Algebraic and model-theoretic methods in constraint satisfaction

4th and last session

Michael Pinsker

Technische Universität Wien / Université Diderot - Paris 7

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Doc-Course, Charles University Prague

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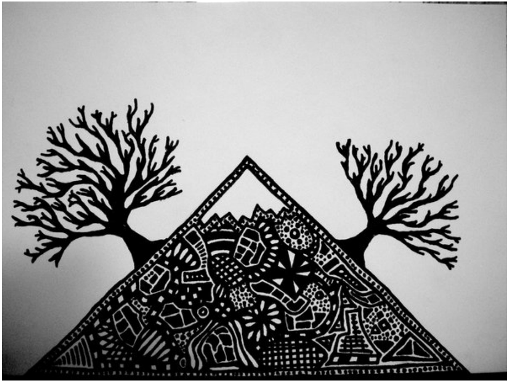
Outline reminder

Part I: CSPs / dividing the world /
pp definitions, polymorphism clones, ω-categoricity

Part II: pp interpretations / topological clones

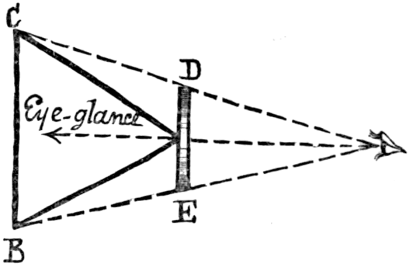
Part III: Canonical functions, Ramsey structures / Graph-SAT

Part IV: Model-complete cores / The infinite tractability conjecture



Part IV:

Model-complete cores / The infinite tractability conjecture



Model-complete cores

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τ-structures Γ, Δ are homomorphically equivalent iff
Γ maps homomorphically into Δ and vice-versa.

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Could possibly obtain hard structure Δ by pp interpretations + homomorphic equivalence, but not by pp interpretations only.

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Note: Property of the topological clone $\mathrm{Pol}(\Delta)$.

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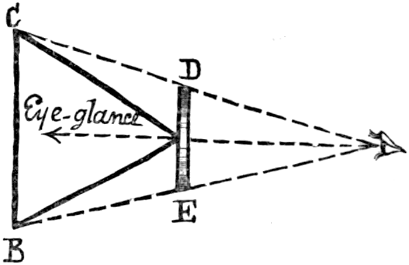
- mc core of $(\mathbb{Q}; \leq)$: one-element poset.



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Let Γ be finite. Let Δ be its mc core expanded by all constants. Then:

- either Pol(Δ) has a homomorphism to **1** (and CSP(Γ) is NP-hard),
- or Pol(Δ) contains a cyclic operation f of arity $n > 1$, i.e.,

$$f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$$

and CSP(Γ) *is in P*.

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- or there are $f(x_1, x_2) \in \mathrm{Pol}(\Gamma)$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{Q}; <)$ such that

$$f(x_1, x_2) = \alpha(f(\beta x_2, \beta x_1))$$

and $\mathrm{CSP}(\Gamma)$ is in P.

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  So they allow for combinatorial analysis and algorithmic use, and "should" satisfy equations.

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Conjecture (Bodirsky + MP '11)

Let Γ be a reduct of a finitely bounded homogeneous structure.

Then CSP(Γ) is in P or NP-complete.

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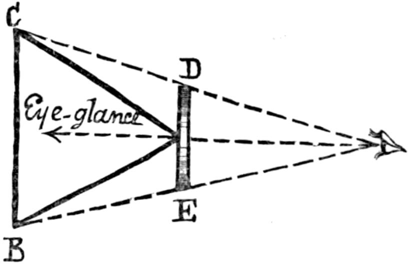
- either there is an expansion Γ' of Γ by finitely many constants such that $\mathrm{Pol}(\Gamma')$ has a continuous homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard);
- $\mathrm{Pol}(\Gamma)$ satisfies a non-trivial equation, and $\mathrm{CSP}(\Gamma)$ is tractable.



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- Clarify relationship between canonical functions and their finite counterparts (algorithmic / equational).

Distress not yourself if you cannot at first understand the deeper mysteries of Spaceland. By degrees they will dawn upon you.