continuous. Therefore, we obtain:

Corollary 4.7 *Every projection covariant operation* $* \colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ *is continuous.*

5. Auxiliary results

We continue in this section with our preparations for the proofs of Theorems 1 and 3. We prove three auxiliary results which will be used at different stages in Section 6. We begin by establishing first constraints on projection covariant operations $*$ on $\mathcal{C}$.

Lemma 5.1 *If* $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ *is projection covariant, then either*

$$K * L \subseteq \mathrm{conv}(K \cup L) \tag{5.1}$$

for all $(K, L) \in \mathcal{C}$ *or*

$$K * L \subseteq -\mathrm{conv}(K \cup L) \tag{5.2}$$

for all $(K, L) \in \mathcal{C}$.

Proof. For $u \in \mathbb{S}^n$, let S_u denote the 0-sphere $\{-u, u\}$. By the projection covariance of $*$, we have

$$(\{u\} * \{u\})|S_u = (\{u\}|S_u) * (\{u\}|S_u) = \{u\} * \{u\}.$$

Thus, $\{u\} * \{u\} \subseteq \{-u, u\}$. However, since $\{u\} * \{u\} \in \mathcal{K}^p(\mathbb{S}^n)$, we must have either $\{u\} * \{u\} = \{u\}$ or $\{u\} * \{u\} = \{-u\}$. Let

$$P = \{u \in \mathbb{S}^n : \{u\} * \{u\} = \{u\}\} \quad \text{and} \quad N = \{u \in \mathbb{S}^n : \{u\} * \{u\} = \{-u\}\}.$$

Clearly, $P \cap N = \emptyset$ and $P \cup N = \mathbb{S}^n$.

Since, by Corollary 4.7, $*$ is continuous, we obtain for every sequence $u_i \in P$ with limit $u \in \mathbb{S}^n$,

$$\{u\} * \{u\} = \{\lim u_i\} * \{\lim u_i\} = \lim \left(\{u_i\} * \{u_i\}\right) = \lim\{u_i\} = \{u\}.$$

Thus, $u \in P$ which shows that P is closed. In the same way, we see that N is closed. Consequently, we have either $P = \mathbb{S}^n$ or $N = \mathbb{S}^n$.

First assume that $P = \mathbb{S}^n$ and let $(K, L) \in \mathcal{C}$. Then there exists $u \in \mathbb{S}^n$ such that K, $L \subset \mathbb{S}^+_u$ or, equivalently, $\mathrm{conv}(K \cup L) \subset \mathbb{S}^+_u$. By the projection covariance of $*$, we have

$$(K * L)|S_u = (K|S_u) * (L|S_u) = \{u\} * \{u\} = \{u\}.$$

Thus, $K * L \subset \mathbb{S}^+_u$ and we conclude that

$$K * L \subseteq \bigcap \{\mathbb{S}^+_u : u \in \mathbb{S}^n \text{ such that } \mathrm{conv}(K \cup L) \subset \mathbb{S}^+_u\} = \mathrm{conv}(K \cup L)$$

for all $(K, L) \in \mathcal{C}$.

Conversely, if $N = \mathbb{S}^n$, then we obtain $(K * L)|S_u = \{-u\}$ and, therefore, $K * L \subset \mathbb{S}_u^- := -\mathbb{S}_u^+$ whenever $\operatorname{conv}(K \cup L) \subset \mathbb{S}_u^+$. This yields

$$K * L \subseteq \bigcap \{\mathbb{S}_u^- : u \in \mathbb{S}^n \text{ such that } \operatorname{conv}(K \cup L) \subset \mathbb{S}_u^+\} = -\operatorname{conv}(K \cup L)$$

for all $(K, L) \in \mathcal{C}$. ■

Our next lemma concerns spherical support functions of a spherical segment contained in an open hemisphere.

Lemma 5.2 *For $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u^+$, $w \in \mathbb{S}_u \cap \mathbb{S}_v$, and $-\frac{\pi}{2} < \alpha \le \beta < \frac{\pi}{2}$ let*

$$I_u^w(\alpha, \beta) = \{u \cos \lambda + w \sin \lambda : \lambda \in [\alpha, \beta]\}.$$

Then,

$$\tan h_v(I_u^w(\alpha, \beta), w) = \frac{\tan \beta}{u \cdot v} \qquad \textit{and} \qquad \tan h_v(I_u^w(\alpha, \beta), -w) = -\frac{\tan \alpha}{u \cdot v}.$$

Proof. First note that by our definition of the spherical support function

$$h_u(I_u^w(\alpha, \beta), w) = \beta \quad \text{and} \quad h_u(I_u^w(\alpha, \beta), -w) = -\alpha.$$

Let

$$A = g_v(I_u^w(\alpha, \alpha)) = \frac{u \cos \alpha + w \sin \alpha}{(u \cdot v) \cos \alpha} - v,$$
$$B = g_v(I_u^w(\beta, \beta)) = \frac{u \cos \beta + w \sin \beta}{(u \cdot v) \cos \beta} - v.$$

By Lemma 4.2 (b), $g_v(I_u^w(\alpha, \beta))$ is the line segment in $\mathbb{R}_v^n$ in direction w with endpoints A and B. Thus, by Lemma 4.3 and the definition of (Euclidean) support functions, we obtain

$$\tan h_v(I_u^w(\alpha, \beta), w) = h(g_v(I_u^w(\alpha, \beta)), w) = w \cdot B = \frac{\tan \beta}{u \cdot v},$$
$$\tan h_v(I_u^w(\alpha, \beta), -w) = h(g_v(I_u^w(\alpha, \beta)), -w) = -w \cdot A = -\frac{\tan \alpha}{u \cdot v}.$$

■

In view of Lemma 5.1, Theorem 4.6, and Theorem 3.3, the following result will be useful in the proof of Theorem 3.

Lemma 5.3 *Let $M \subseteq \mathbb{R}^4$ be closed and convex. If for all $a, b, c, d \in \mathbb{R}$ such that $-a \le b$ and $-c \le d$,*

$$h_M(a, b, c, d) \le \max\{b, d\}, \tag{5.3}$$

then

$$M \subseteq \{(\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in \mathbb{R}^4 : \lambda_1 \in [0, 1], \lambda_2 \le 0, \lambda_3 \le 0\}.$$

Proof. For $z = (-1, 1, -1, 1)$, we obtain from (5.3) that

$$h(M, z) \le 1 \qquad \text{and} \qquad h(M, -z) \le -1.$$

Since $-h(M, -z) \le h(M, z)$, we conclude that $-h(M, -z) = h(M, z) = 1$ or, equivalently,

$$M \subseteq \{x \in \mathbb{R}^4 : -x_1 + x_2 - x_3 + x_4 = 1\}. \tag{5.4}$$

By (5.3), we also have $h_M(1, 0, 0, 0) \le 0$ and $h_M(0, 0, 1, 0) \le 0$. Thus,

$$M \subseteq \{x \in \mathbb{R}^4 : x_1 \le 0, x_3 \le 0\}. \tag{5.5}$$

Finally, we deduce from (5.3) that

$$h_M(-1, 1, 0, 0) \le 1 \qquad \text{and} \qquad h_M(1, -1, 0, 0) \le 0,$$

as well as

$$h_M(0, 0, -1, 1) \le 1 \qquad \text{and} \qquad h_M(0, 0, 1, -1) \le 0.$$

Consequently,

$$M \subseteq \{x \in \mathbb{R}^4 : 0 \le x_2 - x_1 \le 1 \text{ and } 0 \le x_4 - x_3 \le 1\}. \tag{5.6}$$

Combining (5.4), (5.5), and (5.6), completes the proof. ∎

The importance for us of the set

$$E := \{(\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in \mathbb{R}^4 : \lambda_1 \in [0, 1], \lambda_2 \le 0, \lambda_3 \le 0\}$$

follows from

$$h_E(h_{-K}(x), h_K(x), h_{-L}(x), h_L(x)) = h_{\mathrm{conv}(K \cup L)}(x).$$

6. Proofs of Theorems 1 and 3

After these preparations, we are now in a position to first proof Theorem 3 and then complete the proof of Theorem 1. In order to enhance the readability of several formulas below, we write $\tan(x_1,\ldots,x_k)$ for the vector $(\tan x_1,\ldots,\tan x_k)$ and $\arctan(x_1,\ldots,x_k)$ is defined similarly.

Theorem 6.1 *An operation $*\colon \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ is projection covariant if and only if it is either $K * L = \mathrm{conv}(K \cup L)$ or $K * L = -\mathrm{conv}(K \cup L)$ for all $(K,L) \in \mathcal{C}$ or it is trivial, that is, $K * L = K$, or $K * L = -K$, or $K * L = L$, or $K * L = -L$ for all $(K,L) \in \mathcal{C}$.*

Proof. By Lemma 5.1, we may assume that

$$K * L \subseteq \mathrm{conv}(K \cup L) \tag{6.1}$$

holds for all $(K,L) \in \mathcal{C}$ (otherwise, replace $*$ by $*^- : \mathcal{C} \to \mathcal{K}^p(\mathbb{S}^n)$ defined by $K *^- L = -(K * L)$). In particular, for every $u \in \mathbb{S}^n$, the range of the restriction of $*$ to $\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)$ lies in $\mathcal{K}^p_u(\mathbb{S}^n)$.

In the proof of Theorem 4.6 we have seen that, for every $u \in \mathbb{S}^n$, there exists a (unique) projection covariant operation $\overline{*}_u\colon \mathcal{K}(\mathbb{R}^n_u) \times \mathcal{K}(\mathbb{R}^n_u) \to \mathcal{K}(\mathbb{R}^n_u)$ such that

$$\overline{K} \mathbin{\overline{*}_u} \overline{L} = g_u(g_u^{-1}(\overline{K}) * g_u^{-1}(\overline{L}))$$

for all $\overline{K}, \overline{L} \in \mathcal{K}(\mathbb{R}^n_u)$. Thus, by Theorem 3.3, there exists a nonempty closed convex set $M_u \subset \mathbb{R}^4$ such that

$$h_{\overline{K} \mathbin{\overline{*}_u} \overline{L}}(v) = h_{M_u}(h_{\overline{K}}(-v), h_{\overline{K}}(v), h_{\overline{L}}(-v), h_{\overline{L}}(v))$$

for all $v \in \mathbb{S}_u$. Therefore, Lemma 4.3 yields

$$\begin{aligned} \tan h_u(K * L, v) &= h_{g_u(K*L)}(v) = h_{g_u(K) \mathbin{\overline{*}_u} g_u(L)}(v) && (6.2)\\ &= h_{M_u}(h_{g_u(K)}(-v), h_{g_u(K)}(v), h_{g_u(L)}(-v), h_{g_u(L)}(v)) \\ &= h_{M_u}(\tan(h_u(K,-v), h_u(K,v), h_u(L,-v), h_u(L,v))) && (6.3) \end{aligned}$$

for all $K, L \in \mathcal{K}^p_u(\mathbb{S}^n)$. Thus, since $-h_{\overline{K}}(-v) \le h_{\overline{K}}(v)$ for every $\overline{K} \in \mathcal{K}(\mathbb{R}^n_u)$ and every $v \in \mathbb{S}_u$, the restriction of $*$ to $\mathcal{K}^p_u(\mathbb{S}^n) \times \mathcal{K}^p_u(\mathbb{S}^n)$ is completely determined by the values $h_{M_u}(a,b,c,d)$, where $-a \le b$ and $-c \le d$. Next, we want to show that for such $a,b,c,d \in \mathbb{R}$,

$$h_{M_u}(a,b,c,d) = h_{M_v}(a,b,c,d) \tag{6.4}$$

whenever $v \in \mathbb{S}_u^+$. To this end, let $-\frac{\pi}{2} < \alpha \le \beta < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \varphi \le \psi < \frac{\pi}{2}$. For every $u \in \mathbb{S}^n$ and $w \in \mathbb{S}_u$, the u-projection covariance of $*$ implies that there exist σ, τ such that $-\frac{\pi}{2} < \sigma \le \tau < \frac{\pi}{2}$ and

$$I_u^w(\alpha,\beta) * I_u^w(\varphi,\psi) = I_u^w(\sigma,\tau), \tag{6.5}$$

where we have used the notation from Lemma 5.2 for spherical segments I_u^w. Since, for $-\frac{\pi}{2} < \xi \le \zeta < \frac{\pi}{2}$, we have $h_u(I_u^w(\xi,\zeta),-w) = -\xi$ and $h_u(I_u^w(\xi,\zeta),w) = \zeta$, we obtain on the one hand from (6.5), (6.2), and (6.3),

$$\begin{aligned}\tan\tau &= \tan h_u(I_u^w(\sigma,\tau),w) = \tan h_u(I_u^w(\alpha,\beta) * I_u^w(\varphi,\psi),w)\\ &= h_{M_u}(\tan(-\alpha,\beta,-\varphi,\psi)).\end{aligned}$$

For $v \in \mathbb{S}_u^+$ and $w \in \mathbb{S}_u \cap \mathbb{S}_v$, we obtain from Lemma 5.2 and again (6.5), (6.2), and (6.3),

$$\begin{aligned}\tan\tau &= (u\cdot v)\tan h_v(I_u^w(\sigma,\tau),w) = (u\cdot v)\tan h_v(I_u^w(\alpha,\beta) * I_u^w(\varphi,\psi),w)\\ &= (u\cdot v) h_{M_v}\left(\frac{\tan(-\alpha,\beta,-\varphi,\psi)}{u\cdot v}\right) = h_{M_v}(\tan(-\alpha,\beta,-\varphi,\psi))\end{aligned}$$

which proves (6.4). Since $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u^+$, and $\alpha,\beta,\varphi,\psi$ were arbitrary, we conclude from (6.2), (6.3), and (6.4) that there exists a nonempty closed convex set $M \subseteq \mathbb{R}^4$, *independent* of $u \in \mathbb{S}^n$, such that

$$\tan h_u(K*L,v) = h_M(\tan(h_u(K,-v),h_u(K,v),h_u(L,-v),h_u(L,v))) \tag{6.6}$$

for all $K, L \in \mathcal{K}_u^p(\mathbb{S}^n)$ and $v \in \mathbb{S}_u$.

To complete the proof, we have to show that for $-a \le b$ and $-c \le d$, the support function h_M satisfies one of the following three conditions:

(i) $h_M(a,b,c,d) = b$, that is, $K * L = K$ for $(K,L) \in \mathcal{C}$;

(ii) $h_M(a,b,c,d) = d$, that is, $K * L = L$ for $(K,L) \in \mathcal{C}$;

(iii) $h_M(a,b,c,d) = \max\{b,d\}$, that is, $K * L = \mathrm{conv}(K \cup L)$ for $(K,L) \in \mathcal{C}$.

From (6.1) and (6.6), we deduce that

$$h_M(a,b,c,d) \leq \max\{b,d\} \tag{6.7}$$

whenever $-a \leq b$ and $-c \leq d$. Moreover, since $-h_u(K*L,-v) \leq h_u(K*L,v)$ for all $K,L \in \mathcal{K}^p_u(\mathbb{S}^n)$ and $v \in \mathbb{S}_u$, we deduce from (6.6) that

$$-h_M(b,a,d,c) \leq h_M(a,b,c,d). \tag{6.8}$$

Next, we want to show that for all $-\frac{\pi}{2} < \alpha \leq \beta < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi \leq \psi < \frac{\pi}{2}$, and $-\frac{\pi}{2} + \max\{\beta,\psi\} < \eta < \frac{\pi}{2} + \min\{\alpha,\varphi\}$, we have

$$\arctan h_M(\tan\Lambda) = \arctan h_M(\tan(\Lambda+\Theta)) + \eta, \tag{6.9}$$

where $\Lambda = (-\alpha,\beta,-\varphi,\psi)$ and $\Theta = (\eta,-\eta,\eta,-\eta)$. In order to prove (6.9), let $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u$ and define

$$u' = u\cos\eta - v\sin\eta \qquad \text{and} \qquad v' = v\cos\eta + u\sin\eta.$$

Note that u' and v' are rotations of u and v in the plane $\operatorname{span}\{u,v\}$ by an angle $-\eta$. Therefore, for every $\lambda \in [0,2\pi)$,

$$u'\cos\lambda + v'\sin\lambda = u\cos(\lambda-\eta) + v\sin(\lambda-\eta).$$

Hence,

$$I^{v'}_{u'}(\alpha,\beta) = I^v_u(\alpha-\eta,\beta-\eta) \subseteq \mathbb{S}^+_u. \tag{6.10}$$

Now, let

$$\begin{aligned}\sigma &= -h_u(I^v_u(\alpha-\eta,\beta-\eta) * I^v_u(\varphi-\eta,\psi-\eta),-v),\\ \tau &= h_u(I^v_u(\alpha-\eta,\beta-\eta) * I^v_u(\varphi-\eta,\psi-\eta),v),\end{aligned}$$

and

$$\begin{aligned}\sigma' &= -h_{u'}(I^{v'}_{u'}(\alpha,\beta) * I^{v'}_{u'}(\varphi,\psi),-v'),\\ \tau' &= h_{u'}(I^{v'}_{u'}(\alpha,\beta) * I^{v'}_{u'}(\varphi,\psi),v').\end{aligned}$$

By the u-projection covariance and the u'-projection covariance of $*$ and (6.10), we obtain

$$\begin{aligned}I^{v'}_{u'}(\sigma',\tau') &= I^{v'}_{u'}(\alpha,\beta) * I^{v'}_{u'}(\varphi,\psi) = I^v_u(\alpha-\eta,\beta-\eta) * I^v_u(\varphi-\eta,\psi-\eta)\\ &= I^v_u(\sigma,\tau) = I^{v'}_{u'}(\sigma+\eta,\tau+\eta).\end{aligned}$$

Thus, $\tau' = \tau + \eta$. Using (6.6) and the definitions of τ and τ', we obtain (6.9).

From applications of (6.9) with $\Lambda = \pm(-\alpha, \alpha, \alpha, -\alpha)$ and $\eta = \pm\alpha$, where $\alpha \in [0, \frac{\pi}{4})$, we obtain

$$\arctan(h_M(-1,1,1,-1)\tan\alpha) = \arctan(h_M(0,0,1,-1)\tan(2\alpha)) + \alpha, \quad (6.11)$$
$$\arctan(h_M(-1,1,1,-1)\tan\alpha) = \arctan(h_M(-1,1,0,0)\tan(2\alpha)) - \alpha, \quad (6.12)$$

and

$$\arctan(h_M(1,-1,-1,1)\tan\alpha) = \arctan(h_M(0,0,-1,1)\tan(2\alpha)) - \alpha, \quad (6.13)$$
$$\arctan(h_M(1,-1,-1,1)\tan\alpha) = \arctan(h_M(1,-1,0,0)\tan(2\alpha)) + \alpha. \quad (6.14)$$

On the one hand, using (6.11) and (6.12), it is not difficult to show that either

$$h_M(-1,1,1,-1) = 1, \quad h_M(0,0,1,-1) = 0, \quad h_M(-1,1,0,0) = 1, \qquad (6.15)$$

or

$$h_M(-1,1,1,-1) = -1, \quad h_M(0,0,1,-1) = -1, \quad h_M(-1,1,0,0) = 0. \quad (6.16)$$

On the other hand, by (6.13) and (6.14), we have either

$$h_M(1,-1,-1,1) = 1, \quad h_M(0,0,-1,1) = 1, \quad h_M(1,-1,0,0) = 0, \qquad (6.17)$$

or

$$h_M(1,-1,-1,1) = -1, \quad h_M(0,0,-1,1) = 0, \quad h_M(1,-1,0,0) = -1. \quad (6.18)$$

Note that, since $-h_M(1,-1,-1,1) \le h_M(-1,1,1,-1)$, (6.16) and (6.18) cannot both be satisfied. Also recall that by Lemma 5.3, we have

$$M \subseteq E = \{(\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) : \lambda_1 \in [0,1], \lambda_2, \lambda_3 \le 0\}.$$

and let

$$E_0 = \{(\lambda_2, \lambda_2, \lambda_3, 1 + \lambda_3) : \lambda_2 \le 0, \lambda_3 \le 0\},$$
$$E_1 = \{(\lambda_2, 1 + \lambda_2, \lambda_3, \lambda_3) : \lambda_2 \le 0, \lambda_3 \le 0\}.$$

If (6.15) holds, then $h_M(-1,1,0,0) = 1$ and, since $M \subseteq E$, we have

$$1 = \max\{\lambda_1 \in [0,1] : (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}.$$

Thus, there are λ_2, $\lambda_3 \le 0$, such that $(\lambda_2, 1+\lambda_2, \lambda_3, \lambda_3) \in M$ or, equivalently, $M \cap E_1$ is nonempty. Similarly, it follows from (6.17) that $M \cap E_0$ is nonempty.

If (6.16) holds, we have $h_M(-1, 1, 0, 0) = 0$ and we deduce that

$$0 = \max\{\lambda_1 \in [0, 1] : (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}$$

which yields $M \subseteq E_0$. Analogously, (6.18) implies $M \subseteq E_1$.

Next, an application of (6.9) with $\Lambda = (0, \alpha, 0, \alpha)$ and $\eta = \alpha$, where again $\alpha \in [0, \frac{\pi}{4})$, yields

$$\arctan(h_M(0, 1, 0, 1) \tan\alpha) = \arctan(h_M(1, 0, 1, 0) \tan\alpha) + \alpha$$

Clearly, this is possible if and only if either

$$h_M(0, 1, 0, 1) = 1, \quad h_M(1, 0, 1, 0) = 0, \tag{6.19}$$

or

$$h_M(0, 1, 0, 1) = 0, \quad h_M(1, 0, 1, 0) = -1. \tag{6.20}$$

However, (6.20) contradicts (6.8) and is therefore not possible.

From (6.19) and the fact that $M \subseteq E$, we infer

$$0 = \max\{\lambda_2 + \lambda_3 : \lambda_2, \lambda_3 \le 0 \text{ and } (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}$$

which implies

$$M \cap \{(0, \lambda_1, 0, 1 - \lambda_1) : \lambda_1 \in [0, 1]\} \neq \emptyset. \tag{6.21}$$

For the final part of the proof, we distinguish three cases:

(i) (6.15) and (6.18) hold, in particular, $M \subseteq E_1$;

(ii) (6.16) and (6.17) hold, in particular, $M \subseteq E_0$;

(iii) (6.15) and (6.17) hold.

In case (i), $M \subseteq E_1$ and (6.21) imply that $e_2 \in M$. Using (6.7), we conclude that $h_M(a, b, c, d) = b$, that is, $K * L = K$ for $(K, L) \in \mathcal{C}$.

Similarly, in case (ii), $M \subseteq E_0$ and (6.21) imply that $e_4 \in M$. Using again (6.7), we obtain $h_M(a, b, c, d) = d$, that is, $K * L = L$ for $(K, L) \in \mathcal{C}$.

It remains to show that in case (iii), we have e_2, $e_4 \in M$ which, by (6.7), implies that $h_M(a,b,c,d) = \max\{b,d\}$ or $K * L = \mathrm{conv}(K \cup L)$ for $(K,L) \in \mathcal{C}$. To this end, we apply again (6.9) with $\Lambda = (0,\alpha,0,0)$ and $\eta = \alpha$, where $\alpha \in [0,\frac{\pi}{4})$ to obtain

$$\arctan(h_M(0,1,0,0)\tan\alpha) = \arctan(h_M(1,0,1,-1)\tan\alpha) + \alpha$$

This is possible if and only if either

$$h_M(0,1,0,0) = 1, \quad h_M(1,0,1,-1) = 0, \tag{6.22}$$

or

$$h_M(0,1,0,0) = 0, \quad h_M(1,0,1,-1) = -1. \tag{6.23}$$

Assume that (6.23) holds. Then, by (6.7), (6.8), and the subadditivity of h_M, we obtain

$$-1 \le h_M(1,1,1,-1) \le h_M(1,0,1,-1) + h_M(0,1,0,0) = -1 \tag{6.24}$$

Hence, $h_M(1,1,1,-1) = -1$.

Now, consider the convex bodies $\overline{K} = [-e_2,e_2]$ and $\overline{L} = \{e_1\}$ in $\mathbb{R}^n$. Then, $h_{\overline{K}}(x) = |e_2 \cdot x|$ and $h_{\overline{L}}(x) = e_1 \cdot x$ for $x \in \mathbb{R}^n$, and we obtain from (6.15) and $h_M(1,1,1,-1) = -1$,

$$\begin{aligned} h_M(h_{\overline{K}}(e_1), h_{\overline{K}}(-e_1), h_{\overline{L}}(e_1), h_{\overline{L}}(-e_1)) &= h_M(0,0,1,-1) = 0,\\ h_M(h_{\overline{K}}(e_1+e_2), h_{\overline{K}}(-e_1-e_2), h_{\overline{L}}(e_1+e_2), h_{\overline{L}}(-e_1-e_2)) &= h_M(1,1,1,-1) = -1,\\ h_M(h_{\overline{K}}(e_1-e_2), h_{\overline{K}}(e_2-e_1), h_{\overline{L}}(e_1-e_2), h_{\overline{L}}(e_2-e_1)) &= h_M(1,1,1,-1) = -1. \end{aligned}$$

Since $h_M(h_{-\overline{K}}, h_{\overline{K}}, h_{-\overline{L}}, h_{\overline{L}})$ defines a support function of a convex body $\overline{Z}$ in $\mathbb{R}^n$, we infer

$$0 = h_{\overline{Z}}(-2e_1) \ge h_{\overline{Z}}(-e_1-e_2) + h_{\overline{Z}}(-e_1+e_2) = -2$$

which contradicts the subadditivity of $h_{\overline{Z}}$. Thus, (6.23) cannot hold.

Another application of (6.9) with $\Lambda = (0,\alpha,\alpha,0)$ and $\eta = -\alpha$, where $\alpha \in [0,\frac{\pi}{4})$, yields

$$\arctan(h_M(0,1,1,0)\tan\alpha) = \arctan(h_M(\tan(-\alpha,2\alpha,0,\alpha))) - \alpha.$$

Consequently,

$$h_M\left(-\frac{\tan\alpha}{\tan(2\alpha)}, 1, 0, \frac{\tan\alpha}{\tan(2\alpha)}\right) = \frac{\tan(\arctan(h_M(0,1,1,0)\tan\alpha)+\alpha)}{\tan(2\alpha)}.$$

By letting $\alpha \to \frac{\pi}{4}$ and using (6.22), we deduce that $h_M(0,1,1,0) = 1$. Since $M \subseteq E$, this yields

$$1 = \max\{\lambda_1 + \lambda_2 + \lambda_3 : (\lambda_2, \lambda_1 + \lambda_2, \lambda_3, 1 - \lambda_1 + \lambda_3) \in M\}$$

which, in turn, implies that $e_2 \in M$.

The proof that $e_4 \in M$ is now very similar. We first use (6.9) with $\Lambda = (0,0,0,\alpha)$ and $\eta = \alpha$ to deduce that

$$h_M(0,0,0,1) = 1, \quad h_M(1,-1,1,0) = 0.$$

Using this and another application of (6.9) with $\Lambda = (\alpha, 0, 0, \alpha)$ and $\eta = -\alpha$, finally leads to $h_M(1,0,0,1) = 1$. From this and $M \subseteq E$, follows $e_4 \in M$ which completes the proof. ∎

Using Theorem 6.1, we can now also complete the proof of Theorem 1:

Theorem 6.2 *An operation* $*\colon \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n) \to \mathcal{K}^p(\mathbb{S}^n)$ *is projection covariant and continuous if and only if either* $K * L = K$*, or* $K * L = -K$*, or* $K * L = L$*, or* $K * L = -L$ *for all* $K, L \in \mathcal{K}^p(\mathbb{S}^n)$*.*

Proof. By Theorem 6.1, it is sufficient to prove that the convex hull does not admit a continuous extension to a map from $\mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ to $\mathcal{K}^p(\mathbb{S}^n)$. In order to show this, let $u \in \mathbb{S}^n$, $v \in \mathbb{S}_u$, and consider the spherical segments $K = I_u^v(-\frac{\pi}{2}, 0)$ and $L_\varepsilon = I_u^v(0, \frac{\pi}{2} - \varepsilon)$, where $\varepsilon > 0$. Then $(K, L_\varepsilon) \in \mathcal{C}$ converges in the Hausdorff metric to $(K, L_0) \in \mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ as $\varepsilon \to 0^+$. However,

$$\lim_{\varepsilon \to 0^+} \operatorname{conv}(K \cup L_\varepsilon) = \lim_{\varepsilon \to 0^+} I_u^v\left(-\frac{\pi}{2}, \frac{\pi}{2} - \varepsilon\right) = I_u^v\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \notin \mathcal{K}^p(\mathbb{S}^n).$$

∎

We remark, that it is also not difficult to show that the convex hull is *not* continuous as a map from $\mathcal{K}^p(\mathbb{S}^n) \times \mathcal{K}^p(\mathbb{S}^n)$ to $\mathcal{K}(\mathbb{S}^n)$.

7. Section covariant operations

In this final section, first we briefly recall a characterization of rotation and section covariant operations between Euclidean star sets established in [**15**]. Than, we discuss basic properties of spherical star sets in order to eventually prove a corresponding result to Theorem 2 for rotation and section covariant operations between them.

A subset L of $\mathbb{R}^n$ is called *star-shaped* with respect to o if every line through the origin intersects L in a (possibly degenerate) closed line segment. A *star set* in $\mathbb{R}^n$ is a compact set that is star-shaped with respect to o. The radial function $\rho(L,\cdot)=\rho_L:\mathbb{R}^n\backslash\{o\}\to[0,\infty)$ of a star set L is defined by

$$\rho(L,x)=\max\{\lambda\geq 0:\lambda x\in L\},\qquad x\in\mathbb{R}^n\backslash\{o\}.$$

Radial functions are -1-homogeneous, that is, $\rho(L,\lambda x)=\lambda^{-1}\rho(L,x)$ for all $x\in\mathbb{R}^n\backslash\{o\}$ and $\lambda>0$, and are therefore often regarded as functions on $\mathbb{S}^{n-1}$. If $\rho(L,\cdot)$ is positive and continuous, we call L a *star body*. If $K\in\mathcal{K}(\mathbb{R}^n)$ contains the origin in its interior, then K is a star body and we have

$$\rho(K^*,\cdot)=\frac{1}{h(K,\cdot)}\qquad\text{and}\qquad h(K^*,\cdot)=\frac{1}{\rho(K,\cdot)},\tag{7.1}$$

where K^* denotes the *polar body* of K defined by

$$K^*=\{x\in\mathbb{R}^n:x\cdot y\leq 1\text{ for all }y\in K\}.$$

The *radial distance* $\widetilde{\delta}(K,L)$ between two star sets K and L in $\mathbb{R}^n$ is defined by

$$\widetilde{\delta}(K,L)=\|\rho(K,\cdot)-\rho(L,\cdot)\|_\infty.\tag{7.2}$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all star sets in $\mathbb{R}^n$ endowed with the radial distance. The *radial sum* $K\,\widetilde{+}\,L$ of K, $L\in\mathcal{S}(\mathbb{R}^n)$ is defined as the star set such that

$$\rho(K\,\widetilde{+}\,L,\cdot)=\rho(K,\cdot)+\rho(L,\cdot).$$

More generally, for any $p>0$, the L_p *radial sum* $K\,\widetilde{+}_p\,L$ of K, $L\in\mathcal{S}(\mathbb{R}^n)$ is defined by

$$\rho(K\,\widetilde{+}_p\,L,\cdot)^p=\rho(K,\cdot)^p+\rho(L,\cdot)^p.$$

Lutwak [**29**] showed that in the same way as the L_p Minkowski addition leads to the L_p Brunn–Minkowski theory, L_p radial addition leads to a dual L_p Brunn–Minkowski theory (see also [**14**] and the references therein).

While L_p radial addition is *not* projection covariant, the L_p radial sum of star sets is *section covariant*, that is,

$$(K \cap V) \,\widetilde{+}_p\, (L \cap V) = (K \,\widetilde{+}_p\, L) \cap V$$

for every linear subspace V of $\mathbb{R}^n$. It is also $\mathrm{GL}(n)$ covariant and therefore, in particular, covariant with respect to rotations.

A complete classification of all rotation and section covariant binary operations between star sets in $\mathbb{R}^n$ was established by Gardner, Hug, and Weil and can be stated as follows:

Theorem 7.1 **([15])** *An operation* $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is rotation and section covariant if and only if there exists a function* $f : [0,\infty)^4 \to \mathbb{R}$ *such that, for all* K, $L \in \mathcal{S}(\mathbb{R}^n)$,

$$\rho_{K*L}(v) = f(\rho_{-K}(v), \rho_K(v), \rho_{-L}(v), \rho_L(v)), \qquad v \in \mathbb{S}^{n-1}.$$

We turn now to star sets in $\mathbb{S}^n$. We call a subset L of $\mathbb{S}^n$ a *(spherical) star set* with respect to $u \in L$ if $L \cap S_{u,v}$ is a (possibly degenerate) closed spherical segment for all $v \in \mathbb{S}_u$. We denote by $\mathcal{S}_u(\mathbb{S}^n)$ the class of all spherical star sets with respect to u and we write $\mathcal{S}_u^p(\mathbb{S}^n)$ for the subclass of *proper star sets* with respect to u, that is, star sets with respect to u contained in $\mathbb{S}_u^+$.

Definition For $u \in \mathbb{S}^n$ and a proper star set $L \in \mathcal{S}_u^p(\mathbb{S}^n)$, the *spherical radial function* $\rho_u(L, \cdot) : \mathbb{S}_u \to [0, \frac{\pi}{2})$ of L is defined by

$$\rho_u(L, v) = \max\{\alpha \geq 0 : u \cos\alpha + v \sin\alpha \in L\}.$$

Note that, for every $v \in \mathbb{S}_u$, we have

$$u \cos\rho_u(L, v) + v \sin\rho_u(L, v) \in \partial L.$$

The counterparts to Lemma 4.2 (c) and Lemma 4.3 in the setting of spherical star sets are the contents of our next lemma.

Lemma 7.2 *For* $u \in \mathbb{S}^n$*, the following statements hold:*

(a) The gnomonic projection maps $\mathcal{S}_u^p(\mathbb{S}^n)$ *bijectively to* $\mathcal{S}(\mathbb{R}^n_u)$.

(b) For every $L \in \mathcal{S}_u^p(\mathbb{S}^n)$*, we have*

$$\rho(g_u(L), v) = \tan\rho_u(L, v), \qquad v \in \mathbb{S}_u.$$

Proof. Statement (a) is an immediate consequence of Lemma 4.2 (a) and (b). From Lemma 4.2 (a) and the definitions of radial and spherical radial functions, we obtain

$$\begin{aligned}\rho(g_u(L), v) &= \max\{\lambda \geq 0 : \lambda v \in g_u(L)\} \\ &= \max\left\{\lambda \geq 0 : \frac{u + \lambda v}{\|u + \lambda v\|} = \frac{1}{\sqrt{1+\lambda^2}}\, u + \frac{\lambda}{\sqrt{1+\lambda^2}}\, v \in L\right\} \\ &= \tan \max\{\alpha \in [0, \tfrac{\pi}{2}) : u \cos\alpha + v \sin\alpha \in L\} \\ &= \tan \rho_u(L, v)\end{aligned}$$

which proves (b). ∎

By Lemma 7.2 (b), a function $\rho : \mathbb{S}_u \to [0, \frac{\pi}{2})$ is the spherical radial function of a star set $L \in \mathcal{S}_u^p(\mathbb{S}^n)$ if and only if the -1-homogeneous extension of $\tan\rho$ to $\mathbb{R}_u^n$ is the radial function of a star set in $\mathbb{R}_u^n$.

We call a proper star set $L \in \mathcal{S}_u^p(\mathbb{S}^n)$ a *(spherical) star body* with respect to $u \in \mathbb{S}^n$ if $\rho_u(L, \cdot)$ is positive and continuous. Clearly, every proper convex body $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ containing u in its interior is a star body with respect to u. In order to establish a counterpart to (7.1), we recall that, for $K \in \mathcal{K}^p(\mathbb{S}^n)$ with non-empty interior, the polar body $K^\circ \in \mathcal{K}^p(\mathbb{S}^n)$ is defined by

$$K^\circ = \{v \in \mathbb{S}^n : v \cdot w \leq 0 \text{ for all } w \in K\} = \mathbb{S}^n \setminus \operatorname{int} K_{\frac{\pi}{2}}.$$

Note that if $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ contains u in its interior, then $K^\circ \in \mathcal{K}_{-u}^p(\mathbb{S}^n)$ contains $-u$ in its interior.

Proposition 7.3 *If $u \in \mathbb{S}^n$ and $K \in \mathcal{K}_u^p(\mathbb{S}^n)$ contains u in its interior, then*

$$g_u(K)^* = g_{-u}(K^\circ) \tag{7.3}$$

and

$$h_u(K, \cdot) + \rho_{-u}(K^\circ, \cdot) = \frac{\pi}{2}. \tag{7.4}$$

Proof. By the definitions of the Euclidean and spherical polar bodies and the gnomonic projection, we have

$$\begin{aligned}g_u(K)^* &= \{x \in \mathbb{R}_u^n : x \cdot y \leq 1 \text{ for all } y \in g_u(K)\} \\ &= \{x \in \mathbb{R}_u^n : x \cdot g_u(w) \leq 1 \text{ for all } w \in K\} \\ &= \{x \in \mathbb{R}_u^n : w \cdot x \leq w \cdot u \text{ for all } w \in K\} \\ &= \left\{x \in \mathbb{R}_u^n : w \cdot \frac{x - u}{\|x - u\|} \leq 0 \text{ for all } w \in K\right\} \\ &= g_{-u}\left(\{v \in \mathbb{S}^n : v \cdot w \leq 0 \text{ for all } w \in K\}\right) = g_{-u}(K^\circ).\end{aligned}$$

which proves (7.3). Lemma 4.3, (7.1), (7.3), and Lemma 7.2 (b), now yield

$$\tan h_u(K,\cdot) = h(g_u(K),\cdot) = \frac{1}{\rho(g_u(K)^*,\cdot)} = \frac{1}{\rho(g_{-u}(K^\circ),\cdot)} = \frac{1}{\tan \rho_{-u}(K^\circ,\cdot)}$$

which is equivalent to (7.4). ■

Using spherical radial functions, we define a metric $\widetilde{\gamma}_u$ on $\mathcal{S}_u^p(\mathbb{S}^n)$ by

$$\widetilde{\gamma}_u(K,L) = \sup_{v\in\mathbb{S}_u} |\rho_u(K,v) - \rho_u(L,v)|.$$

Note that if $K, L \in \mathcal{S}_u^p(\mathbb{S}^n)$, then by (7.2) and Lemma 7.2 (b),

$$\widetilde{\delta}(g_u(K), g_u(L)) = \sup_{v\in\mathbb{S}_u} |\tan \rho_u(K,v) - \tan \rho_u(L,v)|.$$

Thus, from the continuity of the tangent we obtain the following.

Theorem 7.4 *The gnomonic projection is a homeomorphism between* $(\mathcal{S}_u^p(\mathbb{S}^n), \widetilde{\gamma}_u)$ *and* $(\mathcal{S}(\mathbb{R}_u^n), \widetilde{\delta})$.

For fixed $u \in \mathbb{S}^n$ we call a binary operation $* : \mathcal{S}_u^p(\mathbb{S}^n) \times \mathcal{S}_u^p(\mathbb{S}^n) \to \mathcal{S}_u^p(\mathbb{S}^n)$ *u-section covariant* if for all k-spheres S, $1 \le k \le n-1$, with $u \in S$ and for all $K, L \in \mathcal{S}_u^p(\mathbb{S}^n)$, we have

$$(K \cap S) * (L \cap S) = (K * L) \cap S.$$

The operation $*$ is called *u-rotation covariant* if $(\vartheta K) * (\vartheta L) = \vartheta(K * L)$ for all $\vartheta \in \mathrm{SO}(n+1)$ which fix u. Our next result is a version of Theorem 2 (or Theorem 4.6, respectively) in the setting of star sets.

Theorem 7.5 *For $u \in \mathbb{S}^n$, the gnomonic projection g_u induces a one-to-one correspondence between operations $* : \mathcal{S}_u^p(\mathbb{S}^n) \times \mathcal{S}_u^p(\mathbb{S}^n) \to \mathcal{S}_u^p(\mathbb{S}^n)$ which are u-rotation and u-section covariant and operations $\overline{*} : \mathcal{S}(\mathbb{R}_u^n) \times \mathcal{S}(\mathbb{R}_u^n) \to \mathcal{S}(\mathbb{R}_u^n)$ which are rotation and section covariant. Moreover, any such operation $*$ is continuous if and only if $\overline{*}$ is continuous.*

Proof. First assume that $*$ is u-rotation and u-section covariant and define an operation $\overline{*} : \mathcal{S}(\mathbb{R}_u^n) \times \mathcal{S}(\mathbb{R}_u^n) \to \mathcal{S}(\mathbb{R}_u^n)$ by

$$K \,\overline{*}\, L = g_u(g_u^{-1}(K) * g_u^{-1}(L))$$

for $K, L \in \mathcal{S}(\mathbb{R}^n_u)$. As in the proof of Theorem 4.6, it follows that $\overline{*}$ is section covariant. The rotation covariance of $\overline{*}$ is a consequence of the u-rotation covariance of $*$ and the fact that $\vartheta g_u(L) = g_u(\vartheta L)$ for all $L \in \mathcal{S}^p_u(\mathbb{S}^n)$ and $\vartheta \in \mathrm{SO}(n+1)$ which fix u.

Conversely, if $\overline{*} : \mathcal{S}(\mathbb{R}^n_u) \times \mathcal{S}(\mathbb{R}^n_u) \to \mathcal{S}(\mathbb{R}^n_u)$ is rotation and section covariant, then define $* : \mathcal{S}^p_u(\mathbb{S}^n) \times \mathcal{S}^p_u(\mathbb{S}^n) \to \mathcal{S}^p_u(\mathbb{S}^n)$ by

$$K * L = g_u^{-1}(g_u(K) \,\overline{*}\, g_u(L))$$

for $K, L \in \mathcal{S}^p_u(\mathbb{S}^n)$.

As before, it is easy to show that $*$ is u-rotation and u-section covariant and, by Theorem 7.4, the operation $*$ is continuous if and only if $\overline{*}$ is continuous. ■

We conclude with a corollary to Theorem 7.1 of Gardner, Hug, and Weil and Theorem 7.5.

Corollary 7.6 *For fixed $u \in \mathbb{S}^n$, an operation $* : \mathcal{S}^p_u(\mathbb{S}^n) \times \mathcal{S}^p_u(\mathbb{S}^n) \to \mathcal{S}^p_u(\mathbb{S}^n)$ is u-rotation and u-section covariant if and only if there exists a function $f : [0, \frac{\pi}{2})^4 \to [0, \frac{\pi}{2})$ such that, for all $K, L \in \mathcal{S}^p_u(\mathbb{S}^n)$,*

$$\rho_u(K * L, v) = f(\rho_u(K, -v), \rho_u(K, v), \rho_u(L, -v), \rho_u(L, v)), \qquad v \in \mathbb{S}_u.$$

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