Equations in oligomorphic algebras

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Outline

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I: Finite Taylor algebras & Constraint Satisfaction Problems

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II: Infinite domains: oligomorphicity

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III: Oligomorphic “Taylor" algebras

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IV: Linear equations

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V: Open problems

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ξ clone homomorphism ("ξ preserves equations"):

- ξ preserves arities
- ξ preserves projections
- $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$.

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- **A** has cyclic term $c(x_1, \ldots, x_n) = c(x_2, \ldots, x_n, x_1)$
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$\implies$ **1** $\in$ HSP$^{\text{fin}}$(Pol($\mathbb{A}$)) $\implies$ $\mathbb{A}$ can simulate ("pp-interpret")

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II: Infinite domains: oligomorphicity

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For every $n \geq 1$, there are only finitely many n-tuples
in the algebra / clone / structure modulo the group.

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$(f_i)_{i\in\omega} \to f :\leftrightarrow f_i(\bar{a}) = f(\bar{a})$ eventually, for all $\bar{a}$.

(f_i, f of same arity; “sorts" are clopen sets)

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$(f_i)_{i\in\omega} \to f :\leftrightarrow f_i(\bar{a}) = f(\bar{a})$ eventually, for all $\bar{a}$.

(f_i, f of same arity; "sorts" are clopen sets)

Failure of the above $\Leftrightarrow$ something positive, and algebraic?

Oligomorphicity vs. Idempotency

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Every oligomorphic structure $\mathbb{A}$ is homomorphically equivalent to a unique oligomorphic model-complete core $\mathbb{A}^c$:

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- Can only add finitely many $a \in \mathbb{A}^c$, so no idempotency

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III: Oligomorphic “Taylor" algebras

What is an oligomorphic Taylor algebra?

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- (3) is relevant for CSP $\implies$ our definition of “Taylor algebra"!

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- Criterion positive, algebraic, finite.

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Old Conjecture (reformulated)

Let $\mathbb{A}$ be a reduct of finitely bounded homogeneous structure ($\implies$ oligomorphic).

Then:

- Some stabilizer of $\text{Pol}(\mathbb{A}^c)$ has cont. clone homomorphism to $\mathbf{1}$ ($\implies$ $\text{CSP}(\mathbb{A})$ is NP-complete), or
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- Algebraic criterion in terms of Pol($\mathbb{A}^c$), not Pol($\mathbb{A}$)
- Relies on possibly non-optimal order:
  $\mathbb{A} \implies \mathbb{A}^c \implies$ stabilize $\implies$ pp-interpret

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New Conjecture (Barto + Opršal + P '14)

Let $\mathbb{A}$ be a reduct of finitely bounded homogeneous structure. Then:

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- Criterion in terms of Pol($\mathbb{A}$) rather than Pol($\mathbb{A}^c$).
- Avoids model-complete core $\mathbb{A}^c$.

IV: Linear equations

More oligomorphic Taylor notions

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Two statements for oligomorphic $\mathbb{A}$:

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(1) $\mathrm{Pol}(\mathbb{A})$ has no uniformly cont. h1 clone homomorphism to **1**.

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(1) $\Rightarrow$ **(2)**:
no u.c. h1 clone homomorphism from $\mathrm{Pol}(\mathbb{A}) \implies$
no u.c. (h1) clone homomorphism from any stabilizer of $\mathrm{Pol}(\mathbb{A}^c) \implies$
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Theorem (Barto + Kompatscher + Olšák + Pham + P '16)

For the countable atomless Boolean algebra $\mathbb{A}$:

- $\mathbb{A}$ is oligomorphic model-complete core;
- Pol($\mathbb{A}$) has uniformly cont. h1 clone homomorphism to **1**;
- Pol($\mathbb{A}$) has pseudo-Siggers function.

Orbit growth!

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Theorem (Barto + Kompatscher + Olšák + Pham + P '16)

Let $\mathbb{A}$ be oligomorphic model-complete core such that:

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Proof. Reducts of finitely bounded homogeneous structures have at most exponential orbit growth.

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Remark. Higher-arity structure of $\mathrm{Pol}(\mathbb{A}) \implies$ structure of $\mathrm{Aut}(\mathbb{A})$!

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Theorem (Barto + Kompatscher + Olšák + Pham + P '16)

Let $\mathbb{A}$ be a reduct of finitely bounded homogeneous structure $\mathbb{D}$.

Suppose Pol($\mathbb{A}$) contains function $f(x_1, \ldots, x_k)$ for large enough k such that for all permutations σ of $\{1, \ldots, k\}$

$$u_\sigma\, f(x_1, \ldots, x_k) = v_\sigma\, f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

for unary $u_\sigma, v_\sigma \in \mathrm{End}(\mathbb{D})$.

Then Pol($\mathbb{A}$) satisfies non-trivial linear equations.

Examples

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Theorem (Barto + Kompatscher + Olšák + Pham + P ’16)

If $\mathbb{A}$ is a reduct of any of the above structures, then:

- Pol($\mathbb{A}$) has uniformly cont. h1 clone homomorphism to **1**, and CSP($\mathbb{A}$) is NP-complete, or
- Pol($\mathbb{A}$) satisfies non-trivial linear equations, and CSP($\mathbb{A}$) is in P.

V: Open problems

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If $\text{Pol}(\mathbb{A})$ has an h1 clone homomorphism to $\mathbf{1}$,
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Reference

L. Barto, M. Kompatscher, M. Olšák, T. V. Pham, and M. Pinsker

Equations in oligomorphic clones and the Constraint Satisfaction Problem for ω-categorical structures

Preprint arXiv:1612.07551

Thank you!