Permutations on the random permutation

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Homogeneous structures

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Then there exists a unique (up to isomorphism) countable homogeneous structure Δ whose age is $\mathcal{C}$.

Such a structure is called the Fraïssé limit of $\mathcal{C}$.

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Problem

Classify the reducts of a homogeneous structure up to first-order interdefinability, existential-positive interdefinability, etc.

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- classifying computational complexity of constraint satisfaction problems

Closed groups

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A permutation group $G \leq \mathrm{Sym}(X)$ is closed iff $h \in G$ whenever for all finite $A \subseteq X$ there exists $g \in G$ which agrees with h on A.

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Theorem (Corollary of Ryll-Nardzewski, Engeler, Svenonius)

If Δ is homogeneous in a finite relational language, then

$$\begin{aligned}\{\text{reducts of } \Delta\}/\!\sim \; &\to \{\text{closed supergroups of } \mathrm{Aut}(\Delta)\}\\ \Gamma/\!\sim \; &\mapsto \mathrm{Aut}(\Gamma)\end{aligned}$$

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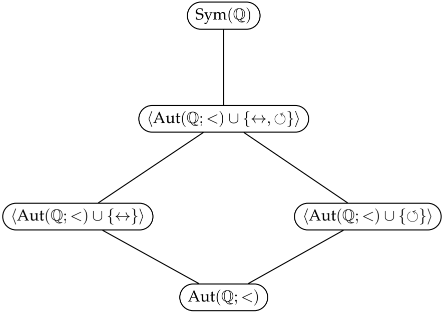

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- a bijection $A \to A$
- a relational structure $(A; <_1, <_2)$

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Question (Cameron, 2002)

What are the closed supergroups of $\mathrm{Aut}(\Pi)$?

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Then $(D; <_1, <_2) \cong \Pi$.

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Theorem (Linman and Pinsker, 2014)

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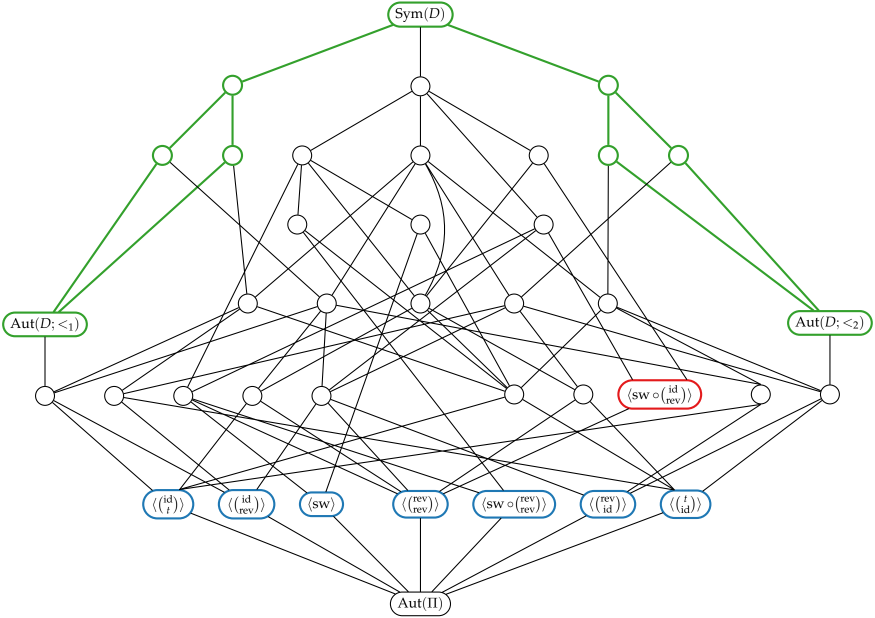
- $\binom{t}{\mathrm{id}}$



$\mathrm{Sym}(D)$
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$\mathrm{Aut}(D;<_2)$
$\langle \mathrm{sw} \circ \binom{\mathrm{id}}{\mathrm{rev}} \rangle$
$\langle \binom{\mathrm{id}}{t} \rangle$
$\langle \binom{\mathrm{id}}{\mathrm{rev}} \rangle$
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Asymmetry in the roles of $\binom{\mathrm{id}}{\mathrm{rev}}$ and $\binom{\mathrm{id}}{t}$

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$$\langle \binom{\mathrm{rev}}{\mathrm{rev}} \rangle = \langle \binom{\mathrm{id}}{\mathrm{rev}} \circ \binom{\mathrm{rev}}{\mathrm{id}} \rangle \subsetneq \langle \binom{\mathrm{id}}{\mathrm{rev}}, \binom{\mathrm{rev}}{\mathrm{id}} \rangle$$

$$\langle \binom{\mathrm{id}}{t} \circ \binom{t}{\mathrm{id}} \rangle = \langle \binom{\mathrm{id}}{t}, \binom{t}{\mathrm{id}} \rangle$$

Closed transformation monoids

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A first-order formula is called existential-positive iff it is of the form

$$\exists x_1, \ldots, x_n \psi_1 \wedge \cdots \wedge \psi_m,$$

where each ψ_i is a disjunction of atomic formulas.

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Theorem (Bodirsky and Pinsker, 2012)

If Δ is countable and ω-categorical, then

$$\begin{aligned} \{\text{reducts of } \Delta\}/\sim &\to \{\text{closed monoids containing } \operatorname{Aut}(\Delta)\} \\ \Gamma/\sim &\mapsto \operatorname{End}(\Gamma) \end{aligned}$$

is an antiisomorphism.

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- $\mathcal{M}$ has a constant operation.
- The permutations in $\mathcal{M}$ form a group which is a dense subset of $\mathcal{M}$ in D^D.

In other words, if Γ is a reduct of Π, either Γ has a constant endomorphism or all endomorphisms of Γ can be interpolated on finite sets by automorphisms of Γ.

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Lemma (Bodirsky and Pinsker, 2012)

A countable ω-categorical structure Δ is model-complete iff $\mathrm{Aut}(\Delta)$ is dense in $\mathrm{Emb}(\Delta)$.

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Lemma (Bodirsky and Pinsker, 2012)

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Corollary (Linman, 2014)

All reducts of Π are model-complete.

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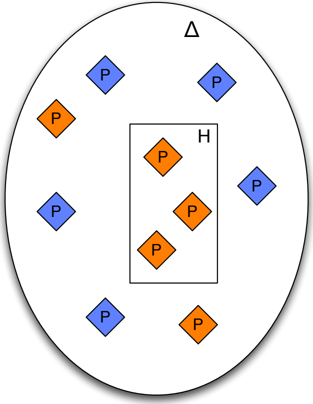

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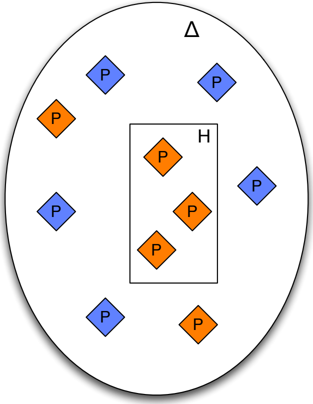

Theorem (Böttcher and Foniok, 2011)

The random permutation is a Ramsey structure.

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- $\binom{\text{id}}{t}$ is canonical from (Π, c) to Π

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We say that $\mathcal{F} \subseteq D^D$ generates a function $g : D \to D$ iff for all finite $A \subseteq D$ there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that $f_1 \circ \cdots \circ f_n$ agrees with g on A.

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Theorem (Bodirsky, Pinsker, Tsankov, 2011)

Let Δ be a structure which is ordered Ramsey and homogeneous in a finite relational language. Let $c_1, \ldots, c_n \in \Delta$ and $f : \Delta \to \Delta$ be a function. Then $\{f\} \cup \mathrm{Aut}(\Delta)$ generates a function which

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- agrees with f on $\{c_1, \ldots, c_n\}$

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- is closed under composition
- contains all projections

Examples

- the projection clone

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- the projection clone
- the polymorphism clone of a structure Δ: the set of homomorphisms $\Delta^n \to \Delta$, for all $n \geq 1$

Closed clones

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A first-order formula is called primitive-positive iff it is of the form

$$\exists x_1, \ldots, x_n \, \psi_1 \wedge \cdots \wedge \psi_m,$$

where each ψ_i is an atomic formula.

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$$\exists x_1, \ldots, x_n \, \psi_1 \wedge \cdots \wedge \psi_m,$$

where each ψ_i is an atomic formula.

Theorem (Bodirsky and Nešetřil, 2006)

If Δ is countable and ω-categorical, then

$$\begin{aligned} \{\text{reducts of } \Delta\}/\!\sim\; &\to \{\text{closed clones containing } \operatorname{Aut}(\Delta)\} \\ \Gamma/\!\sim\; &\mapsto \operatorname{Pol}(\Gamma) \end{aligned}$$

is an antiisomorphism.

Constraint satisfaction problems

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Let Γ be a structure in a finite relational language τ. CSP(Γ) is the computational problem of deciding whether a given primitive-positive τ-sentence holds in Γ.

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Theorem (Bulatov, Krokhin, Jeavons, 2000)

Let $\Gamma = (D; R_1, \ldots, R_n)$ be a structure and let R be a relation with a primitive-positive definition in Γ. Then $\mathrm{CSP}(D; R_1, \ldots, R_n)$ and $\mathrm{CSP}(D; R_1, \ldots, R_n, R)$ are polynomial-time equivalent.

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Therefore, the complexity of $\mathrm{CSP}(\Gamma)$ depends only on $\mathrm{Pol}(\Gamma)$.

Problem

Classify the computational complexity of $\mathrm{CSP}(\Gamma)$ for all reducts Γ of Π.

Related problems

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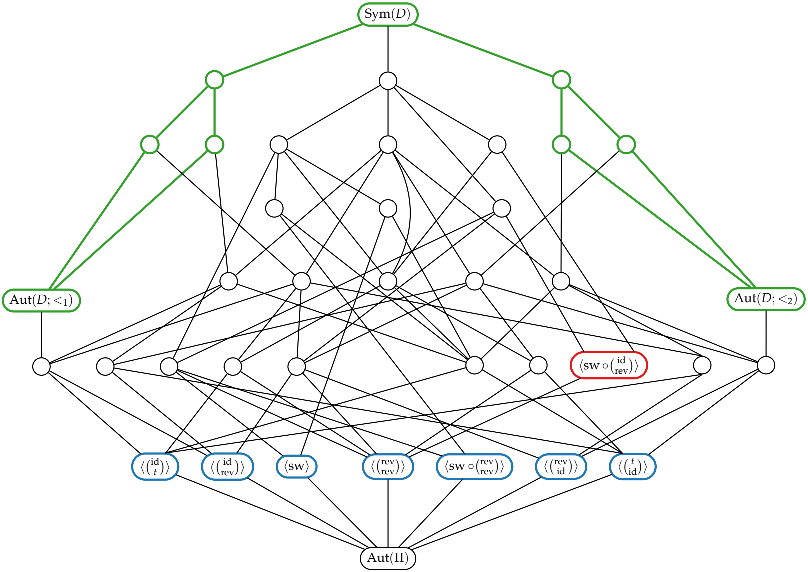
- Does Thomas's conjecture hold for Ramsey structures?
- Does every structure which is homogeneous in a finite relational language have a homogeneous Ramsey expansion?

Thank you!


Sym(D)
Aut(D; $<_1$)
Aut(D; $<_2$)
$\langle sw \circ \binom{id}{rev} \rangle$
$\langle \binom{id}{t} \rangle$
$\langle \binom{id}{rev} \rangle$
$\langle sw \rangle$
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Aut(Π)